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**A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF  
SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS**

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# A UNIQUENESS THEOREM FOR WEAK SOLUTIONS OF SYMMETRIC QUASILINEAR HYPERBOLIC SYSTEMS

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The essentially bounded measurable (vector) function  $u(x, t) = (u_1(x, t), \dots, u_r(x, t))$  is called a weak solution of the initial-value problem for the system

$$\frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0$$

in the upper half-plane  $t \geq 0$  if it satisfies the usual integral identity (defining "weak") together with the condition that, given a compact set  $D$  in  $t \geq 0$ , there exists a function  $K(t) \in L^1_{loc}([0, \infty))$  such that

$$\frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leq K(t)$$

holds a.e. for  $x_1, x_2 \in D$  and  $0 < t < \infty$ . It is shown that, if the matrix  $\partial \mathcal{A} / \partial u$  is symmetric and positive definite (a convexity condition), then weak solutions are uniquely determined by their initial conditions.

In [1] O. A. Oleĭnik established a uniqueness theorem for a rather general class of weak solutions of a quasilinear equation of the form

$$\frac{\partial u}{\partial t} + \frac{\partial \varphi(x, t, u)}{\partial x} + \psi(x, t, u) = 0$$

where the function  $\varphi(x, t, u)$  was subject to a convexity condition in  $u$ , namely,  $\varphi_{uu} \geq 0$ . The purpose of this note is to generalize Oleĭnik's uniqueness result (in the case  $\psi \equiv 0$ ) to certain quasilinear systems which are subject to a symmetry condition (assumption III below) as well as a convexity condition (assumption IV below). In the case of one equation our uniqueness result is slightly less general than Oleĭnik's in that she does not require the function  $K(t)$  occurring in (2) to be locally integrable on  $[0, \infty)$ . The method is the standard variation of Holmgren's method which is employed by Oleĭnik and others, except that we work with mean square rather than sup-norm estimates. Oleĭnik [2] has also established a uniqueness result for a special system of two equations which, however, is not symmetric. Rozhdestvenskii [3] has established a uniqueness theorem for piecewise smooth solutions of certain quasilinear systems but his methods are entirely different from those employed here.

2. In  $D = \{(x, t) : -\infty < x < \infty, 0 \leq t < \infty\}$  we consider the quasi-

linear system of  $r$  equations

$$(1) \quad \frac{\partial u}{\partial t} + \frac{\partial \mathcal{A}(x, t, u)}{\partial x} = 0$$

for the (vector) function  $u(x, t) = (u_1(x, t), \dots, u_r(x, t))$  where

$$\mathcal{A}(x, t, u) = (a_1(x, t, u), \dots, a_r(x, t, u)).$$

The following assumptions will be made:

I. The functions  $a_i(x, t, u)$  possess derivatives  $\partial a_i / \partial u_j$ ,  $\partial^2 a_i / \partial x \partial u_j$  and  $\partial^2 a_i / \partial u_j \partial u_k$  which are bounded subsets of  $(x, t, u)$ -space.

II. Let

$$\frac{\partial a_i(x, t, u)}{\partial u_j} = a_{ij}(x, t, u).$$

Then, if  $u$  is bounded, i.e.,  $\sum u_i^2 \leq M^2$ , there exists a constant  $c$ , depending only on  $M$ , such that

$$-c \sum_{i=1}^r \xi_i^2 \leq \sum_{i,j=1}^r a_{ij}(x, t, u) \xi_i \xi_j \leq c \sum_{i=1}^r \xi_i^2$$

for all vectors  $\xi = (\xi_1, \dots, \xi_r)$ .

III (Symmetry). For all  $x, t$ , and  $u$ ,

$$a_{ij}(x, t, u) = a_{ji}(x, t, u) \quad (i, j = 1, \dots, r)$$

IV (Convexity). For all  $x, t$ , and  $u$ , and each  $k = 1, \dots, r$ , we have

$$\sum_{i,j=1}^r \frac{\partial a_{ij}(x, t, u)}{\partial u_k} \xi_i \xi_j \geq 0$$

for all vectors  $\xi = (\xi_1, \dots, \xi_r)$ .

