A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF
SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS

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The essentially bounded measurable (vector) function
\[ u(x, t) = (u_1(x, t), \ldots, u_r(x, t)) \]
is called a weak solution of the initial-value problem for the system
\[ \frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0 \]
in the upper half-plane \( t \geq 0 \) if it satisfies the usual integral identity (defining "weak") together with the condition that, given a compact set \( D \) in \( t \geq 0 \), there exists a function \( K(t) \in L^1_{\text{loc}}([0, \infty)) \) such that
\[ \frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leq K(t) \]
holds a.e. for \( x_1, x_2 \in D \) and \( 0 < t < \infty \). It is shown that, if the matrix \( \frac{\partial \mathcal{A}}{\partial u} \) is symmetric and positive definite (a convexity condition), then weak solutions are uniquely determined by their initial conditions.

In [1] O. A. Oleinik established a uniqueness theorem for a rather general class of weak solutions of a quasilinear equation of the form
\[ \frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, u)}{\partial x} + \psi(x, t, u) = 0 \]
where the function \( \varphi(x, t, u) \) was subject to a convexity condition in \( u \), namely, \( \varphi_{uu} \geq 0 \). The purpose of this note is to generalize Oleiniks uniqueness result (in the case \( \psi \equiv 0 \)) to certain quasilinear systems which are subject to a symmetry condition (assumption III below) as well as a convexity condition (assumption IV below). In the case of one equation our uniqueness result is slightly less general than Oleiniks in that she does not require the function \( K(t) \) occurring in (2) to be locally integrable on \([0, \infty)\). The method is the standard variation of Holmgren’s method which is employed by Oleinik and others, except that we work with mean square rather than sup-norm estimates. Oleinik [2] has also established a uniqueness result for a special system of two equations which, however, is not symmetric. Rozhdestvenskii [3] has established a uniqueness theorem for piecewise smooth solutions of certain quasilinear systems but his methods are entirely different from those employed here.

2. In \( D = \{(x, t): -\infty < x < \infty, 0 \leq t < \infty\} \) we consider the quasi-
linear system of $r$ equations

$$\frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0$$

for the (vector) function $u(x, t) = (u_1(x, t), \ldots, u_r(x, t))$ where

$$\mathcal{A}(x, t, u) = (a_1(x, t, u), \ldots, a_r(x, t, u)).$$

The following assumptions will be made:

I. The functions $a_i(x, t, u)$ possess derivatives $\frac{\partial a_i}{\partial u_j}, \frac{\partial^2 a_i}{\partial x \partial u_j}$ and $\frac{\partial^2 a_i}{\partial u_j \partial u_k}$ which are bounded subsets of $(x, t, u)$-space.

II. Let

$$\frac{\partial a_i(x, t, u)}{\partial u_j} = a_{ij}(x, t, u).$$

Then, if $u$ is bounded, i.e., $\sum u_i^2 \leq M^2$, there exists a constant $c$, depending only on $M$, such that

$$-c \sum_{i=1}^{r} \xi_i^2 \leq \sum_{i,j=1}^{r} a_{ij}(x, t, u) \xi_i \xi_j \leq c \sum_{i=1}^{r} \xi_i^2$$

for all vectors $\xi = (\xi_1, \ldots, \xi_r)$.

III (Symmetry). For all $x$, $t$, and $u$,

$$a_{ij}(x, t, u) = a_{ji}(x, t, u) \quad (i, j = 1, \ldots, r)$$

IV (Convexity). For all $x$, $t$, and $u$, and each $k = 1, \ldots, r$, we have

$$\sum_{i,j=1}^{r} \frac{\partial a_{ij}(x, t, u)}{\partial u_k} \xi_i \xi_j \geq 0$$

for all vectors $\xi = (\xi_1, \ldots, \xi_r)$.

