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AN INTERPOLATION PROBLEM FOR SUBALGEBRAS OF H^{∞}

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Let E be a closed subset of the unit circle $C = \{z : |z| = 1\}$ and denote by B_E the algebra of all functions which are bounded and continuous on the set $X = \{z : |z| \leq 1, z \notin E\}$ and analytic in the open disc $D = \{z : |z| < 1\}$. An interpolation set for B_{B} is a relatively closed subset S of X with the property that if α is a bounded and continuous function on S (all functions are complex-valued), there is a function f in B_E such that $f(z) = \alpha(z)$ for every $z \in S$. The main result of the paper characterizes the interpolation sets for B_E as those sets S for which $S \cap D$ is an interpolation set for H^{∞} and $S \cap (C-E)$ has Lebesgue measure 0. If, in addition, $S \cap D = \phi$ then S is a peak interpolation set for B_E . Also, through a construction process inspired by recent work of J. P. Kahane, it is shown that the existence of peak points for a sup norm algebra of continuous functions on a compact, connected space implies the existence of infinite interpolation sets relative to the algebra and certain of its weak extensions.

The solution of the interpolation problem in the space $H^{\infty} = B_c$ of bounded analytic functions on D is due to Lennart Carleson [5], and due to A. Beurling and Walter Rudin in the disc algebra $A = B_{\phi}$ [10]. Concerning the latter case see also the notes of Lennart Carleson [5] and the last problem in Hoffman's book [8]. Their results are given by the following two theorems.

THEOREM C. A sequence $\{z_k\}$ of distinct points in D is an interpolation set for H^{∞} if and only if it is uniformly separated¹, that is, if and only if there exists a positive number δ such that

(1)
$$\prod_{j=1: j \neq k}^{\infty} \left| \frac{z_j - z_k}{1 - \overline{z}_j z_k} \right| \geq \delta \qquad (k = 1, 2, \cdots).$$

Whenever this condition holds, a constant $m(\delta)$ exists with the property that for any bounded sequence $\{w_k\}$ there is an f in H^{∞} such that $f(z_k) = w_k$ $(k = 1, 2, \cdots)$ and $||f|| \leq m(\delta) \sup_k |w_k|$.

THEOREM B-R. A closed subset S of \overline{D} is an interpolation set for A if and only if

(i) $S \cap D$ is uniformly separated,

¹ This terminology is due to Professor Peter Duren.

and

(ii) $S \cap C$ has Lebesgue measure 0.

In the terminology introduced above our characterization of the interpolation sets for the algebra B_E takes the following form.

THEOREM 1. The relatively closed subset S of X is an interpolation set for B_E if and only if

(i) $S \cap D$ is uniformly separated, and

(ii) $S \cap (C - E)$ has Lebesgue measure 0.

For example, suppose $E = \{1\}$ and S is the union of the sequences $a_k = 1 - 2^{-k}$ $(k = 1, 2, \cdots)$ with any sequence $\{b_k\}$ of distinct points on C converging to 1 $(b_k \neq 1)$. For a proof that $\{a_k\}$ is uniformly separated, see [8, p. 204]. Our result then applies and asserts that for any pair of bounded sequences $\{\alpha_k\}$ and $\{\beta_k\}$ there exists a function f in H^{∞} , continuous on $\overline{D} - \{1\}$, such that $f(a_k) = \alpha_k$ and $f(b_k) = \beta_k$ $(k = 1, 2, \cdots)$. For $S = \{b_k\}$ alone this is a result of E. L. Stout [11, Lemma 4.1].

Our proof of Theorem 1, presented in §2, depends on Theorem C and the generalized Rudin-Carleson theorem [4]. We also show in §2 that the interpolation sets for B_E which are subsets of C - E have the property that every bounded continuous function α on S has an extension f in B_E with $||f|| = ||\alpha||$ (all norms are supremum norms on the relevant domains). In §3 we present an argument which, in particular, shows that the existence of peak sets for the disc algebra A implies the existence of infinite interpolation sets for H^{∞} .

