SOME RESTRICTED PARTITION FUNCTIONS; CONGRUENCES MODULO 3

D. B. Lahiri
SOME RESTRICTED PARTITION FUNCTIONS:
CONGRUENCES MODULO 3

D. B. LAHIRI

We shall establish in this paper some congruence relations with respect to the modulus 3 for some restricted partition functions. The difference between the unrestricted partition function, \( p(n) \), and these restricted partition functions which we shall denote by

\[ \frac{\tau}{r} p(n) \] with \( r = 3, 6, 12 \),

merely lies in the restriction that no number of the forms \( 27n \), or \( 27n \pm r \), shall be a part of the partitions which are of relevance in the restricted case. Thus to determine the value of \( \frac{\tau}{r} p(n) \) one should count all the unrestricted partitions of \( n \) excepting those which contain a number of any of the above forms as a part. We shall assume \( p(n) \) and \( \frac{\tau}{r} p(n) \) to be unity when \( n \) is zero, and vanishing when the argument is negative. We can now state our theorems.

**Theorem 1.** For almost all values of \( n \)

\[ \frac{\tau}{3} p(n) \equiv \frac{\tau}{6} p(n) \equiv \frac{\tau}{12} p(n) \equiv 0 \pmod{3} \, . \]

**Theorem 2.** For all values of \( n \)

\[ \frac{\tau}{3} p(3n) \equiv \frac{\tau}{6} p(3n + 1) \equiv -\frac{\tau}{12} p(3n + 2) \pmod{3} \, . \]

2. Definitions and notations. We shall use \( m \) to denote an integer positive zero or negative, but \( n \) will stand for a positive or nonnegative integer only.

We define \( u_r \) by

\[ u_0 = 1 \quad \text{and} \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n. \sum_{n=0}^{\infty} p(n)x^n, \quad r > 0 \, , \]

where \( a_n \) is defined by the well-known ‘pentagonal number’ theorem of Euler,

\[ f(x) = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{n=0}^{\infty} (-1)^m x^{3m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n, \]

and \( p(n) \) is the number of unrestricted partitions of \( n \) given by the expansion,

\[ [f(x)]^{-1} = \left[ \prod_{n=1}^{\infty} (1 - x^n) \right]^{-1} = \sum_{n=0}^{\infty} p(n)x^n. \]

We shall use \( v \) to denote the pentagonal numbers,
(4) \[ v = \frac{1}{2} m(3m + 1), \quad m = 0, \pm 1, \pm 2, \cdots; \]

and with each \( v \) there corresponds an 'associated' sign, viz., \((-1)^m\).

We shall come across sums of the type

\[ \sum_v [\mp V(v)] \]

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if \( v \) is \( 1, 2, 12, 15, 35, \cdots \), that is, when it is of the form \((2m + 1)(3m + 1)\), and (b) positive if \( v \) is \( 0, 5, 7, 22, 26 \cdots \), that is, when it is of the form \( m(6m + 1) \). With the above summation notation we can write,

(5) \[ u_r = \sum_v (\mp v^r x^r)/f(x), \]

(6) \[ \sum_v (\mp x^r)/f(x) = 1. \]

We shall also require the functions \( U_i, \quad i = 0, 1, 2 \) which are certain linear functions of \( u_r \)'s, \( r = 0, 1, 2 \) as given below.

\[
\begin{align*}
U_0 &= -u_2 + u_0, \\
U_1 &= -u_2 - u_1, \\
U_2 &= -u_2 + u_1.
\end{align*}
\]

We also need the quadratics \( P_i(v) \) in \( v, \quad i = 0, 1, 2 \) which are obtained by writing \( P_i(v) \) for \( U_i \), and \( v^r \) for \( u_r \). Thus

\[
\begin{align*}
P_0(v) &= -v^2 + 1, \\
P_1(v) &= -v^2 - v, \\
P_2(v) &= -v^2 + v.
\end{align*}
\]

3. Some lemmas. The truth of the following lemma can be easily verified from the expressions for \( P_i(v) \) given in (8).

**Lemma 1.**

\[ P_i(v) \equiv 1 \pmod{3}, \quad \text{if} \quad v \equiv i \pmod{3} \]

\[ \equiv 0 \pmod{3}, \quad \text{if} \quad v \not\equiv i \pmod{3}. \]

If we replace the \( u_r \)'s appearing in the expressions for \( U_i \) in (7) by the right hand expressions in (5) we get

(9) \[ U_i = \sum_v [\mp P_i(v)x^r]/f(x); \]

and then the use of Lemma 1 leads to the next lemma.
**Lemma 2.** $U_i \equiv \sum_{v \equiv i(\mod 3)}(\mp x^v)/f(x) \pmod{3}$, the summation being extended over all pentagonal numbers $v \equiv i(\mod 3)$.

