Pacific Journal of Mathematics

SOME RESTRICTED PARTITION FUNCTIONS; CONGRUENCES MODULO 3

D. B. LAHIRI

Vol. 28, No. 3 May 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 3, 1969

SOME RESTRICTED PARTITION FUNCTIONS: CONGRUENCES MODULO 3

D. B. LAHIRI

We shall establish in this paper some congruence relations with respect to the modulus 3 for some restricted partition functions. The difference between the unrestricted partition function, $p(n)$, and these restricted partition functions which we shall denote by

$$ {}_{r}^{27}p(n) \quad \textbf{with } r = 3, 6, 12 \,,$$

merely lies in the restriction that no number of the forms $27n$, or $27n \pm r$, shall be a part of the partitions which are of relevance in the restricted case. Thus to determine the value of ${}_{r}^{27}p(n)$ one should count all the unrestricted partitions of n excepting those which contain a number of any of the above forms as a part. We shall assume $p(n)$ and ${}_{r}^{27}p(n)$ to be unity when n is zero, and vanishing when the argument is negative. We can now state our theorems.

THEOREM 1. **For almost all values of n**

$${}_{3}^{27}p(n) \equiv {}_{6}^{27}p(n) \equiv {}_{12}^{27}p(n) \equiv 0 \pmod 3 \,.$$

THEOREM 2. **For all values of n**

$${}_{3}^{27}p(3n) \equiv {}_{6}^{27}p(3n+1) \equiv -{}_{12}^{27}p(3n+2) \pmod 3 \,.$$

2. **Definitions and notations.** We shall use m to denote an integer positive zero or negative, but n will stand for a positive or nonnegative integer only.

We define u_r by

$$(1) \qquad u_0 = 1 \quad \text{and} \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n,\ r > 0 \,,$$

where a_n is defined by the well-known 'pentagonal number' theorem of Euler,

$$(2) \qquad f(x) = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{-\infty}^{+\infty} (-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n \,,$$

and $p(n)$ is the number of unrestricted partitions of n given by the expansion,

$$(3) \qquad [f(x)]^{-1} = \left[\prod_{n=1}^{\infty} (1 - x^n)\right]^{-1} = \sum_{n=0}^{\infty} p(n) x^n \,.$$

We shall use v to denote the pentagonal numbers,

$$(4) \qquad v = \frac{1}{2} m(3m + 1), \ m = 0, \pm 1, \pm 2, \cdots ;$$

and with each v there corresponds an 'associated' sign, viz., $(-1)^m$. We shall come across sums of the type

$$\sum_v [\mp V(v)]$$

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if v is 1, 2, 12, 15, 35, $\cdots$, that is, when it is of the form $(2m + 1)(3m + 1)$, and (b) positive if v is 0, 5, 7, 22, 26 $\cdots$, that is, when it is of the form $m(6m + 1)$. With the above summation notation we can write,

$$(5) \qquad u_r = \sum_v (\mp v^r x^v)/f(x) ,$$

$$(6) \qquad \sum_v (\mp x^v)/f(x) = 1 .$$

We shall also require the functions U_i, $i = 0, 1, 2$ which are certain linear functions of u_r's, $r = 0, 1, 2$ as given below.

$$(7) \qquad \begin{cases} U_0 = - u_2 + u_0 , \\ U_1 = - u_2 - u_1 , \\ U_2 = - u_2 + u_1 . \end{cases}$$

We also need the quadratics $P_i(v)$ in v, $i = 0, 1, 2$ which are obtained by writing $P_i(v)$ for U_i, and v^r for u_r. Thus

$$(8) \qquad \begin{cases} P_0(v) = - v^2 + 1 , \\ P_1(v) = - v^2 - v , \\ P_2(v) = - v^2 + v . \end{cases}$$

3. Some lemmas. The truth of the following lemma can be easily verified from the expressions for $P_i(v)$ given in (8).

LEMMA 1.

$$\begin{aligned} P_i(v) &\equiv 1 \pmod 3 , \quad \textit{if } v \equiv i \pmod 3 \\ &\equiv 0 \pmod 3 , \quad \textit{if } v \not\equiv i \pmod 3 . \end{aligned}$$

If we replace the u_r's appearing in the expressions for U_i in (7) by the right hand expressions in (5) we get

$$(9) \qquad U_i = \sum_v [\mp P_i(v) x^v]/f(x) ;$$

and then the use of Lemma 1 leads to the next lemma.

LEMMA 2. *$U_i \equiv \sum_{v\equiv i}(\mp x^v)/f(x)$ (mod 3), the summation being extended over all pentagonal numbers $v \equiv i$ (mod 3).*

The truth of the following lemma can be verified without much difficulty by writing $3m + j$, with $j = 0$; -1; and 1 respectively, in place of m in the expression $\frac{1}{2}m(3m + 1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered that $\frac{1}{2}(3m - 1)(9m - 2)$ and $\frac{1}{2}(3m + 1)(9m + 2)$ represent the same set of numbers.

