REPRESENTABLE DISTRIBUTIVE NOETHER LATTICES

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Recently, Bogart showed that a certain class of distributive Noether lattices, namely regular local ones, are embeddable in the lattice of ideals of an appropriate Noetherian ring. In this paper a characterization of the distributive Noether lattices which are representable as the complete lattice of ideals of a Noetherian ring is obtained.

We observe that if \( L(R) \) is the lattice of ideals of a ring \( R \) (commutative with 1) and if \( A, B \) and \( C \) are elements of \( L(R) \) with \( A \not\leq B \) and \( A \not\leq C \), then there exists a principal element \( E \in L(R) \) with \( E \leq A, E \not\leq B \) and \( E \not\leq C \). If a Noether lattice \( L \) has this property, then we will say that \( L \) satisfies the weak union condition. (The term union condition has been used elsewhere for a stronger property.) With this definition, then, the main result of this paper is that a distributive Noether lattice \( L \) is representable as the lattice of ideals of a Noetherian ring if, and only if, \( L \) satisfies the weak union condition.

We adopt the terminology of [2] and we assume throughout that \( L \) is a Noether lattice.

**Lemma 0.** If \( L \) is local, and if the maximal element \( P \in L \) is principal, then every element \( A \neq 0 \) of \( L \) is a power \( P^n(0 \leq n) \) of \( P \).

**Proof.** If \( A \neq 0 \), then by the Intersection Theorem [2] there exists a largest integer \( n \) such that \( A \leq P^n \). Then

\[
A = A \land P^n = (A: P^n)P^n,
\]

so since \( A \not\leq P^{n+1} \), it follows that \( A: P^n = I \), and therefore that \( A = P^n \).

**Lemma 1.** Assume \( L \) is distributive and satisfies the weak union condition. If \( L \) is local and if the maximal element of \( L \) is principal, or if \( 0 \) is prime and every element \( A \neq 0 \) has a primary decomposition involving only powers of maximal primes, then \( L \) is representable as the lattice of ideals of a Noetherian ring.

**Proof.** Assume \( L \) is local with maximal element \( P \), and that \( P \) is principal. Let \((R, M)\) be a regular local ring of altitude one. If \( 0 \) is prime in \( L \), then the powers of \( P \) are distinct, and \( L \) is isomorphic to the lattice of ideals of \( R \). If \( 0 \) is not prime in \( L \), and if \( k \) is the least positive integer such that \( P^k = P^{k+1} \), then \( L \) is isomorphic to the lattice of ideals of \( R | M^k \).
Now, assume that 0 is prime and that every element \( A \neq 0 \) has a primary decomposition \( P_i^{e_i} \cap \cdots \cap P_k^{e_k} \), where each \( P_i \) is maximal. Then every prime \( P \neq 0 \) is maximal, so the \( P_i \) in any decomposition \( A = P_i^{e_i} \cap \cdots \cap P_k^{e_k} \) are just the minimal primes over \( A \). Since 0 is prime in \( L \), it follows that distinct powers of maximal primes are distinct. Then by the comaximality of distinct primes, it follows that every element \( A \neq 0 \) has a factorization as a product of primes \([2]\), and since the primes involved are maximal, the factorizations are unique.

Now, let \( \alpha \) be the cardinality of the collection \( \mathscr{P} \) of maximal primes in \( L \), and let \( K \) be a field of cardinality \( \beta \geq \alpha \). Let \( A \) be a subset of \( K \) of cardinality \( \alpha \), and let \( S \) be the complement in \( K[x] \) of the union of the prime ideals \( (a+x) \), \( a \in A \). Then \( S \) is a multiplicatively closed subset of \( K[x] \) which doesn't meet any of the prime ideals \( (a+x) \), and which meets every other prime ideal. Hence \( K[x]_S \) is a Dedekind Domain with \( \alpha \) maximal primes \([3]\).

