SOME RESTRICTED PARTITION FUNCTIONS;
CONGRUENCES MODULO 3

D. B. LAHIRI
SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 3

D. B. LAHIRI

We shall establish in this paper some congruence relations with respect to the modulus 3 for some restricted partition functions. The difference between the unrestricted partition function, \( p(n) \), and these restricted partition functions which we shall denote by 

\[ \tau_r p(n) \] with \( r = 3, 6, 12 \),

merely lies in the restriction that no number of the forms \( 27n \), or \( \pm 7n \), shall be a part of the partitions which are of relevance in the restricted case. Thus to determine the value of \( \tau_r p(n) \) one should count all the unrestricted partitions of \( n \) excepting those which contain a number of any of the above forms as a part. We shall assume \( p(n) \) and \( \tau_r p(n) \) to be unity when \( n \) is zero, and vanishing when the argument is negative. We can now state our theorems.

**Theorem 1.** For almost all values of \( n \)

\[ \tau_3 p(n) \equiv \tau_6 p(n) \equiv \tau_{12} p(n) \equiv 0 \pmod{3} . \]

**Theorem 2.** For all values of \( n \)

\[ \tau_3 p(3n) \equiv \tau_6 p(3n + 1) \equiv -\tau_{12} p(3n + 2) \pmod{3} . \]

2. Definitions and notations. We shall use \( m \) to denote an integer positive zero or negative, but \( n \) will stand for a positive or nonnegative integer only.

We define \( u_r \) by

\[ u_0 = 1 \quad \text{and} \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n . \sum_{n=0}^{\infty} p(n)x^n , \quad r > 0 , \]

where \( a_n \) is defined by the well-known `pentagonal number' theorem of Euler,

\[ f(x) = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=0}^{\infty} (-1)^n x^{\frac{1}{3}m(3m + 1)} = \sum_{n=0}^{\infty} a_n x^n , \]

and \( p(n) \) is the number of unrestricted partitions of \( n \) given by the expansion,

\[ [f(x)]^{-1} = \left[ \prod_{n=1}^{\infty} (1 - x^n) \right]^{-1} = \sum_{n=0}^{\infty} p(n)x^n . \]

We shall use \( v \) to denote the pentagonal numbers,
\( v = \frac{1}{2} m(3m + 1), m = 0, \pm 1, \pm 2, \cdots; \)

and with each \( v \) there corresponds an 'associated' sign, viz., \((-1)^m\).

We shall come across sums of the type

\[
\sum_v \left[ \mp V(v) \right]
\]

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if \( v \) is 1, 2, 12, 15, \cdots, that is, when it is of the form \((2m + 1)(3m + 1)\), and (b) positive if \( v \) is 0, 5, 7, 22, 26 \cdots, that is, when it is of the form \( m(6m + 1) \). With the above summation notation we can write,

\[
(5) \quad u_r = \sum_v \left( \mp v'x' \right) / f(x),
\]

\[
(6) \quad \sum_v \left( \mp x' \right) / f(x) = 1.
\]

We shall also require the functions \( U_i, i = 0, 1, 2 \) which are certain linear functions of \( u_r \)'s, \( r = 0, 1, 2 \) as given below.

\[
(7) \quad \begin{cases} 
U_0 &= -u_2 + u_0, \\
U_1 &= -u_2 - u_1, \\
U_2 &= -u_2 + u_1.
\end{cases}
\]

We also need the quadratics \( P_i(v) \) in \( v, i = 0, 1, 2 \) which are obtained by writing \( P_i(v) \) for \( U_i \), and \( v' \) for \( u_r \). Thus

\[
(8) \quad \begin{cases} 
P_0(v) &= -v^2 + 1, \\
P_1(v) &= -v^2 - v, \\
P_2(v) &= -v^2 + v.
\end{cases}
\]

3. Some lemmas. The truth of the following lemma can be easily verified from the expressions for \( P_i(v) \) given in (8).

