

Pacific Journal of Mathematics

ON THE JOIN OF SUBNORMAL ELEMENTS IN A LATTICE

ROBERT LEROY KRUSE

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Of fundamental importance to the study of subnormal subgroups is the following result of Wielandt:

Let A and B be subnormal subgroups of a group G such that A is normal in $A \cup B$. Then $A \cup B$ is subnormal in G .

The usual proof of Wielandt's result depends on the construction by conjugation of a special subnormal series from A to G . It would be of interest to obtain a proof which uses only the given subnormal series, without explicit dependence on conjugation, and valid in algebraic systems other than groups.

This note presents, in the more general context of a lattice with the normality relation introduced by R. A. Dean, a proof of the analogous result in case either A or B has defect three or less.

We begin with the definition of a lattice normality relation from [1].

DEFINITION. A reflexive relation \triangleleft on a lattice \mathfrak{L} is called a *normality relation* if, for all $a, b, c, d \in \mathfrak{L}$:

- (1) $a \triangleleft b$ implies $a \leq b$,
- (2) $a \triangleleft b, c \triangleleft d$ implies $a \cap c \triangleleft b \cap d$,
- (3) $a \triangleleft b, a \triangleleft c$ implies $a \triangleleft b \cup c$,
- (4) $a \triangleleft b, c \triangleleft d$ implies $a \cup c \triangleleft a \cup c \cup (b \cap d)$,
- (5) $a \leq b$ and either $a \triangleleft a \cup c$ or $c \triangleleft a \cup c$ implies

$$a \cup (b \cap c) = b \cap (a \cup c).$$

An element a of a lattice \mathfrak{L} is called *subnormal* in $b \in \mathfrak{L}$, denoted $a \triangleleft \triangleleft b$, if there exists a chain of elements $a_i \in \mathfrak{L}, i = 0, 1, \dots, n$, such that

$$a = a_n \triangleleft a_{n-1} \triangleleft \dots \triangleleft a_0 = b.$$

The length of the shortest such chain is called the *defect* of a in b . Suppose $a \triangleleft \triangleleft u$ and $b_3 \triangleleft b_2 \triangleleft b_1 \triangleleft u$. We shall prove:

THEOREM 1. *If $b_3 \triangleleft a \cup b_3$, then $a \cup b_3 \triangleleft \triangleleft u$.*

THEOREM 2. *If $a \triangleleft a \cup b_3$, then $a \cup b_3 \triangleleft \triangleleft u$.*

The following results will be needed in the proofs.

LEMMA A. *If $x \triangleleft \triangleleft u, y \triangleleft \triangleleft u$, and x has defect 2 or less in u , then $x \cup y \triangleleft \triangleleft u$.*

LEMMA B. *If $a \leq x \leq b$ and $a \triangleleft b$, then $a \triangleleft x$.*

Lemma A is proved in [1], while Lemma B is an immediate consequence of (2).

Proof of Theorem 1. Since $b_3 \triangleleft a \cup b_3$ and $b_3 \triangleleft b_2$, by (3),

$$b_3 \triangleleft (a \cup b_3) \cup b_2 = a \cup b_2 .$$

By intersection of subnormal chains $a \triangleleft \triangleleft a \cup b_2$. Then, by Lemma A, $a \cup b_3 \triangleleft \triangleleft a \cup b_2$, and $a \cup b_2 \triangleleft \triangleleft u$. Thus $a \cup b_3 \triangleleft \triangleleft u$.

Proof of Theorem 2. Let the given subnormal chain from a to u be

$$a = a_n \triangleleft a_{n-1} \triangleleft \dots \triangleleft a_0 = u .$$

Define, for $m = 0, 1, \dots, n$,

$$x_m = a \cup b_3 \cup (a_m \cap b_2) .$$

By a finite induction it will be shown that $x_m \triangleleft \triangleleft x_{m-1}, 1 \leq m \leq n$. But $x_n = a \cup b_3$, and $x_0 = a \cup b_2$, so, by Lemma A, $x_0 \triangleleft \triangleleft u$. $a \cup b_3 \triangleleft \triangleleft u$ thus follows from transitivity of subnormality. Since the relation $a \cup (a_0 \cap b_2) = a_0 \cap x_0$ is trivial, the proof of Theorem 2 will be complete upon verification of the induction step:

LEMMA C. *Suppose $a \cup (a_{m-1} \cap b_2) = a_{m-1} \cap x_{m-1}$. Then $x_m \triangleleft \triangleleft x_{m-1}$ and $a \cup (a_m \cap b_2) = a_m \cap x_m$.*