DEFINITION. Let  $\psi(x)$  be an essentially bounded measurable function defined on  $-\infty < x < \infty$ . An essentially bounded measurable function  $u(x, t)$  is called a weak solution of (1) in  $D$  with initial conditions  $\psi(x)$  if,

(a) for every test function  $\varphi(x, t)$  which is continuously differentiable with compact support in the  $(x, t)$ -plane we have

$$(2) \quad \int_D \left[ \left\langle u, \frac{\partial \varphi}{\partial t} \right\rangle + \left\langle A(t, x, u), \frac{\partial \varphi}{\partial x} \right\rangle \right] dx dt + \int_{-\infty}^{\infty} \langle \varphi(x, 0), \psi(x) \rangle dx = 0$$

where  $\langle, \rangle$  is the inner product in Euclidean  $r$ -space;

(b) given any compact subset of  $D$  there is a corresponding function  $K(t) \in L^1_{loc}([0, \infty))$  such that

$$(3) \quad \frac{u_i(x_1, t) - u_i(x_2, t)}{x_1 - x_2} \leq K(t)$$

( $i = 1, \dots, r$ ) holds a.e. for  $x_1$  and  $x_2$  in the compact subset, and  $0 < t < \infty$ .

**THEOREM.** *Weak solutions of (1) are uniquely determined by their initial conditions.*

*Proof.* Let  $u_1(x, t)$  and  $u_2(x, t)$  be two weak solutions of (1) with the same initial conditions  $\psi(x)$ . We will show that, if  $F(x, t) = (F_1(x, t), \dots, F_r(x, t))$  is any smooth (vector) function with compact support contained in  $t > 0$ , then

$$\int_D \langle u_1 - u_2, F \rangle dx dt = 0,$$

thus proving that  $u_1 = u_2$  a.e. in  $D$ .

Let  $\omega^n$  be the usual Gaussian averaging kernel with support contained in the sphere  $x^2 + t^2 \leq 1/n^2$ . Given a function  $\varphi(x, t) \in L^2_{loc}(D)$  we define the averaged function  $\varphi^n(x, t)$  by convolution;  $\varphi^n = \varphi * \omega^n$ . By a familiar argument we see that  $u_{i,k}^n \rightarrow u_{ik}$  ( $i = 1, 2$  and  $k = 1, \dots, r$ ) in mean square on compact subsets of  $D$ . From (3) it follows (see [1]) that

$$(4) \quad \frac{\partial u_{i,k}^n}{\partial x} \leq K(t) \quad (i = 1, 2 \text{ and } k = 1, \dots, r)$$

on compact subsets of  $D$ .

We now define the functions

$$\begin{aligned} \alpha_{ij}(x, t) &= \int_0^1 a_{ij}(x, t, \tau u_1 + (1 - \tau)u_2) d\tau \\ \alpha_{ij}^n(x, t) &= \int_0^1 a_{ij}(x, t, \tau u_1^n + (1 - \tau)u_2^n) d\tau \end{aligned}$$

( $i, j = 1, \dots, r$  and  $n = 1, 2, \dots$ ) and the associated matrices  $A(x, t) = (\alpha_{ij}(x, t))$  and  $A^n(x, t) = (\alpha_{ij}^n(x, t))$ .

It is immediate that

$$\mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2) = A(x, t)(u_1 - u_2).$$

Also

$$| \alpha_{ij}^n(x, t) - \alpha_{ij}(x, t) | \leq \text{const.} [ | u_1^n - u_1 | + | u_2^n - u_2 | ]$$

on compact subsets of  $D$ , from which it follows that  $\alpha_{ij}^n \rightarrow \alpha_{ij}$  in mean square on compact subsets of  $D$ . From II we see that

$$(5) \quad -c\langle \xi, \xi \rangle \leq \langle A^n(x, t)\xi, \xi \rangle \leq c\langle \xi, \xi \rangle$$

for some constant  $c > 0$  and all real vectors  $\xi$ . Finally we note that

$$\begin{aligned} \frac{\partial \alpha_{ij}^n}{\partial x} &= \int_0^1 \left\{ \frac{\partial a_{ij}}{\partial x}(x, t, \tau u_1^n + (1 - \tau)u_2^n) \right. \\ &\quad \left. + \sum_{k=1}^r \frac{\partial a_{ij}}{\partial u_k}(x, t, \tau u_1^n + (1 - \tau)u_2^n) \left[ \frac{\tau \partial u_{1,k}^n}{\partial x} + (1 - \tau) \frac{\partial u_{2,k}^n}{\partial x} \right] \right\} dt . \end{aligned}$$

Using I, IV and (4) it follows that

$$\left\langle \frac{\partial A^n}{\partial x} \xi, \xi \right\rangle \leq K_1(t) \langle \xi, \xi \rangle$$

on compact subsets of  $D$  for every vector  $\xi$ , where  $K_1(t) \in L_{loc}^1([0, \infty))$ .