**DEFINITION.** Let $\psi(x)$ be an essentially bounded measurable function defined on $-\infty < x < \infty$. An essentially bounded measurable function $u(x, t)$ is called a weak solution of (1) in $D$ with initial conditions $\psi(x)$ if,

(a) for every test function $\varphi(x, t)$ which is continuously differentiable with compact support in the $(x, t)$-plane we have

$$\left( \int_D \left[ \left\langle u, \frac{\partial \varphi}{\partial t} \right\rangle + \left\langle A(t, x, u), \frac{\partial \varphi}{\partial x} \right\rangle \right] dx dt + \int_{-\infty}^{\infty} \langle \varphi(x, 0), \psi(x) \rangle dx = 0 \right)$$

where $\langle , \rangle$ is the inner product in Euclidean $r$-space;

(b) given any compact subset of $D$ there is a corresponding function $K(t) \in L_{\infty}(0, \infty)$ such that
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\( (i = 1, \ldots, r) \) holds a.e. for \( x_1 \) and \( x_2 \) in the compact subset, and \( 0 < t < \infty \).

**Theorem.** Weak solutions of (1) are uniquely determined by their initial conditions.

**Proof.** Let \( u_1(x, t) \) and \( u_2(x, t) \) be two weak solutions of (1) with the same initial conditions \( \psi(x) \). We will show that, if \( F(x, t) = (F_1(x, t), \ldots, F_r(x, t)) \) is any smooth (vector) function with compact support contained in \( t > 0 \), then

\[
\int_D \langle u_1 - u_2, F \rangle dxdt = 0,
\]

thus proving that \( u_1 = u_2 \) a.e. in \( D \).

Let \( \omega^n \) be the usual Gaussian averaging kernel with support contained in the sphere \( x^2 + t^2 \leq 1/n^2 \). Given a function \( \varphi(x, t) \in L_{loc}^2(D) \) we define the averaged function \( \varphi^n(x, t) \) by convolution; \( \varphi^n = \varphi * \omega^n \). By a familiar argument we see that \( u_{ik}^n \to u_{ik} \) (\( i = 1, 2 \) and \( k = 1, \ldots, r \)) in mean square on compact subsets of \( D \). From (3) it follows (see [1]) that

\[
\frac{\partial u_{ik}^n}{\partial x} \leq K(t) \quad (i = 1, 2 \text{ and } k = 1, \ldots, r)
\]

on compact subsets of \( D \).

We now define the functions

\[
\alpha_{ij}(x, t) = \int_0^t a_{ij}(x, t, \tau u_1 + (1 - \tau)u_2) d\tau
\]

\[
\alpha_{ij}^n(x, t) = \int_0^t a_{ij}(x, t, \tau u_1^n + (1 - \tau)u_2^n) d\tau
\]

(\( i, j = 1, \ldots, r \) and \( n = 1, 2, \ldots \)) and the associated matrices \( A(x, t) = (\alpha_{ij}(x, t)) \) and \( A^n(x, t) = (\alpha_{ij}^n(x, t)) \).

It is immediate that

\[
\mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2) = A(x, t)(u_1 - u_2).
\]

Also

\[
|\alpha_{ij}^n(x, t) - \alpha_{ij}(x, t)| \leq \text{const.}[|u_1^n - u_1| + |u_2^n - u_2|]
\]

on compact subsets of \( D \), from which it follows that \( \alpha_{ij}^n \to \alpha_{ij} \) in mean square on compact subsets of \( D \). From II we see that
for some constant $c > 0$ and all real vectors $\xi$. Finally we note that
\[
\frac{\partial a_{ij}^n}{\partial x} = \int_0^1 \left\{ \frac{\partial a_{ij}}{\partial x} (x, t, \tau u_1^n + (1 - \tau)u_2^n) \right.
\]
\[
+ \sum_{k=1}^r \frac{\partial a_{ij}}{\partial u_k} (x, t, \tau u_1^n + (1 - \tau)u_2^n) \left[ \frac{\tau \partial u_1^n}{\partial x} + (1 - \tau) \frac{\partial u_2^n}{\partial x} \right] \right\} dt .
\]
Using I, IV and (4) it follows that
\[
\left\langle \frac{\partial A^n}{\partial x} \xi, \xi \right\rangle \leq K(t) \left\langle \xi, \xi \right\rangle
\]
on compact subsets of $D$ for every vector $\xi$, where $K(t) \in L^1_{loc}([0, \infty))$.