2. Interpolation in B_{E} . First we shall deal with those interpolation sets for B_{E} which are contained in D. Naturally, such sets are countable.

LEMMA 1. A sequence $\{z_k\}$ of distinct points in D is an interpolation set for B_E if and only if it is uniformly separated and all of its limit points belong to E. If this condition is satisfied then there is a constant $m(\delta/2)$ such that if $\{w_k\}$ is a bounded sequence there exists an f in B_E such that

- (i) $f(z_k) = w_k \ (k = 1, 2, \cdots),$
- (ii) $||f|| \leq m(\delta/2) \sup_k |w_k|.$

Proof. If $\{z_k\}$ is an interpolation set for B_E it is certainly one also for H^{∞} and is therefore uniformly separated by Theorem C. And if $e^{i\theta}$ is a limit point of $\{z_k\}$ there is a function in B_E which is dis-

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continuous there; hence E contains all the limit points of $\{z_k\}$.

Now suppose that the sequence $\{z_k\}$ of distinct points in D is uniformly separated with relevant constant δ and has all its limit points in E. It is no restriction to suppose in addition that no z_k is zero. (To avoid the situation covered by Theorem C we assume that E is a proper subset of the unit circle.) The Blaschke product

(2)
$$B(z) = \prod_{k=1}^{\infty} \frac{\overline{z}_k}{|z_k|} \frac{z_k - z}{1 - \overline{z}_k z}$$

and each of its subproducts represent functions analytic in the complement of the compact set K consisting of E together with the points $1/\overline{z}_k$ [8]. Thus B is analytic and of unit modulus at each point which belongs to one of the arcs β_1, β_2, \cdots in C complementary to E. This means that each point of β_n is the center of a small disc contained in the complement of K and on which B is analytic and $|B(z)| \leq 2$. Cover β_n by a countable and locally finite (relative to β_n) collection of such discs and let β'_n be that part of the boundary of the union of these discs which lies outside D. The set β'_n is a Jordan arc having the same endpoints as β_n , and, except for these endpoints it is contained in the complement of \overline{D} . Now let D^* be the simply connected domain containing D whose boundary is E together with the nonintersecting arcs β'_n $(n = 1, 2, \cdots)$. Clearly $|B(z)| \leq 2$ for $z \in D^*$.

Let $B(D^*)$ denote the space of functions bounded and analytic on D^* . If we can show that $\{z_k\}$ is an interpolation set for $B(D^*)$ the proof will be complete since $B(D^*) \subset B_E$. (In this connection compare Stout's general characterization of interpolation sets [11, Th. 5.9].) To this end choose a conformal map ϕ from D^* onto D and set

$$\phi(z_k) = y_k, f_k = B_k \circ \phi^{-1} (k = 1, 2, \cdots)$$

where B_k is the Blaschke product B with the k^{th} factor removed. For each $k, f_k \in H^{\infty}$, $||f_k|| \leq 2$, $|f_k(y_k)| \geq \delta$ (see (1)) and

$$|f_k(y_j)| = 0 \ (j \neq k)$$
.

If C_{ks} is the finite Blaschke product (see (2)) associated with the points $y_1, y_2, \dots, y_{k-1}, y_{k+1}, \dots, y_s$ $(1 \le k \le s, s = 2, 3, \dots)$, we have

$$rac{\delta}{\mid C_{ks}(y_k)\mid} \leqq rac{\mid f_k(y_k)\mid}{\mid C_{ks}(y_k)\mid} \leqq \left\| rac{f_k}{C_{ks}}
ight\| \leqq 2 \; ,$$

that is,

$$\prod_{j=1:\,j\neq k}^{s} \left| \frac{y_j - y_k}{1 - y_j y_k} \right| \geq \delta/2 \; .$$

This proves that $\{y_k\}$ is uniformly separated in D. Hence if $\{w_k\}$ is

a bounded sequence, there exists an f in H^{∞} such that $f(y_k) = w_k$ $(k = 1, 2, \dots)$ and $||f|| \leq m(\delta/2) \sup |w_k|$. The function $f \circ \phi$ is bounded and analytic on D^* , $f \circ \phi(z_k) = w_k$ and $||f \circ \phi|| \leq m(\varepsilon/2) \sup |w_k|$. This completes the proof.