The truth of the following lemma can be verified without much difficulty by writing $3m + j$, with $j = 0$; $-1$; and $1$ respectively, in place of $m$ in the expression $\frac{1}{3} m (3m + 1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered that $\frac{1}{3}(3m - 1)(9m - 2)$ and $\frac{1}{3}(3m + 1)(9m + 2)$ represent the same set of numbers.

**Lemma 3.** The solutions of

$$v \equiv i \pmod{3}, \quad i = 0, 1, 2$$

are as noted below, (the associated signs are also shown).

<table>
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<th>$i$</th>
<th>solutions</th>
<th>sign</th>
</tr>
</thead>
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<tr>
<td>0</td>
<td>$\frac{1}{2}(27m^2 + 3m)$</td>
<td>$(-1)^m$</td>
</tr>
<tr>
<td>1</td>
<td>$\frac{1}{2}(27m^2 + 15m) + 1$</td>
<td>$(-1)^{m+1}$</td>
</tr>
<tr>
<td>2</td>
<td>$\frac{1}{2}(27m^2 + 21m) + 2$</td>
<td>$(-1)^{m+1}$</td>
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The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [3, p. 283] viz.,

$$\prod_{n=0}^{\infty} [(1 - x^{27n+k-l})(1 - x^{27n+k+l})(1 - x^{27n+2k})] = \sum_{m=-\infty}^{\infty} (-1)^m x^{k(27m^2 + 1)m}.$$

In establishing this lemma $k$ and $l$ are given values which are in conformity with the quadratic expressions in $m$ given in Lemma 3. As an illustration we have

$$\sum_{v=2} \mp x^v = \sum_{-\infty}^{\infty} (-1)^{n+1} x^{\frac{1}{3}(27n^2 + 21n) + 2}$$

$$= -x^2 \prod_{n=0}^{\infty} [(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})].$$

**Lemma 4.** Writing $v \equiv i$ simply for $v \equiv i(\mod 3)$

$$\sum_{v=0}^{\infty} (\mp x^v) = \prod_{n=0}^{\infty} [(1 - x^{27n+12})(1 - x^{27n+15})(1 - x^{27n+27})]$$

$$\sum_{v=1}^{\infty} (\mp x^v) = -x \prod_{n=0}^{\infty} [(1 - x^{27n+6})(1 - x^{27n+21})(1 - x^{27n+27})].$$

$$\sum_{v=2} (\mp x^v) = -x^2 \prod_{n=0}^{\infty} [(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})].$$

Lemma 5, given below is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v=1}(\mp x^v)$ as given in...
the above lemma. The following fact also is to be taken into consideration.

\begin{equation}
\prod_{n=0}^{\infty} (1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27}) \bigg/ f(x) = \prod_{n=0}^{\infty} [(1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27})] \bigg/ [(1 - x)(1 - x^3)(1 - x^3)\cdots] = \sum_{n=0}^{\infty} 27p(n)x^n.
\end{equation}

**Lemma 5.**

\begin{align*}
U_0 &\equiv \sum_{n=0}^{\infty} 27p(n)x^n \pmod{3} \\
U_1 &\equiv -\sum_{n=0}^{\infty} 27p(n-1)x^n \pmod{3} \\
U_2 &\equiv -\sum_{n=0}^{\infty} 27p(n-2)x^n \pmod{3}.
\end{align*}

We require another set of congruences which are obtained from the classical result, due to Catalan [1, p. 290].

\begin{equation}
p(n - 1) + 2p(n - 2) - 5p(n - 5) - 7p(n - 7) + \cdots = \sigma(n),
\end{equation}

and another result due to Glaisher [1, p. 312]

\begin{equation}
p(n - 1) + 2^5p(n - 2) - 5^2p(n - 5) - 7^2p(n - 7) + \cdots = -\frac{1}{12} [5\sigma_3(n) - (18n - 1)\sigma(n)].
\end{equation}

These results can be rewritten according to our notation as

\begin{equation}
\sum_v [\mp vp(n - v)] = -\sigma(n),
\end{equation}

\begin{equation}
\sum_v [\mp v^5p(n - v)] = \frac{1}{12} [5\sigma_3(n) - (18n - 1)\sigma(n)].
\end{equation}

Now from (5) we have

\begin{equation}
u_r = \sum_v (\mp v^r x^r) \bigg/ f(x) = \sum_v (\mp v^r x^r) \cdot \sum_{n=0}^{\infty} p(n)x^n = \sum_{n=1}^{\infty} \left\{ \sum_v [\mp v^r p(n - v)] \right\} x^n, \quad r > 0.
\end{equation}

It is now easy to establish the validity of the following lemma from the above three relations (15), (16) and (17).
LEMMA 6.

\[ u_1 = - \sum_{n=1}^{\infty} \sigma(n)x^n \]
\[ u_2 = \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n. \]

The next lemma can be easily obtained by the substitution of the above values of \( u_1 \) and \( u_2 \) in (7).