Proof of lemma. Define

$$(i) \quad y = b_1 \cap [a \cup (a_m \cap b_2)] .$$

We shall begin by proving

$$(ii) \quad b_3 \cup y \triangleleft x_{m-1} .$$

To prove (ii) let us first observe that, by (2),

$$(iii) \quad y \triangleleft a \cup (a_m \cap b_2) .$$

From $b_2 \triangleleft b_1 \geq y \cup b_2$ Lemma B gives $b_2 \triangleleft y \cup b_2$. This, with

$$a_m \cap b_2 \leq y \leq a_m ,$$

implies by (5)

$$(iv) \quad y = y \cup (a_m \cap b_2) = a_m \cap (y \cup b_2) .$$

Since $a_m \triangleleft a_{m-1}$, (2) then gives $y \triangleleft a_{m-1} \cap (y \cup b_2)$, and (5) implies $a_{m-1} \cap (y \cup b_2) = y \cup (a_{m-1} \cap b_2)$. Next, by (3) let us combine

$$y \triangleleft y \cup (a_{m-1} \cap b_2)$$

with (iii) to obtain $y \triangleleft a \cup (a_{m-1} \cap b_2)$. Therefore, by the hypothesis of the lemma,

$$(v) \quad y \triangleleft a_{m-1} \cap x_{m-1} .$$

Hence, with $b_3 \triangleleft b_2$, (4) gives

$$(vi) \quad b_3 \cup y \triangleleft b_3 \cup y \cup (b_2 \cap a_{m-1}) .$$

In addition, $a \triangleleft a \cup b_3$ implies

$$\begin{aligned} b_3 \cup (a \cap b_1) &= b_1 \cap (a \cup b_3) && \text{by (5)} \\ &\triangleleft a \cup b_3 && \text{by (2)} . \end{aligned}$$

Since $a \cap b_1 \leq y$, (4) and (v) imply

$$\begin{aligned} b_3 \cup y &= \{b_3 \cup (a \cap b_1)\} \cup y \\ &\triangleleft b_3 \cup y \cup [(a \cup b_3) \cap a_{m-1} \cap x_{m-1}] \geq a , \end{aligned}$$

so Lemma B gives $b_3 \cup y \triangleleft b_3 \cup y \cup a$. Finally, by (3), let us combine this with (vi) to obtain

$$b_3 \cup y \triangleleft b_3 \cup y \cup a \cup (a_{m-1} \cap b_2) = x_{m-1} .$$

Thus (ii) is proved.

We next establish $x_m \triangleleft \triangleleft x_{m-1}$. From $b_1 \triangleleft u \geq a \cup b_1$ Lemma B yields $b_1 \triangleleft a \cup b_1$. Hence

$$\begin{aligned} x_m &= b_3 \cup a \cup (a_m \cap b_2) \\ &= b_3 \cup \{[a \cup (a_m \cap b_2)] \cap (a \cup b_1)\} && \text{by absorption} \\ &= b_3 \cup \{a \cup \{b_1 \cap [a \cup (a_m \cap b_2)]\}\} && \text{by (5)} \\ &= a \cup b_3 \cup y && \text{by (i)} . \end{aligned}$$

But $b_3 \cup y \triangleleft x_{m-1}$ and $a \triangleleft \triangleleft x_{m-1}$, so Lemma A gives

$$x_m = a \cup (b_3 \cup y) \triangleleft \triangleleft x_{m-1} .$$

Finally, we prove $a \cup (a_m \cap b_2) = a_m \cap x_m$. By (ii) $b_3 \cup y \triangleleft x_{m-1}$, and $a \cup (a_m \cap b_2) \leq x_m \leq x_{m-1}$, so Lemma B gives

$$b_3 \cup y \triangleleft (b_3 \cup y) \cup [a \cup (a_m \cap b_2)] .$$

Thus,

$$\begin{aligned}
\alpha_m \cap x_m &= \alpha_m \cap \{b_3 \cup a \cup (\alpha_m \cap b_2)\} && \text{by definition of } x_m \\
&= \alpha_m \cap \{(b_3 \cup y) \cup [a \cup (\alpha_m \cap b_2)]\} && \text{since, by (i), } y \leq a \cup (\alpha_m \cap b_2) \\
&= [a \cup (\alpha_m \cap b_2)] \cup \{\alpha_m \cap (b_3 \cup y)\} && \text{by (5)} \\
&\leq a \cup [\alpha_m \cap (b_2 \cup y)] \\
&= a \cup y && \text{by (iv)} \\
&\leq a \cup (\alpha_m \cap b_2) && \text{by (i) .}
\end{aligned}$$

The reverse containment is obvious. Thus $\alpha_m \cap x_m = a \cup (\alpha_m \cap b_2)$, and the proof is complete.

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