**Lemma 1.**

\[
P_i(v) \equiv 1 \pmod{3}, \quad \text{if } v \equiv i \pmod{3}
\]

\[
\equiv 0 \pmod{3}, \quad \text{if } v \not\equiv i \pmod{3}.
\]

If we replace the \( u_r \)'s appearing in the expressions for \( U_i \) in (7) by the right hand expressions in (5) we get

\[
(9) \quad U_i = \sum_v \left[ \mp P_i(v)x' \right] / f(x);
\]

and then the use of Lemma 1 leads to the next lemma.
**Lemma 2.** \( U_i \equiv \sum_{v=i}(-1)^v/f(x) \pmod{3} \), the summation being extended over all pentagonal numbers \( v \equiv i \pmod{3} \).

The truth of the following lemma can be verified without much difficulty by writing \( 3m + j \), with \( j = 0; -1; \) and 1 respectively, in place of \( m \) in the expression \( \frac{1}{2}m(3m + 1) \) for the pentagonal numbers, and in \((-1)^m\) its associated sign. It is also to be remembered that \( \frac{1}{2}(3m - 1)(9m - 2) \) and \( \frac{1}{2}(3m + 1)(9m + 2) \) represent the same set of numbers.

**Lemma 3.** The solutions of
\[ v \equiv i \pmod{3}, \quad i = 0, 1, 2 \]
are as noted below, (the associated signs are also shown).

<table>
<thead>
<tr>
<th>( i )</th>
<th>solutions</th>
<th>sign</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \frac{1}{2}(27m^2 + 3m) )</td>
<td>((-1)^m)</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{2}(27m^2 + 15m) + 1 )</td>
<td>((-1)^{m+1})</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{1}{2}(27m^2 + 21m) + 2 )</td>
<td>((-1)^{m+1})</td>
</tr>
</tbody>
</table>

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [3, p. 283] viz.,

\[ \prod_{n=0}^m [(1 - x^{2kn+k-1})(1 - x^{2kn+k+1})(1 - x^{2kn+2k})] = \sum_{n=-m}^m (-1)^m x^{k(2n+\bar{k})+m}. \]

In establishing this lemma \( k \) and \( l \) are given values which are in conformity with the quadratic expressions in \( m \) given in Lemma 3. As an illustration we have

\[ \sum_{v=0}^\infty (-1)^v \prod_{n=0}^\infty [(1 - x^{27n+\bar{v}})(1 - x^{27n+21})(1 - x^{27n+27})]. \]

**Lemma 4.** Writing \( v \equiv i \) simply for \( v \equiv i \pmod{3} \)

\[ \sum_{v=0}^\infty (-1)^v \prod_{n=0}^\infty [(1 - x^{27n+12})(1 - x^{27n+15})(1 - x^{27n+27})] \]
\[ \sum_{v=1}^\infty (-1)^v \prod_{n=0}^\infty [(1 - x^{27n+6})(1 - x^{27n+21})(1 - x^{27n+27})] . \]
\[ \sum_{v=2}^\infty (-1)^v \prod_{n=0}^\infty [(1 - x^{27n+9})(1 - x^{27n+24})(1 - x^{27n+27})] . \]

Lemma 5, given below is derived from Lemma 2 after the substitution in it of the product expressions for \( \sum_{v=i}(-1)^v \) as given in
the above lemma. The following fact also is to be taken into
consideration.

\[(12) \prod_{n=0}^{\infty} (1 - x^{27n+r})(1 - x^{27n+27})(1 - x^{27n+27})/f(x) = \prod_{n=0}^{\infty} [(1 - x^{27n+r})(1 - x^{27n+27})(1 - x^{27n+27})]/[(1 - x)(1 - x^7)(1 - x^7) \cdots] = \sum_{n=0}^{\infty} p(n)x^n.
\]

**Lemma 5.**

\[
U_0 \equiv \sum_{n=0}^{\infty} p(n)x^n \pmod{3}
\]

\[
U_1 \equiv -\sum_{n=0}^{\infty} p(n - 1)x^n \pmod{3}
\]

\[
U_2 \equiv -\sum_{n=0}^{\infty} p(n - 2)x^n \pmod{3}.
\]

We require another set of congruences which are obtained from
the classical result, due to Catalan [1, p. 290].

