ON THE JOIN OF SUBNORMAL ELEMENTS IN A LATTICE

ROBERT LEROY KRUSE
ON THE JOIN OF SUBNORMAL ELEMENTS
IN A LATTICE

ROBERT L. KRUSE

Of fundamental importance to the study of subnormal subgroups is the following result of Wielandt:

Let $A$ and $B$ be subnormal subgroups of a group $G$ such that $A$ is normal in $A \cup B$. Then $A \cup B$ is subnormal in $G$.

The usual proof of Wielandt's result depends on the construction by conjugation of a special subnormal series from $A$ to $G$. It would be of interest to obtain a proof which uses only the given subnormal series, without explicit dependence on conjugation, and valid in algebraic systems other than groups.

This note presents, in the more general context of a lattice with the normality relation introduced by R. A. Dean, a proof of the analogous result in case either $A$ or $B$ has defect three or less.

We begin with the definition of a lattice normality relation from [1].

**Definition.** A reflexive relation $\triangleleft$ on a lattice $\mathcal{L}$ is called a normality relation if, for all $a, b, c, d \in \mathcal{L}$:

1. $a \triangleleft b$ implies $a \leq b$,
2. $a \triangleleft b, c \triangleleft d$ implies $a \cap c \triangleleft b \cap d$,
3. $a \triangleleft b, a \triangleleft c$ implies $a \triangleleft b \cup c$,
4. $a \triangleleft b, c \triangleleft d$ implies $a \cup c \triangleleft a \cup c \cup (b \cap d)$,
5. $a \leq b$ and either $a \triangleleft a \cup c$ or $c \triangleleft a \cup c$ implies $a \cup (b \cap c) = b \cap (a \cup c)$.

An element $a$ of a lattice $\mathcal{L}$ is called subnormal in $b \in \mathcal{L}$, denoted $a \triangleleft \triangleleft b$, if there exists a chain of elements $a_i \in \mathcal{L}, i = 0, 1, \ldots, n$, such that

$$a = a_n \triangleleft a_{n-1} \triangleleft \cdots \triangleleft a_0 = b.$$ 

The length of the shortest such chain is called the defect of $a$ in $b$.

Suppose $a \triangleleft \triangleleft u$ and $b_3 \triangleleft b_2 \triangleleft b_1 \triangleleft u$. We shall prove:

**Theorem 1.** If $b_3 \triangleleft a \cup b_3$, then $a \cup b_3 \triangleleft \triangleleft u$.

**Theorem 2.** If $a \triangleleft a \cup b_3$, then $a \cup b_3 \triangleleft \triangleleft u$.

The following results will be needed in the proofs.
LEMMA A. If \( x \triangleleft \triangleleft u, y \triangleleft \triangleleft u \), and \( x \) has defect 2 or less in \( u \), then \( x \cup y \triangleleft \triangleleft u \).

LEMMA B. If \( a \leq x \leq b \) and \( a \triangleleft b \), then \( a \triangleleft x \).

Lemma A is proved in [1], while Lemma B is an immediate consequence of (2).

**Proof of Theorem 1.** Since \( b_3 \triangleleft a \cup b_3 \) and \( b_3 \triangleleft b_3 \), by (3),
\[ b_3 \triangleleft (a \cup b_3) \cup b_2 = a \cup b_2. \]
By intersection of subnormal chains \( a \triangleleft \triangleleft a \cup b_3 \). Then, by Lemma A, \( a \cup b_3 \triangleleft \triangleleft a \cup b_2 \), and \( a \cup b_2 \triangleleft \triangleleft u \). Thus \( a \cup b_3 \triangleleft \triangleleft u \).

