

Pacific Journal of Mathematics

**A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF
SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS**

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A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS

A. E. HURD

The essentially bounded measurable (vector) function $u(x, t) = (u_1(x, t), \dots, u_r(x, t))$ is called a weak solution of the initial-value problem for the system

$$\frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0$$

in the upper half-plane $t \geq 0$ if it satisfies the usual integral identity (defining "weak") together with the condition that, given a compact set D in $t \geq 0$, there exists a function $K(t) \in L^1_{loc}([0, \infty))$ such that

$$\frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leq K(t)$$

holds a.e. for $x_1, x_2 \in D$ and $0 < t < \infty$. It is shown that, if the matrix $\partial \mathcal{A} / \partial u$ is symmetric and positive definite (a convexity condition), then weak solutions are uniquely determined by their initial conditions.

In [1] O. A. Oleinik established a uniqueness theorem for a rather general class of weak solutions of a quasilinear equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, u)}{\partial x} + \psi(x, t, u) = 0$$

where the function $\varphi(x, t, u)$ was subject to a convexity condition in u , namely, $\varphi_{uu} \geq 0$. The purpose of this note is to generalize Oleinik's uniqueness result (in the case $\psi \equiv 0$) to certain quasilinear systems which are subject to a symmetry condition (assumption III below) as well as a convexity condition (assumption IV below). In the case of one equation our uniqueness result is slightly less general than Oleinik's in that she does not require the function $K(t)$ occurring in (2) to be locally integrable on $[0, \infty)$. The method is the standard variation of Holmgren's method which is employed by Oleinik and others, except that we work with mean square rather than sup-norm estimates. Oleinik [2] has also established a uniqueness result for a special system of two equations which, however, is not symmetric. Rozhdestvenskii [3] has established a uniqueness theorem for piecewise smooth solutions of certain quasilinear systems but his methods are entirely different from those employed here.

2. In $D = \{(x, t): -\infty < x < \infty, 0 \leq t < \infty\}$ we consider the quasi-

linear system of r equations

$$(1) \quad \frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0$$

for the (vector) function $u(x, t) = (u_1(x, t), \dots, u_r(x, t))$ where

$$\mathcal{A}(x, t, u) = (a_1(x, t, u), \dots, a_r(x, t, u)) .$$

The following assumptions will be made:

I. The functions $a_i(x, t, u)$ possess derivatives $\partial a_i / \partial u_j$, $\partial^2 a_i / \partial x \partial u_j$ and $\partial^2 a_i / \partial u_j \partial u_k$ which are bounded subsets of (x, t, u) -space.

II. Let

$$\frac{\partial a_i(x, t, u)}{\partial u_j} = a_{ij}(x, t, u) .$$

Then, if u is bounded, i.e., $\sum u_i^2 \leq M^2$, there exists a constant c , depending only on M , such that

$$-c \sum_{i=1}^r \xi_i^2 \leq \sum_{i,j=1}^r a_{ij}(x, t, u) \xi_i \xi_j \leq c \sum_{i=1}^r \xi_i^2$$

for all vectors $\xi = (\xi_1, \dots, \xi_r)$.

III (Symmetry). For all x, t , and u ,

$$a_{ij}(x, t, u) = a_{ji}(x, t, u) \quad (i, j = 1, \dots, r)$$

IV (Convexity). For all x, t , and u , and each $k = 1, \dots, r$, we have

$$\sum_{i,j=1}^r \frac{\partial a_{ij}(x, t, u)}{\partial u_k} \xi_i \xi_j \geq 0$$

for all vectors $\xi = (\xi_1, \dots, \xi_r)$.