A remark is in order concerning Lemma 1. In [1] Akutowicz and Carleson considered the general question of analytic continuation of interpolating functions. In the course of their work it was shown that if $\{z_k\}$ is an interpolation set for H^{∞} which clusters on the closed set E, then there exists a solution to the interpolation problem which has an analytic continuation to a larger domain obtained by pushing out through proper subarcs of finitely many of the complementary arcs β_1, β_2, \cdots [1, Th. 4]. Note that the interpolation function $f \circ \phi$ of the preceding argument is analytic in a domain which contains all the complementary arcs β_n .

For a proof of the following lemma see [2, Th. 1.2].

LEMMA 2. Let $T: X \to Y$ be a linear and continuous map from the Banach space X into the normed linear space Y. Suppose there exist constants $\delta < 1$ and M such that for each $y \in Y$ with $||y|| \leq 1$, there exists an $x \in X$ such that

$$|| Tx - y || \leq \delta, || x || \leq M.$$

Then TX = Y. If $||y|| \leq 1$, there exists an x such that Tx = y and $||x|| \leq M(1 - \delta)^{-1}$.

LEMMA 3. The relatively closed subset K of C - E is an interpolation set for B_E if, and only if, K has measure 0.

Proof. Clearly every such interpolation set for B_E must be of measure 0.

For the converse we need to know that any relatively closed subset K of C - E of measure 0 can be written as the disjoint union of compact sets

$$K = igcup_{n=1}^{\infty} K_n$$

in such a way that there exist disjoint open sets $O_n \subset C - E$ which satisfy the inclusions

$$K_n \subset O_n \ (n = 1, 2, \cdots)$$
.

Because K is nowhere dense in C - E it is possible to replace any finite disjoint collection of open arcs J_1, J_2, \dots, J_s which cover E by another collection of open arcs $I_p \subset J_p$ $(p = 1, 2, \dots, s)$ which cover

E and have all their endpoints in $C - E \bigcup K$. Hence there exists a sequence $G_1 \supset G_2 \supset G_3 \supset \cdots$ such that $E = \bigcap_{n=1}^{\infty} G_n$ and each G_n is a finite disjoint collection of open arcs, all of whose endpoints lie in $C - E \bigcup K$. Define

$$K_{1} = K \bigcap (C - G_{1}), O_{1} = C - \overline{G}_{1}$$

and, for n > 1,

$$K_n=Kigcap_{n-1}-G_n$$
), $O_n=G_{n-1}-ar{G}_n$.

Now let α be a bounded, complex-valued continuous function on K with $||\alpha|| = 1$. Denote the restriction of α to K_n by α_n and fix δ , $0 < \delta < 1$. According to the general Rudin-Carleson interpolation theorem [4] we may choose positive continuous functions Δ_n $(n = 1, 2, \dots)$ on C such that

(a) $\Delta_n = |\alpha_n| + \delta/2^n$ on K_n ,

(b) $\Delta_n = \delta/2^n$ on $C - O_n$,

(c) $0 < \Delta_n \leq ||\alpha|| + \delta/2^n$ everywhere;

then select functions $f_n \in A$ (the disc algebra) having the following properties:

(d) $f_n = \alpha_n$ on K_n ,

(e) $|f_n| \leq \Delta_n$ on C.

For the function f defined by

(3)
$$f(z) = \sum_{n=1}^{\infty} f_n(z) \qquad (z \in X) ,$$

we make the following claims:

- (i) $f \in B_E$,
- (ii) $||f|| \leq 1 + \delta$,
- (iii) $\sup_{z \in K} |f(z) \alpha(z)| \leq \delta$.