We now construct for each $n = 1, 2, \ldots$ the vector function $\varphi^n(x, t)$ satisfying the linear system
\[
\frac{\partial \varphi^n}{\partial t} + A^n(x, t) \frac{\partial \varphi^n}{\partial x} = F(x, t)
\]
and vanishing on $t = T$, where the support of $F$ is assumed to be below $t = T$. This is achieved by solving the system
\[
\frac{\partial \tilde{\varphi}^n}{\partial t} - A^n(x, T - t) \frac{\partial \tilde{\varphi}^n}{\partial x} = F(x, T - t)
\]
for the vector function $\tilde{\varphi}^n(x, t)$ in $D$, with the initial conditions $\tilde{\varphi}^n(x, 0) = 0$, and then putting $\varphi^n(x, t) = \tilde{\varphi}(x, T - t)$. The classical existence theory guarantees that $\varphi^n(x, t)$ exists, is smooth, and, by (5), has support contained in a compact set which is independent of $n$, and so is a legitimate test function.

Using (2) we obtain
\[
\int_D \left\langle u_1 - u_2, \frac{\partial \varphi^n}{\partial t} \right\rangle dx dt = -\int_D \left\langle \mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt
\]
\[
= -\int_D \left\langle A(x, t)(u_1 - u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt .
\]
Thus
\[
\int_D \left\langle u_1 - u_2, F \right\rangle dx dt = \int_D \left\langle u_1 - u_2, (A^n - A) \frac{\partial \varphi^n}{\partial x} \right\rangle dx dt .
\]
Using the facts that (i) the supports of the $\varphi^n$ lie in a fixed compact subset of $D$, (ii) the $u_i$ are essentially bounded and (iii) the coefficients of $A^n$ converge in the mean square on compact subsets of $D$ to the coefficients of $A$, we see immediately that the right hand side of (6) approaches zero as $n \to \infty$, as long as the mean square norms of the
A uniqueness theorem for weak solutions 

Let \( \partial \tilde{\phi}^n / \partial x = v^n, A^n(x, T - t) = \tilde{A}^n(x, t) \) and \( F(x, T - t) = \tilde{F}(x, t) \). Then \( v^n \) satisfies the equation

\[
\frac{\partial v^n}{\partial t} - \tilde{A}^n \frac{\partial v^n}{\partial x} - \frac{\partial \tilde{A}^n}{\partial x} v^n = \frac{\partial \tilde{F}}{\partial x}
\]

in \( 0 \leq t \leq T \), and the initial conditions \( v^n(x, 0) = 0 \). We may suppose that the supports of the \( v^n(n = 1, 2, \cdots) \) in \( 0 \leq t \leq T \) are all strictly contained in some interval \( a < x < b \). Then

\[
\frac{\partial}{\partial t} \langle v^n, v^n \rangle - \frac{\partial}{\partial x} \langle \tilde{A}^n v^n, v^n \rangle = 2 \langle \frac{\partial \tilde{F}}{\partial x}, v^n \rangle + \langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \rangle.
\]

Using Green’s formula

\[
\int_a^b \langle v^n(x, t), v^n(x, t) \rangle dx \leq \int_0^t \int_a^b 2 \langle \frac{\partial \tilde{F}}{\partial x}, v^n \rangle dx dt + \int_0^t \int_a^b \langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \rangle dx dt
\]

\[
\leq \int_0^t \int_a^b \left( \frac{\partial \tilde{F}}{\partial x} \frac{\partial \tilde{F}}{\partial x} \right) dx dt + \int_0^t (1 + K_1(s)) \left[ \int_a^b \langle v(x, s), v(x, s) \rangle dx \right] ds
\]

from which it follows by Gronwall’s Lemma that

\[
\int_0^T \int_a^b \langle v^n, v^n \rangle dx dt \leq \text{constant,}
\]

the constant depending on the \( L^2 \)-norm of \( \partial \tilde{F} / \partial x \) and \( \int_0^T K_1(t) dt \), but not on \( n \). This completes the proof.

BIBLIOGRAPHY


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