DEFINITION. Let $\psi(x)$ be an essentially bounded measurable function defined on $-\infty < x < \infty$. An essentially bounded measurable function $u(x, t)$ is called a weak solution of (1) in D with initial conditions $\psi(x)$ if,

(a) for every test function $\varphi(x, t)$ which is continuously differentiable with compact support in the (x, t) -plane we have

$$(2) \quad \int_D \left[\left\langle u, \frac{\partial \varphi}{\partial t} \right\rangle + \left\langle A(t, x, u), \frac{\partial \varphi}{\partial x} \right\rangle \right] dx dt + \int_{-\infty}^{\infty} \langle \varphi(x, 0), \psi(x) \rangle dx = 0$$

where \langle, \rangle is the inner product in Euclidean r -space;

(b) given any compact subset of D there is a corresponding function $K(t) \in L^1_{loc}([0, \infty))$ such that

$$(3) \quad \frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leqq K(t)$$

($i = 1, \dots, r$) holds a.e. for x_1 and x_2 in the compact subset, and $0 < t < \infty$.

THEOREM. *Weak solutions of (1) are uniquely determined by their initial conditions.*

Proof. Let $u_1(x, t)$ and $u_2(x, t)$ be two weak solutions of (1) with the same initial conditions $\psi(x)$. We will show that, if $F(x, t) = (F_1(x, t), \dots, F_r(x, t))$ is any smooth (vector) function with compact support contained in $t > 0$, then

$$\int_D \langle u_1 - u_2, F \rangle dx dt = 0,$$

thus proving that $u_1 = u_2$ a.e. in D .

Let ω^n be the usual Gaussian averaging kernel with support contained in the sphere $x^2 + t^2 \leqq 1/n^2$. Given a function $\varphi(x, t) \in L^2_{loc}(D)$ we define the averaged function $\varphi^n(x, t)$ by convolution; $\varphi^n = \varphi * \omega^n$. By a familiar argument we see that $u^n_{i,k} \rightarrow u_{ik}$ ($i = 1, 2$ and $k = 1, \dots, r$) in mean square on compact subsets of D . From (3) it follows (see [1]) that

$$(4) \quad \frac{\partial u^n_{i,k}}{\partial x} \leqq K(t) \quad (i = 1, 2 \text{ and } k = 1, \dots, r)$$

on compact subsets of D .

We now define the functions

$$\begin{aligned} \alpha_{ij}(x, t) &= \int_0^1 a_{ij}(x, t, \tau u_1 + (1 - \tau)u_2) d\tau \\ \alpha^n_{ij}(x, t) &= \int_0^1 a_{ij}(x, t, \tau u^n_1 + (1 - \tau)u^n_2) d\tau \end{aligned}$$

($i, j = 1, \dots, r$ and $n = 1, 2, \dots$) and the associated matrices $A(x, t) = (\alpha_{ij}(x, t))$ and $A^n(x, t) = (\alpha^n_{ij}(x, t))$.

It is immediate that

$$\mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2) = A(x, t)(u_1 - u_2).$$

Also

$$| \alpha^n_{ij}(x, t) - \alpha_{ij}(x, t) | \leqq \text{const.} [|u^n_1 - u_1| + |u^n_2 - u_2|]$$

on compact subsets of D , from which it follows that $\alpha^n_{ij} \rightarrow \alpha_{ij}$ in mean square on compact subsets of D . From II we see that

$$(5) \quad -c\langle \xi, \xi \rangle \leq \langle A^n(x, t)\xi, \xi \rangle \leq c\langle \xi, \xi \rangle$$

for some constant $c > 0$ and all real vectors ξ . Finally we note that

$$\begin{aligned} \frac{\partial \alpha_{ij}^n}{\partial x} &= \int_0^1 \left\{ \frac{\partial a_{ij}}{\partial x}(x, t, \tau u_1^n + (1 - \tau)u_2^n) \right. \\ &\quad \left. + \sum_{k=1}^r \frac{\partial a_{ij}}{\partial u_k}(x, t, \tau u_1^n + (1 - \tau)u_2^n) \left[\frac{\tau \partial u_{1,k}^n}{\partial x} + (1 - \tau) \frac{\partial u_{2,k}^n}{\partial x} \right] \right\} dt . \end{aligned}$$

Using I, IV and (4) it follows that

$$\left\langle \frac{\partial A^n}{\partial x} \xi, \xi \right\rangle \leq K_1(t) \langle \xi, \xi \rangle$$

on compact subsets of D for every vector ξ , where $K_1(t) \in L^1_{loc}([0, \infty))$.