\[(13) p(n - 1) + 2p(n - 2) - 5p(n - 5) - 7p(n - 7) + \cdots = \sigma(n),
\]

and another result due to Glaisher [1, p. 312]

\[(14) p(n - 1) + 2p(n - 2) - 5p(n - 5) - 7p(n - 7) + \cdots = -\frac{1}{12}[5\sigma_5(n) - (18n - 1)\sigma(n)].
\]

These results can be rewritten according to our notation as

\[(15) \sum_{\nu} [\nu p(n - \nu)] = -\sigma(n),
\]

\[(16) \sum_{\nu} [\nu^2 p(n - \nu)] = \frac{1}{12} [5\sigma_5(n) - (18n - 1)\sigma(n)].
\]

Now from (5) we have

\[
u_r = \sum_{\nu} (\nu^r x^\nu)/f(x) = \sum_{\nu} (\nu^r x^\nu) \cdot \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=1}^{\infty} \{\sum_{\nu} [\nu^r p(n - \nu)]x^n, \ r > 0 \}.
\]

It is now easy to establish the validity of the following lemma from
the above three relations (15), (16) and (17).
Lemma 6.

\[ u_2 = \sum_{w=1}^{\infty} \left[ 5\sigma_3(w) - (18w - 1)\sigma(n) \right] x^n. \]

The next lemma can be easily obtained by the substitution of the above values of \( u_1 \) and \( u_2 \) in (7).

Lemma 7.

\[ U_0 - 1 = \sum_{w=1}^{\infty} \left[ 5\sigma_3(w) - (18w - 1)\sigma(n) \right] x^n, \]
\[ U_1 = \sum_{w=1}^{\infty} \left[ 5\sigma_3(w) - (18w + 11)\sigma(n) \right] x^n, \]
\[ U_2 = \sum_{w=1}^{\infty} \left[ 5\sigma_3(w) - (18w - 13)\sigma(n) \right] x^n. \]

The congruences given in Lemma 8 are elementary and can be readily proved.

Lemma 8.

\[ \sigma(3n - 1) \equiv 0 \pmod{3}, \]
\[ \sigma(3\lambda n) \equiv \sigma(n) \pmod{3}, \quad \lambda \geq 0. \]

4. Proof of the theorems. By comparing the coefficients of like powers of \( x \) in the expressions (modulo 3) for \( U_i \) given in Lemmas 5 and 7 we obtain the following congruences for \( n > 0 \).

\[ \sigma_3 p(n) \equiv -\frac{1}{12} \left[ 5\sigma_3(n) - (18n - 1)\sigma(n) \right] \pmod{3} \]
\[ \sigma_3 p(n - 1) \equiv -\frac{1}{12} \left[ 5\sigma_3(n) - (18n + 11)\sigma(n) \right] \pmod{3} \]
\[ \sigma_3 p(n - 2) \equiv -\frac{1}{12} \left[ 5\sigma_3(n) - (18n - 13)\sigma(n) \right] \pmod{3}. \]

Remembering the well-known congruence, [4 \( ; 2 \), p. 167],

\[ \sigma_k(n) \equiv 0 \pmod{M} \text{ for almost all } n \]

for arbitrarily fixed \( M \) and odd \( k \), it is a straightforward matter to
deduce Theorem 1 from the above congruences.

To establish Theorem 2 we obtain by a process of addition or subtraction of (18), (19) and (20) in pairs the following.

\[(22) \quad -\psi_{14}p(n) - \psi_{15}p(n - 1) \equiv \psi_{14}p(n) + \psi_{15}p(n - 2) \\
\equiv \psi_{14}p(n - 1) - \psi_{15}p(n - 2) \equiv \sigma(n) \pmod{3} .\]

Now writing $3n + 2$ for $n$ in (22) and making use of the first relation of Lemma 8 we obtain the theorem immediately.