It follows from (b), (c) and (e) that the series (3) converges for every $z \in C$ and that its partial sums are bounded by δ for

$$\pmb{z} \in C \, - \, \bigcup_{n=1}^{\infty} O_n$$

and by $1 + \delta$ if $z \in O_n$ for some positive integer *n*. Therefore the series converges pointwise on X to an H^{∞} function with norm satisfying (ii). Further, (b) and (e) show that convergence in (3) is uniform on any compact subset of C - E because such a set misses all but a finite number of the sets O_n .

Thus f is continuous on C - E, hence continuous on X and (i) holds. In order to establish (iii), suppose $z \in K$; then $z \in K_p$ for some positive integer p so, by (d), $f(z) - \alpha(z) = \sum_{n \neq p} f_n(z)$ and, by (b) and (e), $|f(z) - \alpha(z)| \leq \sum_{n \neq p} \delta 2^{-n} < \delta$ as required.

Let C(K) be the Banach space of bounded continuous functions on K, and let T be the restriction mapping from B_E into C(K). Conditions (i), (ii) and (iii) above show that Lemma 2 applies. Hence if $\delta < 1$ and $\alpha \in C(K)$, there exists an f in B_E such that $f = \alpha$ on the set K and $||f|| \leq (1 + \delta)(1 - \delta)^{-1} ||\alpha||$. This is the desired conclusion.

LEMMA 4. Let K be a relatively closed subset of C - E of measure 0. Then the ideal

$$J(K) = \{ f \in B_E : f(K) = 0 \}$$

has an approximate unit.

Proof. The implication is that there exists a net $\{e_{\tau}\}$ in J(K) such that $||e_{\tau}|| \leq 1$ and $e_{\tau} \to 1$ uniformly on closed subsets of X disjoint from the set K.

We assume the notation and decomposition of Lemma 3 except that each of the sets O_n is replaced by

$${V}_{\scriptscriptstyle n} = \{re^{i heta} \colon e^{i heta} \in {O}_{\scriptscriptstyle n}, \, 1 - rac{1}{n} < r \leqq 1\}$$
 .

The sets V_n are pairwise disjoint, open in X, and $K_n \subset V_n$. Choose numbers $c_n, 0 < c_n < 1$, so that $\sum_{n=1}^{\infty} c_n < \infty$ and functions $g_n \in A$ such that $||g_n|| \leq 1, g_n$ vanishes exactly on K_n and $|1 - g_n| < c_n$ on $X - V_n$ (the existence of such functions follows immediately from the construction given in [8, Chapter 6, p-80]). Define g by

$$g(z) = \prod_{n=1}^{\infty} g_n(z)$$
 $(z \in X)$

The inequality

$$ig| 1 - \prod_{n=N}^{N+p} g_n(z) ig| \leq \prod_{n=N}^{N+p} (1 + |1 - g_n(z)|) - 1 \ \leq \exp \sum_{n=N}^{N+p} c_n - 1 \;,$$

valid for $z \notin \bigcup_{n=N}^{\infty} V_n$, shows that the product defining g converges uniformly on compact subsets of X. Thus $g \in B_E$ and

$$|1 - g(z)| \leq \exp\sum_{n=1}^{\infty} c_n - 1$$

for $z \in X - \bigcup_{n=1}^{\infty} V_n$. This argument shows that if $\varepsilon > 0$ and S is a closed subset of X disjoint from K, then proper choices for V_n and c_n yield a g in J(K) such that $||g|| \leq 1$, g vanishes precisely on K in

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X and $|1 - g(z)| < \varepsilon (z \in S)$.

Proof of Theorem 1. If the relatively closed subset S of X is an interpolation set for B_E , then clearly $S \cap D$ is countable and $S \cap C$ has measure 0. The proof that $S \cap D$ is uniformly separated is identical with the corresponding proof for H^{∞} [8, p. 196].