We now construct for each $n = 1, 2, \dots$ the vector function $\varphi^n(x, t)$ satisfying the linear system

$$\frac{\partial \varphi^n}{\partial t} + A^n(x, t) \frac{\partial \varphi^n}{\partial x} = F(x, t)$$

and vanishing on $t = T$, where the support of F is assumed to be below $t = T$. This is achieved by solving the system

$$\frac{\partial \tilde{\varphi}^n}{\partial t} - A^n(x, T - t) \frac{\partial \tilde{\varphi}^n}{\partial x} = F(x, T - t)$$

for the vector function $\tilde{\varphi}^n(x, t)$ in D , with the initial conditions $\tilde{\varphi}^n(x, 0) = 0$, and then putting $\varphi^n(x, t) = \tilde{\varphi}^n(x, T - t)$. The classical existence theory guarantees that $\varphi^n(x, t)$ exists, is smooth, and, by (5), has support contained in a compact set which is independent of n , and so is a legitimate test function.

Using (2) we obtain

$$\begin{aligned} \int_D \left\langle u_1 - u_2, \frac{\partial \varphi^n}{\partial t} \right\rangle dx dt &= - \int \left\langle \mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt \\ &= - \int_D \left\langle A(x, t)(u_1 - u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt . \end{aligned}$$

Thus

$$(6) \quad \int_D \langle u_1 - u_2, F \rangle dx dt = \int_D \left\langle u_1 - u_2, (A^n - A) \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt .$$

Using the facts that (i) the supports of the φ^n lie in a fixed compact subset of D , (ii) the u_i are essentially bounded and (iii) the coefficients of A^n converge in the mean square on compact subsets of D to the coefficients of A , we see immediately that the right hand side of (6) approaches zero as $n \rightarrow \infty$, as long as the mean square norms of the

$\partial\varphi_i^r/\partial x$ (on compact subsets of D) are uniformly bounded. The proof will be completed by establishing this fact.

Let $\partial\tilde{\varphi}^n/\partial x = v^n$, $A^n(x, T-t) = \tilde{A}^n(x, t)$ and $F(x, T-t) = \tilde{F}(x, t)$. Then v^n satisfies the equation

$$\frac{\partial v^n}{\partial t} - \tilde{A}^n \frac{\partial v^n}{\partial x} - \frac{\partial \tilde{A}^n}{\partial x} v^n = \frac{\partial \tilde{F}}{\partial x}$$

in $0 \leq t \leq T$, and the initial conditions $v^n(x, 0) = 0$. We may suppose that the supports of the v^n ($n = 1, 2, \dots$) in $0 \leq t \leq T$ are all strictly contained in some interval $a < x < b$. Then

$$\frac{\partial}{\partial t} \langle v^n, v^n \rangle - \frac{\partial}{\partial x} \langle \tilde{A}^n v^n, v^n \rangle = 2 \left\langle \frac{\partial \tilde{F}}{\partial x}, v^n \right\rangle + \left\langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \right\rangle.$$

Using Green's formula

$$\begin{aligned} \int_a^b \langle v^n(x, t), v^n(x, t) \rangle dx &\leq \int_0^t \int_a^b 2 \left\langle \frac{\partial \tilde{F}}{\partial x}, v^n \right\rangle dx dt + \int_0^t \int_a^b \left\langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \right\rangle dx dt \\ &\cong \int_0^t \int_a^b \left\langle \frac{\partial \tilde{F}}{\partial x}, \frac{\partial \tilde{F}}{\partial x} \right\rangle dx dt + \int_0^t (1 + K_1(s)) \left[\int_a^b \langle v(x, s), v(x, s) \rangle dx \right] ds \end{aligned}$$

from which it follows by Gronwall's Lemma that

$$\int_0^T \int_a^b \langle v^n, v^n \rangle dx dt \leq \text{constant},$$

the constant depending on the L^2 -norm of $\partial\tilde{F}/\partial x$ and $\int_0^T K_1(t) dt$, but not on n . This completes the proof.

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Received April 10, 1967, and in revised form May 28, 1968. The preparation of this paper was sponsored in part by NSF Grant GP-5279.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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