LEMMA 7.

\[ U_0 - 1 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n, \]
\[ U_1 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n + 11)\sigma(n)]x^n, \]
\[ U_2 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 13)\sigma(n)]x^n. \]

The congruences given in Lemma 8 are elementary and can be readily proved.

LEMMA 8.

\[ \sigma(3n - 1) \equiv 0 \pmod{3} . \]
\[ \sigma(3^\lambda n) \equiv \sigma(n) \pmod{3} , \lambda \geq 0 . \]

4. Proof of the theorems. By comparing the coefficients of like powers of \( x \) in the expressions (modulo 3) for \( U_i \) given in Lemmas 5 and 7 we obtain the following congruences for \( n > 0 \).

\[ \frac{1}{12} p(n) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n - 1)\sigma(n)] \pmod{3} \]
\[ \frac{1}{12} p(n - 1) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n + 11)\sigma(n)] \pmod{3} \]
\[ \frac{1}{12} p(n - 2) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n - 13)\sigma(n)] \pmod{3} . \]

Remembering the well-known congruence, [4 ; 2, p. 167],

\[ \sigma_k(n) \equiv 0 \pmod{M} \] for almost all \( n \)

for arbitrarily fixed \( M \) and odd \( k \), it is a straightforward matter to
deduce Theorem 1 from the above congruences.

To establish Theorem 2 we obtain by a process of addition or subtraction of (18), (19) and (20) in pairs the following.

\begin{equation}
- \frac{\tau}{12} p(n) - \frac{\tau}{6} p(n - 1) \equiv \frac{\tau}{12} p(n) + \frac{\tau}{3} p(n - 2) \\
\equiv \frac{\tau}{6} p(n - 1) - \frac{\tau}{3} p(n - 2) \equiv \sigma(n) \pmod{3}.
\end{equation}

Now writing $3n + 2$ for $n$ in (22) and making use of the first relation of Lemma 8 we obtain the theorem immediately.

To derive a generalization from (22) we write $3^k n$ for $n$ in it and make use of the last congruence of Lemma 8 to obtain,

\begin{equation}
- \frac{\tau}{12} p(3^k n) - \frac{\tau}{6} p(3 \cdot n - 1) \equiv \frac{\tau}{12} p(3^k n) + \frac{\tau}{3} p(3^k n - 2) \\
\equiv \frac{\tau}{6} p(3^k n - 1) - \frac{\tau}{3} p(3^k n - 2) \equiv \sigma(n) \pmod{3}.
\end{equation}

We need write $3n - 1$ for $n$ in (23) and use the first congruence of Lemma 8 to arrive at the more general Theorem 3.

**THEOREM 3.** With respect to the modulus 3

\begin{equation}
- \frac{\tau}{12} p(3^{k+1} n - 3^2) \equiv \frac{\tau}{6} p(3^{k+1} n - 3^2 - 1) \equiv \frac{\tau}{3} p(3^{k+1} n - 3^2 - 2).
\end{equation}

Finally, it might be of interest to note that the three restricted partition functions $\frac{\tau}{r} p(n)$, $r = 3, 6$ and 12, are connected by the identical relation,

\begin{equation}
\frac{\tau}{12} p(n) = \frac{\tau}{6} p(n - 1) + \frac{\tau}{3} p(n - 2), \quad n > 0.
\end{equation}

This is seen to be true by a joint consideration of (6), Lemma 4, and (12). The first relation gives

\begin{equation}
\sum_{t=0}^{2} \sum_{r=1}^{\infty} (\mp x^r)/f(x) = 1.
\end{equation}

We substitute the values of $\sum_{r=1}^{\infty} (\mp x^r)$ in the product form as given in Lemma 4, and then make use of (12) in order to express the left hand side of (25) as a power series in $x$ whose coefficients are simple linear functions of the restricted partition functions. Now (24) is obtained directly by equating to zero the coefficient of $x^n$, $n > 0$.

**REFERENCES**


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