For the converse let the relatively closed subset S of X be the union of a uniformly separated sequence $\{z_k\}$ in D and a subset K of C - E of measure 0. Let α be a bounded, continuous function on S with $||\alpha|| \leq 1$. By Lemma 3 there exists an f_1 in B_E such that $f_1 = \alpha$ on K and $||f_1|| \leq 3/2$; hence $\alpha_1 = \alpha - f_1$ vanishes on K and $||\alpha_1|| \leq 5/2$. Lemma 1 guarantees the existence of a constant c, depending only on the points z_k , and a function h in $B_{E \cup K}$ such that

$$h(z_k)=lpha_{\scriptscriptstyle 1}(z_k) \hspace{0.2cm} (k=1,\,2,\,\cdots), \hspace{0.2cm} ||\hspace{0.1cm} h\hspace{0.1cm}|| \leq c \cdot 5/2$$
 .

Since α_1 is continuous on S and vanishes on K, we may choose open sets U_n in X such that $K_n \subset U_n \subset V_n$ $(n = 1, 2, \dots)$ (we assume the decomposition and notation of Lemma 4) and $|\alpha_1| < 1/4$ on the set $\bigcup_{n=1}^{\infty} U_n$. Now choose, by Lemma 4, a function g in J(K) such that $||g|| \leq 1$ and |1 - g(z)| < 1/4 for $z \notin \bigcup_{n=1}^{\infty} U_n$; hence the product ghbelongs to B_E and for all k

$$||g(z_k)h(z_k)-lpha_{_1}\!(z_k)|=||(g(z_k)-1)lpha_{_1}\!(z_k)|< 3/4$$
 ,

since $|g(z_k) - 1| < 1/4$, $|\alpha_1(z_k)| \leq 5/2$ for $z_k \notin \bigcup U_n$ and

 $| \, 1 \, - \, g(z_{\scriptscriptstyle k}) \, | \, \leq 2$, $| \, lpha_{\scriptscriptstyle 1}(z_{\scriptscriptstyle k}) \, | \, < 1/4$

otherwise. This proves that the function $f = f_1 + gh$ has the following properties:

 $egin{array}{lll} ({f i}\)& f\in B_{\scriptscriptstyle E}\ ({f ii}\)& ||f||\leq 3/2+5/2{f \cdot}c=M\ ({f iii}\)& \sup_{z\in S}|f(z)-lpha(z)|\leq 3/4.\ {f In view of Lemma 2 the proof is complete.} \end{array}$

THEOREM 2. Let K be the closed subset of C - E of measure 0. Then any bounded continuous function α on K has an extension g in B_E with precisely the same norm as α ; in fact g can be chosen in B_E so that $g = \alpha$ on K and

$$|g(z)| < ||\alpha||, z \in X - K$$
.

Proof. We have already established the following weaker result (see Lemma 3): if $\varepsilon > 0$, α has an extension g_{ε} in B_E such that $||g_{\varepsilon}|| \leq (1 + \varepsilon) ||\alpha||$. In addition, Lemma 4 asserts that K is a strong hull in the Banach function algebra B_E [7], that is, for each closed subset S of X disjoint from K and each $\varepsilon > 0$ there exists a function

f in β_E such that f(K) = 0, $||f|| \leq 1$, and $|1 - f(S)| < \varepsilon$. The conditions guarantee the existence of the required function g [see 7, Th. 4.6].

3. Peak points and interpolating sequences. Previous to Carleson's paper [5], Gleason and Newman had constructed examples (unpublished) proving the existence of infinite interpolation sets for H^{∞} . In this section we present a process, depending only on the existence of peak points in the underlying algebra, which constructs infinite interpolation sets in some rather general H^{∞} spaces. According to Bishop's minimal boundary theorem [3] peak points always exist for a sup norm algebra defined on a compact metric space.

Let A be a sup norm algebra on the compact Hausdorff space X and suppose that the function $F \in A$ peaks at x, that is,

$$F(x) = 1 \text{ and } |F(y)| < 1$$
 $(y \in X, y \neq x)$.