**Proof of Theorem 2.** Let the given subnormal chain from \( a \) to \( u \) be
\[ a = a_n \triangleleft a_{n-1} \triangleleft \cdots \triangleleft a_0 = u. \]
Define, for \( m = 0, 1, \cdots, n \),
\[ x_m = a \cup b_3 \cup (a_m \cap b_2). \]
By a finite induction it will be shown that \( x_m \triangleleft \triangleleft x_{m-1}, 1 \leq m \leq n \). But \( x_n = a \cup b_3 \), and \( x_0 = a \cup b_2 \), so, by Lemma A, \( x_0 \triangleleft \triangleleft u \). \( a \cup b_3 \triangleleft \triangleleft u \) thus follows from transitivity of subnormality. Since the relation \( a \cup (a_0 \cap b_2) = a_0 \cap x_0 \) is trivial, the proof of Theorem 2 will be complete upon verification of the induction step:

**LEMMA C.** Suppose \( a \cup (a_{m-1} \cap b_2) = a_{m-1} \cap x_{m-1} \). Then \( x_m \triangleleft \triangleleft x_{m-1} \) and \( a \cup (a_m \cap b_2) = a_m \cap x_m \).  

**Proof of lemma.** Define
\[ (i) \quad y = b_1 \cap [a \cup (a_m \cap b_2)]. \]
We shall begin by proving
\[ (ii) \quad b_3 \cup y \triangleleft \triangleleft x_{m-1}. \]
To prove (ii) let us first observe that, by (2),
\[ (iii) \quad y \triangleleft \triangleleft a \cup (a_m \cap b_2). \]
From \( b_2 \triangleleft b_1 \geq y \cup b_2 \) Lemma B gives \( b_2 \triangleleft \triangleleft y \cup b_2 \). This, with
\[ a_m \cap b_2 \leq y \leq a_m, \]
implies by (5)
(iv) \[ y = y \cup (a_m \cap b_2) = a_m \cap (y \cup b_2). \]

Since \( a_m \triangleleft a_{m-1}, \) (2) then gives \( y \triangleleft a_{m-1} \cap (y \cup b_2), \) and (5) implies \( a_{m-1} \cap (y \cup b_2) = y \cup (a_{m-1} \cap b_2). \) Next, by (3) let us combine

\[ y \triangleleft y \cup (a_{m-1} \cap b_2) \]

with (iii) to obtain \( y \triangleleft a \cup (a_{m-1} \cap b_2). \) Therefore, by the hypothesis of the lemma,

(v) \[ y \triangleleft a_{m-1} \cap x_{m-1}. \]

Hence, with \( b_3 \triangleleft b_2, \) (4) gives

(vi) \[ b_3 \cup y \triangleleft b_3 \cup y \cup (b_2 \cap a_{m-1}). \]

In addition, \( a \triangleleft a \cup b_3 \) implies

\[ b_3 \cup (a \cap b_3) = b_1 \cap (a \cup b_3) \quad \text{by (5)} \]

\[ \triangleleft a \cup b_3 \quad \text{by (2)}. \]

Since \( a \cap b_1 \leq y, \) (4) and (v) imply

\[
\begin{align*}
b_3 \cup y &= \{b_3 \cup (a \cap b_3)\} \cup y \\
&\leq b_3 \cup y \cup [(a \cup b_3) \cap a_{m-1} \cap x_{m-1}] \geq a,
\end{align*}
\]

so Lemma B gives \( b_3 \cup y \leq b \cup y \cup a. \) Finally, by (3), let us combine this with (vi) to obtain

\[ b_3 \cup y \leq b_3 \cup y \cup a \cup (a_{m-1} \cap b_2) = x_{m-1}. \]

Thus (ii) is proved.

We next establish \( x_m \triangleleft x_{m-1}. \) From \( b_1 \triangleleft a \cup b_1 \) Lemma B yields \( b_1 \triangleleft a \cup b_1. \) Hence

\[
\begin{align*}
x_m &= b_3 \cup a \cup (a_m \cap b_2) \\
&= b_3 \cup \{[a \cup (a_m \cap b_2)] \cap (a \cup b_1)\} \quad \text{by absorption} \\
&= b_3 \cup \{a \cup \{b_1 \cap [a \cup (a_m \cap b_2)]\}\} \quad \text{by (5)} \\
&= a \cup b_3 \cup y \quad \text{by (i)}. 
\end{align*}
\]

But \( b_3 \cup y \leq x_{m-1} \) and \( a \leq x_{m-1}, \) so Lemma A gives

\[ x_m = a \cup (b_3 \cup y) \triangleleft x_{m-1}. \]