To derive a generalization from (22) we write $3\lambda n$ for $n$ in it and make use of the last congruence of Lemma 8 to obtain,

\[(23) \quad -\psi_{14}p(3\lambda n) - \psi_{15}p(3\lambda n - 1) \equiv \psi_{14}p(3\lambda n) + \psi_{15}p(3\lambda n - 2) \\
\equiv \psi_{14}p(3\lambda n - 1) - \psi_{15}p(3\lambda n - 2) \equiv \sigma(n) \pmod{3} .\]

We need write $3n - 1$ for $n$ in (23) and use the first congruence of Lemma 8 to arrive at the more general Theorem 3.

**Theorem 3.** With respect to the modulus 3

\[ -\psi_{14}p(3^{i+1}n - 3^i) \equiv \psi_{14}p(3^{i+1}n - 3^i - 1) \equiv \psi_{14}p(3^{i+1}n - 3^i - 2) .\]

Finally, it might be of interest to note that the three restricted partition functions $\psi_r p(n)$, $r = 3, 6$ and 12, are connected by the identical relation,

\[(24) \quad \psi_{14}p(n) = \psi_{14}p(n - 1) + \psi_{15}p(n - 2) , \quad n > 0 .\]

This is seen to be true by a joint consideration of (6), Lemma 4, and (12). The first relation gives

\[(25) \quad \sum_{i=0}^{3} \sum_{x=1}^{n} \psi_{14}(x^n)/f(x) = 1 .\]

We substitute the values of $\sum_{x=1}^{n} (x^n)$ in the product form as given in Lemma 4, and then make use of (12) in order to express the left hand side of (25) as a power series in $x$ whose coefficients are simple linear functions of the restricted partition functions. Now (24) is obtained directly by equating to zero the coefficient of $x^n$, $n > 0$.

**References**


Received October 3, 1967, and in revised from May 27, 1968.

Indian Statistical Institute
Calcutta
Pacific Journal of Mathematics
Vol. 28, No. 3 May, 1969

Jon F. Carlson, Automorphisms of groups of similitudes over \( F_3 \) .......... 485
W. Wistar (William) Comfort, Neil Hindman and Stelios A. Negrepontis, \( F' \)-spaces and their product with \( P \)-spaces ................. 489
Archie Gail Gibson, Triples of operator-valued functions related to the unit circle ................................................................. 503
David Saul Gillman, Free curves in \( E^3 \) ........................................ 533
E. A. Heard and James Howard Wells, An interpolation problem for subalgebras of \( H^\infty \) ......................................................... 543
Albert Emerson Hurd, A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems .................................................. 555
E. W. Johnson and J. P. Lediaev, Representable distributive Noether lattices ................................................................. 561
David G. Kendall, Incidence matrices, interval graphs and seriation in archeology ................................................................. 565
Robert Leroy Kruse, On the join of subnormal elements in a lattice ........ 571
D. B. Lahiri, Some restricted partition functions; Congruences modulo 3 ... 575
Norman D. Lane and Kamla Devi Singh, Strong cyclic, parabolic and conical differentiability .................................................. 583
William Franklin Lucas, Games with unique solutions that are nonconvex ................................................................. 599
Eugene A. Maier, Representation of real numbers by generalized geometric series ................................................................. 603
Daniel Paul Maki, A note on recursively defined orthogonal polynomials . 611
Mark Mandelker, \( F' \)-spaces and \( z \)-embedded subspaces ......................... 615
James R. McLaughlin and Justin Jesse Price, Comparison of Haar series with gaps with trigonometric series ........................................ 623
Ernest A. Michael and A. H. Stone, Quotients of the space of irrationals .... 629
William H. Mills and Neal Zierler, On a conjecture of Golomb ................ 635
J. N. Pandey, An extension of Haimo’s form of Hankel convolutions .......... 641
Terence John Reed, On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles ......................... 653
Haskell Paul Rosenthal, A characterization of the linear sets satisfying Herz’s criterion ................................................................. 663
George Thomas Sallee, The maximal set of constant width in a lattice ...... 669
I. H. Sheth, On normaloid operators ............................................. 675
James D. Stasheff, Torsion in BBSO ............................................. 677
Billy Joe Thorne, A – \( P \) congruences on Baer semigroups ..................... 681
Robert Breckenridge Warfield, Jr., Purity and algebraic compactness for modules ................................................................. 699
Joseph Zaks, On minimal complexes ............................................. 721