LEMMA 3. *The solutions of*

$$v \equiv i \pmod 3, \quad i = 0, 1, 2$$

are as noted below, (the associated signs are also shown).

i	*solutions*	*sign*
0	$\frac{1}{2}(27m^2 + 3m)$	$(-1)^m$
1	$\frac{1}{2}(27m^2 + 15m) + 1$	$(-1)^{m+1}$
2	$\frac{1}{2}(27m^2 + 21m) + 2$	$(-1)^{m+1}$.

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [3, p. 283] viz.,

$$\prod_{n=0}^{\infty}[(1 - x^{2kn+k-l})(1 - x^{2kn+k+l})(1 - x^{2kn+2k})] = \sum_{-\infty}^{+\infty}(-1)^m x^{km^2+lm} . \tag{10}$$

In establishing this lemma k and l are given values which are in conformity with the quadratic expressions in m given in Lemma 3. As an illustration we have

$$\begin{aligned}\sum_{v=2}(\mp x^v) &= \sum_{-\infty}^{+\infty}(-1)^{m+1}x^{\frac{1}{2}(27m^2+21m)+2} \\ &= -x^2\prod_{n=0}^{\infty}[(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})] .\end{aligned} \tag{11}$$

LEMMA 4. *Writing $v \equiv i$ simply for $v \equiv i$ (mod 3)*

$$\sum_{v\equiv 0}(\mp x^v) = \prod_{n=0}^{\infty}[(1 - x^{27n+12})(1 - x^{27n+15})(1 - x^{27n+27})]$$

$$\sum_{v\equiv 1}(\mp x^v) = -x\prod_{n=0}^{\infty}[(1 - x^{27n+6})(1 - x^{27n+21})(1 - x^{27n+27})] .$$

$$\sum_{v\equiv 2}(\mp x^v) = -x^2\prod_{n=0}^{\infty}[(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})] .$$

Lemma 5, given below is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v\equiv i}(\mp x^v)$ as given in the above lemma. The following fact also is to be taken into consideration.

$$(12)\qquad \prod_{n=0}^{\infty}(1-x^{27n+r})(1-x^{27n+27-r})(1-x^{27n+27})]/f(x)$$
$$=\prod_{n=0}^{\infty}[(1-x^{27n+r})(1-x^{27n+27-r})(1-x^{27n+27})]/[(1-x)(1-x^2)(1-x^3)\cdots]$$
$$=\sum_{n=0}^{\infty}{}_{r}^{27}p(n)x^n .$$

LEMMA 5.

$$U_0 \equiv \sum_{n=0}^{\infty}{}_{12}^{27}p(n)x^n \pmod 3$$
$$U_1 \equiv -\sum_{n=0}^{\infty}{}_{6}^{27}p(n-1)x^n \pmod 3$$
$$U_2 \equiv -\sum_{n=0}^{\infty}{}_{3}^{27}p(n-2)x^n \pmod 3 .$$

We require another set of congruences which are obtained from the classical result, due to Catalan [1, p. 290].

$$(13)\qquad p(n-1)+2p(n-2)-5p(n-5)-7p(n-7)+\cdots=\sigma(n) ,$$

and another result due to Glaisher [1, p. 312]

$$(14)\qquad p(n-1)+2^2p(n-2)-5^2p(n-5)-7^2p(n-7)+\cdots$$
$$=-\frac{1}{12}[5\sigma_3(n)-(18n-1)\sigma(n)] .$$

These results can be rewritten according to our notation as

$$(15)\qquad \sum_{v}[\mp vp(n-v)]=-\sigma(n) ,$$

$$(16)\qquad \sum_{v}[\mp v^2p(n-v)]=\frac{1}{12}[5\sigma_3(n)-(18n-1)\sigma(n)] .$$

Now from (5) we have

$$u_r=\sum_{v}(\mp v^r x^v)/f(x)$$
$$(17)\qquad =\sum_{v}(\mp v^r x^v)\,.\,\sum_{n=0}^{\infty}p(n)x^n$$
$$=\sum_{n=1}^{\infty}\{\sum_{v}[\mp v^r p(n-v)]\}x^n ,\quad r>0 .$$

It is now easy to establish the validity of the following lemma from the above three relations (15), (16) and (17).

LEMMA 6.

$$u_1 = -\sum_{n=1}^{\infty} \sigma(n)x^n$$

$$u_2 = \frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n .$$

The next lemma can be easily obtained by the substitution of the above values of u_1 and u_2 in (7).