Let S_x be the set of all bounded and continuous functions f on $X - \{x\}$ for which there exists a constant m and a sequence $\{f_n\}$ in A with $||f_n|| \leq m$ and such that $f_n \to f$ uniformly on compact subsets of $X - \{x\}$. B_x is the uniform closure of S_x .

THEOREM 3. Suppose P is a connected subset of X and $x \in \overline{P} - P$. Then there is an infinite sequence $\{z_k\}$ of distinct points in P which interpolates for B_x , that is, the map $T: B_x \to l^{\infty}$ defined by $Tf = \{f(z_k)\}$ is an onto map.

Proof. Choose δ , $0 < \delta < 1/4$, so that the closed set

 $U_1 = \{y \colon |F(y) - 1| \ge \delta\}$

intersects P. Set $F_1 = F$ and n(1) = 1. We wish to construct an increasing sequence $n(1) < n(2) < \cdots$ of integers which obey

$$(4)$$
 $n(k+1) > kn(k)$ $(k = 1, 2, \cdots)$

and for which the sets

$$egin{array}{lll} U_k = \{y \colon |\, F_k(y) - 1\,| \ge \delta/2^{k-1}\}\ ,\ V_k = \{y \colon |\, F_k(y)\,| < \delta/2^{k-1}\} \end{array},$$

associated with the functions

$$F_k = F^{n(k)}$$

satisfy

$$(7) U_1 \subset V_2 \subset U_2 \subset V_3 \cdots$$

and

(8)
$$X - \{x\} = \bigcup_{i=1}^{\infty} U_i$$

In order to construct F_2 let n(2) > 1 be an integer so large that

 $|F^{n(2)}|<\delta/2$

on U_1 and define $F_2 = F^{n(2)}$. Notice that $U_1 \subset V_2 \subset U_2$. Suppose $n(1), n(2), \dots, n(k)$ have been chosen. Choose

$$n(k+1) > kn(k)$$

so large that $|F^{n(k+1)}| < \delta/2^k$ on the closed set U_k and define $F_{k+1} = F^{n(k+1)}$. Clearly $U_k \subset V_{k+1} \subset U_{k+1}$. The existence of the required sequence $\{n(k)\}$ follows by induction. If a point y belongs to none of the sets V_k then, by (4) and (6), $|F(y)| > \delta^{1/n(k+1)} 2^{-k/n(k+1)} \to 1$ as $k \to \infty$ showing that y = x. Hence (8) holds also.

For each integer k choose a point z_k from the set $P \bigcap (V_{k+1} - U_k)$, this being possible because U_k and $X - V_{k+1}$ are disjoint closed sets both of which intersect the connected set P. Fix a bounded sequence $\{w_k\}, ||w|| \leq 1$, and define g by

(9)
$$g = \sum_{p=1}^{\infty} w_p (F_p - F_{p+1})$$
.

The series converges uniformly on compact subsets of $X - \{x\}$ since any such set is eventually captured by a V_k . In order to establish bounds on the partial sums for the series (9) notice that

$$X - \{x\} = U_1 \bigcup (U_2 - U_1) \bigcup (U_3 - U_2) \bigcup \cdots$$

(the sets in the union being pairwise disjoint) and for any point y,

$$egin{aligned} g(y) &= \sum\limits_{p=1}^{k-1} w_p(F_p(y) - F_{p+1}(y)) + \, w_k(F_k(y) - F_{k+1}(y)) \ &+ \, w_{k+1}(F_{k+1}(y) - F_{k+2}(y)) + \sum\limits_{p=k+2}^\infty w_p(F_p(y) - F_{p+1}(y)) \end{aligned}$$

If $y \in U_{k+1} - U_k$, we have the inequalities

$$egin{aligned} ext{(A)} & \left|\sum\limits_{p=1}^{k-1} w_p(F_p(y)-F_{p+1}(y))
ight| &\leq \sum\limits_{p=1}^{k-1} \left(|F_p(y)-1|+|F_{p+1}(y)-1|
ight) \ &< \sum\limits_{p=1}^{k-1} \left(\delta/2^{p-1}+\delta/2^p
ight) \ ; \end{aligned}$$