We let \( \phi \) be a one-one correspondence between the maximal primes of \( L \) and the maximal primes of \( K[x]_S \), and extend \( \phi \) to a map of \( L \) onto the lattice of ideals of \( K[x]_S \) by taking \( 0 \) to \( 0 \) and products to products. Then since \( L \) is distributive and distinct nonzero primes are comaximal, we have

\[
(i) \quad \left( \prod_1^n P_i^{e_i} \right) \cdot \left( \prod_1^n P_i^{f_i} \right) = \prod_1^n P_i^{e_i+f_i}
\]

\[
(ii) \quad \left( \prod_1^n P_i^{e_i} \right) \wedge \left( \prod_1^n P_i^{f_i} \right) = \left( \bigwedge_1^n P_i^{e_i} \right) \wedge \left( \bigwedge_1^n P_i^{f_i} \right)
\]

\[
= \bigwedge_1^n P_i^{\max(e_i,f_i)} = \prod_1^n P_i^{\max(e_i,f_i)}, \quad \text{and}
\]

\[
(iii) \quad \left( \prod_1^n P_i^{e_i} \right) \lor \left( \prod_1^n P_i^{f_i} \right) = \left( \bigwedge_1^n P_i^{e_i} \right) \lor \left( \bigwedge_1^n P_i^{f_i} \right)
\]

\[
= \bigwedge_1^n P_i^{\min(e_i,f_i)} = \prod_1^n P_i^{\min(e_i,f_i)},
\]

for distinct primes \( P_i \) and for \( e_i, f_i \geq 0 \). Since the lattice of ideals of a Dedekind domain also has these properties \([3]\), it follows that \( \phi \) is an isomorphism of \( L \) onto the lattice of ideals of \( K[x]_S \).

To reduce the general case to the cases covered by Lemma 1, we require the following lemmas.

**Lemma 2.** If \( L \) is distributive and satisfies the weak union condition, and if \( D \in L \), then \( L \upharpoonright D \) and \( L_D \) are distributive and satisfy the weak union condition.

**Proof.** The proof is immediate for \( L \upharpoonright D \), as is the distributivity of \( L_D \). If \( \{A\}, \{B\} \) and \( \{C\} \) are elements of \( L_D \) with \( \{A\} \not\leq \{B\} \) and
\{A\} \subseteq \{C\}, then \(A \not\subseteq B_p\) and \(A \not\subseteq C_p\). So there exists a principal element \(E \in L\) with \(E \subseteq A_p, E \not\subseteq B_p\) and \(E \not\subseteq C_p\). Then \(\{E\}\) is principal with \(\{E\} \subseteq \{A\}, \{E\} \not\subseteq \{B\}\) and \(\{E\} \not\subseteq \{C\}\).

**Lemma 3.** If \(L\) is a distributive local Noether lattice which satisfies the weak union condition, then the maximal element \(P\) of \(L\) is principal.

**Proof.** Let \(A_1, \ldots, A_k\) be a minimal collection of principal elements with join \(P\). If \(k > 1\), then \(P \not\subseteq A_1 \lor \cdots \lor A_{k-1}\) and \(P \not\subseteq A_k\), so there exists a principal element \(A \subseteq P\) with \(A \not\subseteq A_1 \lor \cdots \lor A_{k-1}\) and \(A \not\subseteq A_k\). Then

\[
A = A \land P = A \land [(A_1 \lor \cdots \lor A_{k-1}) \lor A_k]
\]

\[
= ((A_1 \lor \cdots \lor A_{k-1}) \land A) \lor (A_k \land A)
\]

\[
= ((A_1 \lor \cdots \lor A_{k-1}) : A) \lor (A_k : A) A.
\]

Since \(A \neq 0\), it follows from the Intersection Theorem [2] that

\[
(A_1 \lor \cdots \lor A_{k-1}) : A \lor A_k : A = I,
\]

which is a contradiction since \(L\) is local. Hence \(k = 1\).

We are now ready to prove the following

**Theorem 4.** If \(L\) is a distributive Noether lattice, then \(L\) is representable as the lattice of ideals of a Noetherian ring if and only if, \(L\) satisfies the weak union condition.

**Proof.** Since the lattice of ideals of any ring satisfies the weak union condition, the "only if" is clear. Hence, assume \(L\) is a distributive Noether lattice which satisfies the weak union condition. Let

\[
0 = Q_1 \cap \cdots \cap Q_s \cap \cdots \cap Q_k
\]

be a normal decomposition of 0 in which \(Q_i\) is \(P_i\)-primary. We assume that \(P_1, \ldots, P_s\) are nonmaximal elements of \(L\) and that \(P_{s+1}, \ldots, P_k\) are maximal.

By Lemmas 2 and 3 and the Principal Ideal Theorem [2], if \(P\) is any prime in \(L\), then \(P\) has height no greater than one, so every prime is either maximal or minimal. Further, if \(P' < P\) are primes, then by Lemma 0, 0 is prime in \(L_P\), so \(O_P = P' = \Lambda_i P_i\). It follows from this that 0 has no embedded primes, that the primaries \(Q_i, 1 \leq i \leq s\), are the \(P_i\), and that no prime \(P\) contains two distinct minimal primes. Further, since every element, except possibly 0, of \(L_P\) is a power of the maximal element, we have that the \(P\)-primary elements of the
maximal primes $P$ are precisely the powers $P^n$ of $P$.