Finally, we prove \( a \cup (a_m \cap b_2) = a_m \cap x_m. \) By (ii) \( b_3 \cup y \leq x_{m-1}, \) and \( a \cup (a_m \cap b_2) \leq x_m \leq x_{m-1}, \) so Lemma B gives

\[ b_3 \cup y \leq (b_3 \cup y) \cup [a \cup (a_m \cap b_2)]. \]

Thus,
\[ a_m \cap x_m = a_m \cap \{ b_3 \cup a \cup (a_m \cap b_2) \} \quad \text{by definition of } x_m \]
\[ = a_m \cap \{ (b_3 \cup y) \cup [a \cup (a_m \cap b_2)] \} \quad \text{since, by (i), } y \leq a \cup (a_m \cap b_2) \]
\[ = [a \cup (a_m \cap b_2)] \cup \{ a_m \cap (b_3 \cup y) \} \quad \text{by (5)} \]
\[ \leq a \cup [a_m \cap (b_2 \cup y)] \]
\[ = a \cup y \quad \text{by (iv)} \]
\[ \leq a \cup (a_m \cap b_2) \quad \text{by (i)} . \]

The reverse containment is obvious. Thus \( a_m \cap x_m = a \cup (a_m \cap b_2) \), and the proof is complete.

The author wishes to thank the Sandia Corporation for the use of an electronic computer by which partial results pertaining to this paper were first found, and the referee for suggesting the inclusion of several details to clarify the proofs.

**References**


Received March 19, 1968. The author wishes to thank the United States Atomic Energy Commission for financial support.

SANDIA LABORATORY
ALBUQUERQUE, NEW MEXICO
The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the Pacific Journal of Mathematics should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. 36, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The Pacific Journal of Mathematics is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
Jon F. Carlson, *Automorphisms of groups of similitudes over $F_3$* .......................... 485
Archie Gail Gibson, *Triples of operator-valued functions related to the unit circle* ................................................................. 503
David Saul Gillman, *Free curves in $E^3$* ................................................................. 533
E. A. Heard and James Howard Wells, *An interpolation problem for subalgebras of $H^\infty$* ................................................................. 543
Albert Emerson Hurd, *A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems* ........................................... 555
E. W. Johnson and J. P. Lediaev, *Representable distributive Noether lattices* ................................................................. 561
David G. Kendall, *Incidence matrices, interval graphs and seriation in archeology* ................................................................. 565
Robert Leroy Kruse, *On the join of subnormal elements in a lattice* ................. 571
D. B. Lahiri, *Some restricted partition functions; Congruences modulo 3* .......... 575
Norman D. Lane and Kamla Devi Singh, *Strong cyclic, parabolic and conical differentiability* ................................................................. 583
William Franklin Lucas, *Games with unique solutions that are nonconvex* .................. 599
Eugene A. Maier, *Representation of real numbers by generalized geometric series* ................................................................. 603
Daniel Paul Maki, *A note on recursively defined orthogonal polynomials* ............ 611
Mark Mandelker, *$F'$-spaces and $z$-embedded subspaces* ...................................... 615
James R. McLaughlin and Justin Jesse Price, *Comparison of Haar series with gaps with trigonometric series* ................................................................. 623
Ernest A. Michael and A. H. Stone, *Quotients of the space of irrationals* ............... 629
William H. Mills and Neal Zierler, *On a conjecture of Golomb* ............................ 635
J. N. Pandey, *An extension of Haimo’s form of Hankel convolutions* .................. 641
Terence John Reed, *On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles* ................................................................. 653
Haskell Paul Rosenthal, *A characterization of the linear sets satisfying Herz’s criterion* ................................................................. 663
George Thomas Sallee, *The maximal set of constant width in a lattice* ................. 669
I. H. Sheth, *On normaloid operators* ................................................................. 675
James D. Stasheff, *Torsion in BBSO* ................................................................. 677
Billy Joe Thorne, *A – P congruences on Baer semigroups* ..................................... 681
Robert Breckenridge Warfield, Jr., *Purity and algebraic compactness for modules* ................................................................. 699
Joseph Zaks, *On minimal complexes* ..............................