We now construct for each  $n = 1, 2, \dots$  the vector function  $\varphi^n(x, t)$  satisfying the linear system

$$\frac{\partial \varphi^n}{\partial t} + A^n(x, t) \frac{\partial \varphi^n}{\partial x} = F(x, t)$$

and vanishing on  $t = T$ , where the support of  $F$  is assumed to be below  $t = T$ . This is achieved by solving the system

$$\frac{\partial \tilde{\varphi}^n}{\partial t} - A^n(x, T - t) \frac{\partial \tilde{\varphi}^n}{\partial x} = F(x, T - t)$$

for the vector function  $\tilde{\varphi}^n(x, t)$  in  $D$ , with the initial conditions  $\tilde{\varphi}^n(x, 0) = 0$ , and then putting  $\varphi^n(x, t) = \tilde{\varphi}^n(x, T - t)$ . The classical existence theory guarantees that  $\varphi^n(x, t)$  exists, is smooth, and, by (5), has support contained in a compact set which is independent of  $n$ , and so is a legitimate test function.

Using (2) we obtain

$$\begin{aligned} \int_D \left\langle u_1 - u_2, \frac{\partial \varphi^n}{\partial t} \right\rangle dxdt &= - \int \left\langle \mathcal{A}(x, t, u_1) - \mathcal{A}(x, t, u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dxdt \\ &= - \int_D \left\langle A(x, t)(u_1 - u_2), \frac{\partial \varphi^n}{\partial x} \right\rangle dxdt . \end{aligned}$$

Thus

$$(6) \quad \int_D \langle u_1 - u_2, F \rangle dxdt = \int_D \left\langle u_1 - u_2, (A^n - A) \frac{\partial \varphi^n}{\partial x} \right\rangle dxdt .$$

Using the facts that (i) the supports of the  $\varphi^n$  lie in a fixed compact subset of  $D$ , (ii) the  $u_i$  are essentially bounded and (iii) the coefficients of  $A^n$  converge in the mean square on compact subsets of  $D$  to the coefficients of  $A$ , we see immediately that the right hand side of (6) approaches zero as  $n \rightarrow \infty$ , as long as the mean square norms of the

$\partial\varphi_i^*/\partial x$  (on compact subsets of  $D$ ) are uniformly bounded. The proof will be completed by establishing this fact.

Let  $\partial\tilde{\varphi}^n/\partial x = v^n, A^n(x, T-t) = \tilde{A}^n(x, t)$  and  $F(x, T-t) = \tilde{F}(x, t)$ . Then  $v^n$  satisfies the equation

$$\frac{\partial v^n}{\partial t} - \tilde{A}^n \frac{\partial v^n}{\partial x} - \frac{\partial \tilde{A}^n}{\partial x} v^n = \frac{\partial \tilde{F}}{\partial x}$$

in  $0 \leq t \leq T$ , and the initial conditions  $v^n(x, 0) = 0$ . We may suppose that the supports of the  $v^n (n = 1, 2, \dots)$  in  $0 \leq t \leq T$  are all strictly contained in some interval  $a < x < b$ . Then

$$\frac{\partial}{\partial t} \langle v^n, v^n \rangle - \frac{\partial}{\partial x} \langle \tilde{A}^n v^n, v^n \rangle = 2 \left\langle \frac{\partial \tilde{F}}{\partial x}, v^n \right\rangle + \left\langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \right\rangle.$$

Using Green's formula

$$\begin{aligned} \int_a^b \langle v^n(x, t), v^n(x, t) \rangle dx &\leq \int_0^t \int_a^b 2 \left\langle \frac{\partial \tilde{F}}{\partial x}, v^n \right\rangle dx dt + \int_0^t \int_a^b \left\langle \frac{\partial \tilde{A}^n}{\partial x} v^n, v^n \right\rangle dx dt \\ &\leq \int_0^t \int_a^b \left\langle \frac{\partial \tilde{F}}{\partial x}, \frac{\partial \tilde{F}}{\partial x} \right\rangle dx dt + \int_0^t (1 + K_1(s)) \left[ \int_a^b \langle v(x, s), v(x, s) \rangle dx \right] ds \end{aligned}$$

from which it follows by Gronwall's Lemma that

$$\int_0^T \int_a^b \langle v^n, v^n \rangle dx dt \leq \text{constant},$$

the constant depending on the  $L^2$ -norm of  $\partial\tilde{F}/\partial x$  and  $\int_0^T K_1(t) dt$ , but not on  $n$ . This completes the proof.

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