LEMMA 7.

$$U_0 - 1 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n ,$$

$$U_1 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n + 11)\sigma(n)]x^n ,$$

$$U_2 = -\frac{1}{12}\sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 13)\sigma(n)]x^n .$$

The congruences given in Lemma 8 are elementary and can be readily proved.

LEMMA 8.

$$\sigma(3n - 1) \equiv 0 \pmod 3 .$$

$$\sigma(3^\lambda n) \equiv \sigma(n) \pmod 3 , \quad \lambda \geq 0 .$$

4. Proof of the theorems. By comparing the coefficients of like powers of x in the expressions (modulo 3) for U_i given in Lemmas 5 and 7 we obtain the following congruences for $n > 0$.

(18) $$ {}^{27}_{12}p(n) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n - 1)\sigma(n)] \pmod 3$$

(19) $$ -{}^{27}_{6}p(n - 1) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n + 11)\sigma(n)] \pmod 3$$

(20) $$ -{}^{27}_{3}p(n - 2) \equiv -\frac{1}{12}[5\sigma_3(n) - (18n - 13)\sigma(n)] \pmod 3 .$$

Remembering the well-known congruence, [4 ; 2, p. 167],

(21) $$\sigma_k(n) \equiv 0 \pmod M \text{ for almost all } n$$

for arbitrarily fixed M and odd k, it is a straightforward matter to deduce Theorem 1 from the above congruences.

To establish Theorem 2 we obtain by a process of addition or subtraction of (18), (19) and (20) in pairs the following.

$$(22)\qquad \begin{aligned} -\,{}^{27}_{12}p(n) - {}^{27}_{6}p(n-1) &\equiv {}^{27}_{12}p(n) + {}^{27}_{3}p(n-2) \\ &\equiv {}^{27}_{6}p(n-1) - {}^{27}_{3}p(n-2) \equiv \sigma(n) \pmod 3 . \end{aligned}$$

Now writing $3n+2$ for n in (22) and making use of the first relation of Lemma 8 we obtain the theorem immediately.

To derive a generalization from (22) we write $3^\lambda n$ for n in it and make use of the last congruence of Lemma 8 to obtain,

$$(23)\qquad \begin{aligned} -\,{}^{27}_{12}p(3^\lambda n) - {}^{27}_{6}p(3^\lambda n-1) &\equiv {}^{27}_{12}p(3^\lambda n) + {}^{27}_{3}p(3^\lambda n-2) \\ &\equiv {}^{27}_{6}p(3^\lambda n-1) - {}^{27}_{3}p(3^\lambda n-2) \\ &\equiv \sigma(n) \pmod 3 . \end{aligned}$$

We need write $3n-1$ for n in (23) and use the first congruence of Lemma 8 to arrive at the more general Theorem 3.

THEOREM 3. *With respect to the modulus* 3

$$-\,{}^{27}_{12}p(3^{\lambda+1}n - 3^\lambda) \equiv {}^{27}_{6}p(3^{\lambda+1}n - 3^\lambda - 1) \equiv {}^{27}_{3}p(3^{\lambda+1}n - 3^\lambda - 2) .$$

Finally, it might be of interest to note that the three restricted partition functions ${}^{27}_{r}p(n)$, $r = 3, 6$ and 12, are connected by the identical relation,

$$(24)\qquad {}^{27}_{12}p(n) = {}^{27}_{6}p(n-1) + {}^{27}_{3}p(n-2) , \quad n > 0 .$$

This is seen to be true by a joint consideration of (6), Lemma 4, and (12). The first relation gives

$$(25)\qquad \sum_{i=0}^{2} \sum_{v \equiv i} (\mp x^v)/f(x) = 1 .$$

We substitute the values of $\sum_{v\equiv i}(\mp x^v)$ in the product form as given in Lemma 4, and then make use of (12) in order to express the left hand side of (25) as a power series in x whose coefficients are simple linear functions of the restricted partition functions. Now (24) is obtained directly by equating to zero the coefficient of x^n, $n > 0$.

REFERENCES

1. L. E. Dickson, *History of the theory of numbers*, Vol. I, Chelsea Publishing Company, New York, 1952.
2. G. H. Hardy, *Ramanujan*, Cambridge, 1940.
3. G. H. Hardy and E. M. Wright, *An Introduction to the theory of numbers*, Oxford, 4th ed., 1960.

4. G. N. Watson, *Über Ramanujansche Kongruenzeigenshaften der Zerfäl ungsanzahlen*, Math. Z. **39** (1935), 712–731.

Received October 3, 1967, and in revised from May 27, 1968.

INDIAN STATISTICAL INSTITUTE
CALCUTTA

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 28, No. 3 May, 1969