(B)
$$|w_p(F_p(y) - F_{p+1}(y))| \le 2$$
 $(p = k, k+1)$;

(C)
$$\left|\sum_{p=k+2}^{\infty} w_p(F_p(y) - F_{p+1}(y))\right| < \sum_{p=k+2}^{\infty} (\delta/2^{p-1} + \delta/2^p)$$
.

Hence $4 + 2\delta \sum 2^{-p} = 4 + 4\delta$ is a bound for the partial sums. Inequalities (B) and (C) give the same bounds when $y \in U_1$. Therefore $g \in S_x$.

In order to estimate $g(z_k) - w_k$ subtract w_k from both members of (9) and replace y by z_k in (A) and (C). In place of (B) we have the inequalities

$$\begin{array}{l} (\mathrm{B'}) \qquad \qquad | \ w_k(F_k(z_k)-1-F_{k+1}(z_k)) \ | \le \delta/2^{k-1}+\delta/2^k \ , \\ | \ w_{k+1}(F_{k+1}(z_k)-F_{k+2}(z_k)) \ | \le \delta/2^k+\delta/2^{k+1} \end{array}$$

because $z_k \in V_{k+1} - U_k$ implies $|F_{k+1}(z_k)| < \delta/2^k$, $|F_{k+2}(z_k)| < \delta/2^{k+1}$ and $|F_k(z_k) - 1| < \delta/2^{k-1}$. Addition of (A), (B') and (C) gives

$$||g(z_k)-w_k|<4\delta$$
 $(k=1,\,2,\,\cdots)$.

In summary, we have shown that for any $w = \{w_n\} \in l^{\infty}$ with $||w|| \leq 1$ there exists a function g in S_x with $||g|| \leq 4 + 4\delta$ and

$$\sup_{k} |g(z_k) - w_k| \leq 4\delta.$$

Since $\delta < 1/4$ Lemma 2 applies; hence there exists a function f in B_x such that

$$f(\boldsymbol{z}_k) = w_k$$
 $(k = 1, 2, \cdots)$.

This completes the proof.

The preceding argument shows that the series (9) converges absolutely and has uniformly bounded partial sums for every bounded sequence $\{w_k\}$ and will therefore converge uniformly on X provided $\lim w_k = 0$. This means that we may again apply Lemma 2, this time with T identified as the map $f \rightarrow \{f(z_k)\}$ from A into the space c of convergent sequences.

COROLLARY. Let A be a sup norm function algebra on the compact Hausdorff space, let P be a connected subset of X and let $x \in \overline{P} - P$ be a peak point for A. Then there exists an infinite sequence of distinct points in P which converges to x and has the property that for every convergent sequence $\{c_k\}$ there exists an f in A such that $f(z_k) = c_k \ (k = 1, 2, \cdots)$.

Let *m* be a positive Baire measure on *X* which is multiplicative on the sup norm algebra *A* and not equal to point evaluation at *x*. Clearly, the functions in S_x are elements of $H^2(dm)$, the closure of *A* in $L^2(dm)$, and therefore

$$S_x \subset H^\infty(dm) = L^\infty(dm) \bigcap H^2(dm)$$
 .

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Since the norm in $H^{\infty}(dm)$ is the essential supremum norm relative to m, it follows that $B_x \subset H^{\infty}(dm)$. Thus, under the assumptions of Theorem 3, we can make the following rather weak statement: there exist infinite interpolation sets for $H^{\infty}(dm)$ whenever point evaluations on the set P extend to homomorphisms of $H^{\infty}(dm)$.

Finally, we remark that in so far as we know the Carleson corona theorem, the question of whether D is dense in the maximal ideal space of B_E , is open in case E is a proper nonempty subset of C.

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