Then for each $i, s + 1 \leq i \leq k$, there exists a positive integer $e_i$ with $Q_i = P^{e_i}$. Hence $0 = P_1 \cap \cdots \cap P_s \cap P_{s+1} \cap \cdots \cap P_k$. Then since the $P_i$ are pairwise comaximal we have

$$L \cong L | P_1 \oplus \cdots \oplus L | P_s \oplus L | P_{s+1}^{e_{s+1}} \oplus \cdots \oplus L | P_k^{e_k},$$

where each summand is of the type considered in Lemma 1.

Since the lattice of ideals of a direct sum $R_1 \oplus \cdots \oplus R_n$ of rings is isomorphic to the direct sum of the lattices of ideals of the rings, the result now follows.

It is easily seen from the decomposition

$$L \cong L | P_1 \oplus \cdots \oplus L | P_s \oplus L | P_{s+1}^{e_{s+1}} \oplus \cdots \oplus L | P_k^{e_k},$$

in the proof of Theorem 4 that every element of $L$ is a product of primes and that the maximal elements of $L$ are meet principal (in fact that every element is principal). Also, it is seen that the decomposition above characterizes the distributive Noether lattices which are representable as the lattice of ideals of a Noetherian ring. These observations lead to the following theorem which is stated without proof since the proof is similar to that of Theorem 4.

**Theorem 5.** The following are equivalent for a Noether lattice $L$:

(i) $L$ is distributive and representable as the lattice of ideals of a Noetherian ring

(ii) $L$ is distributive and satisfies the weak union condition

(iii) For every maximal element $P$, $L_P$ is linear

(iv) Every element $A$ of $L$ different from $I$ is a product of primes

(v) Every maximal element $P$ of $L$ satisfies the condition $A \land P = (A : P)P$, for all $A$ in $L$

(vi) $L$ is the direct sum $L = L_1 \oplus \cdots \oplus L_n$ of Noether lattices $L_i$, where for each $i$, either $L_i$ is local with a principal maximal element, or $0$ is prime in $L_i$ and every element $A \neq I$ is a (unique) product of primes.

**References**


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UNIVERSITY OF IOWA
Jon F. Carlson, *Automorphisms of groups of similitudes over $F_3$* 485
Archie Gail Gibson, *Triples of operator-valued functions related to the unit circle* 503
David Saul Gillman, *Free curves in $E^3$* 533
E. A. Heard and James Howard Wells, *An interpolation problem for subalgebras of $H^\infty$* 543
Albert Emerson Hurd, *A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems* 555
E. W. Johnson and J. P. Lediaev, *Representable distributive Noether lattices* 561
David G. Kendall, *Incidence matrices, interval graphs and seriation in archeology* 565
Robert Leroy Kruse, *On the join of subnormal elements in a lattice* 571
D. B. Lahiri, *Some restricted partition functions; Congruences modulo 3* 575
Norman D. Lane and Kamla Devi Singh, *Strong cyclic, parabolic and conical differentiability* 583
William Franklin Lucas, *Games with unique solutions that are nonconvex* 599
Eugene A. Maier, *Representation of real numbers by generalized geometric series* 603
Daniel Paul Maki, *A note on recursively defined orthogonal polynomials* 611
Mark Mandelker, *$F'$-spaces and $z$-embedded subspaces* 615
James R. McLaughlin and Justin Jesse Price, *Comparison of Haar series with gaps with trigonometric series* 623
Ernest A. Michael and A. H. Stone, *Quotients of the space of irrationals* 629
William H. Mills and Neal Zierler, *On a conjecture of Golomb* 635
J. N. Pandey, *An extension of Haimo's form of Hankel convolutions* 641
Terence John Reed, *On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles* 653
Haskell Paul Rosenthal, *A characterization of the linear sets satisfying Herz’s criterion* 663
George Thomas Sallee, *The maximal set of constant width in a lattice* 669
I. H. Sheth, *On normaloid operators* 675
James D. Stasheff, *Torsion in BBSO* 677
Billy Joe Thorne, A – $P$ congruences on Baer semigroups 681
Robert Breckenridge Warfield, Jr., *Purity and algebraic compactness for modules* 699
Joseph Zaks, *On minimal complexes* 721