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**SOME RESTRICTED PARTITION FUNCTIONS;
CONGRUENCES MODULO 3**

D. B. LAHIRI

SOME RESTRICTED PARTITION FUNCTIONS : CONGRUENCES MODULO 3

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We shall establish in this paper some congruence relations with respect to the modulus 3 for some restricted partition functions. The difference between the unrestricted partition function, $p(n)$, and these restricted partition functions which we shall denote by

$${}_{r}^{27}p(n) \quad \text{with } r = 3, 6, 12,$$

merely lies in the restriction that no number of the forms $27n$, or $27n \pm r$, shall be a part of the partitions which are of relevance in the restricted case. Thus to determine the value of ${}_{r}^{27}p(n)$ one should count all the unrestricted partitions of n excepting those which contain a number of any of the above forms as a part. We shall assume $p(n)$ and ${}_{r}^{27}p(n)$ to be unity when n is zero, and vanishing when the argument is negative. We can now state our theorems.

THEOREM 1. For almost all values of n

$${}_{3}^{27}p(n) \equiv {}_{6}^{27}p(n) \equiv {}_{12}^{27}p(n) \equiv 0 \pmod{3}.$$

THEOREM 2. For all values of n

$${}_{3}^{27}p(3n) \equiv {}_{6}^{27}p(3n + 1) \equiv -{}_{12}^{27}p(3n + 2) \pmod{3}.$$

2. Definitions and notations. We shall use m to denote an integer positive zero or negative, but n will stand for a positive or nonnegative integer only.

We define u_r by

$$(1) \quad u_0 = 1 \quad \text{and} \quad u_r = \sum_{n=0}^{\infty} n^r a_n x^n \cdot \sum_{n=0}^{\infty} p(n) x^n, \quad r > 0,$$

where a_n is defined by the well-known 'pentagonal number' theorem of Euler,

$$(2) \quad f(x) = \prod_{n=1}^{\infty} (1 - x^n) = \sum_{m=-\infty}^{+\infty} (-1)^m x^{\frac{1}{2}m(3m+1)} = \sum_{n=0}^{\infty} a_n x^n,$$

and $p(n)$ is the number of unrestricted partitions of n given by the expansion,

$$(3) \quad [f(x)]^{-1} = \left[\prod_{n=1}^{\infty} (1 - x^n) \right]^{-1} = \sum_{n=0}^{\infty} p(n) x^n.$$

We shall use v to denote the pentagonal numbers,

$$(4) \quad v = \frac{1}{2} m(3m + 1), \quad m = 0, \pm 1, \pm 2, \dots;$$

and with each v there corresponds an 'associated' sign, viz., $(-1)^m$. We shall come across sums of the type

$$\sum_v [\mp V(v)]$$

where it is understood that the sign to be prefixed is the 'associated' one, which would thus be (a) negative if v is 1, 2, 12, 15, 35, \dots , that is, when it is of the form $(2m + 1)(3m + 1)$, and (b) positive if v is 0, 5, 7, 22, 26 \dots , that is, when it is of the form $m(6m + 1)$. With the above summation notation we can write,

$$(5) \quad u_r = \sum_v (\mp v^r x^v) / f(x),$$

$$(6) \quad \sum_v (\mp x^v) / f(x) = 1.$$

We shall also require the functions U_i , $i = 0, 1, 2$ which are certain linear functions of u_r 's, $r = 0, 1, 2$ as given below.

$$(7) \quad \begin{cases} U_0 = -u_2 + u_0, \\ U_1 = -u_2 - u_1, \\ U_2 = -u_2 + u_1. \end{cases}$$

We also need the quadratics $P_i(v)$ in v , $i = 0, 1, 2$ which are obtained by writing $P_i(v)$ for U_i , and v^r for u_r . Thus

$$(8) \quad \begin{cases} P_0(v) = -v^2 + 1, \\ P_1(v) = -v^2 - v, \\ P_2(v) = -v^2 + v. \end{cases}$$

3. **Some lemmas.** The truth of the following lemma can be easily verified from the expressions for $P_i(v)$ given in (8).

LEMMA 1.

$$\begin{aligned} P_i(v) &\equiv 1 \pmod{3}, & \text{if } v &\equiv i \pmod{3} \\ &\equiv 0 \pmod{3}, & \text{if } v &\not\equiv i \pmod{3}. \end{aligned}$$

If we replace the u_r 's appearing in the expressions for U_i in (7) by the right hand expressions in (5) we get

$$(9) \quad U_i = \sum_v [\mp P_i(v)x^v] / f(x);$$

and then the use of Lemma 1 leads to the next lemma.

LEMMA 2. $U_i \equiv \sum_{v \equiv i} (\mp x^v) / f(x) \pmod{3}$, the summation being extended over all pentagonal numbers $v \equiv i \pmod{3}$.

The truth of the following lemma can be verified without much difficulty by writing $3m + j$, with $j = 0; -1; \text{ and } 1$ respectively, in place of m in the expression $\frac{1}{2}m(3m + 1)$ for the pentagonal numbers, and in $(-1)^m$ its associated sign. It is also to be remembered that $\frac{1}{2}(3m - 1)(9m - 2)$ and $\frac{1}{2}(3m + 1)(9m + 2)$ represent the same set of numbers.

LEMMA 3. The solutions of

$$v \equiv i \pmod{3}, \quad i = 0, 1, 2$$

are as noted below, (the associated signs are also shown).

i	solutions	sign
0	$\frac{1}{2}(27m^2 + 3m)$	$(-1)^m$
1	$\frac{1}{2}(27m^2 + 15m) + 1$	$(-1)^{m+1}$
2	$\frac{1}{2}(27m^2 + 21m) + 2$	$(-1)^{m+1}$.

The identities given in the next lemma are simple applications of a special case of a famous identity of Jacobi [3, p. 283] viz.,

$$(10) \quad \prod_{n=0}^{\infty} [(1 - x^{2kn+k-l})(1 - x^{2kn+k+l})(1 - x^{2kn+2k})] = \sum_{m=-\infty}^{+\infty} (-1)^m x^{km^2+lm}.$$

In establishing this lemma k and l are given values which are in conformity with the quadratic expressions in m given in Lemma 3. As an illustration we have

$$(11) \quad \sum_{v=2} (\mp x^v) = \sum_{m=-\infty}^{+\infty} (-1)^{m+1} x^{\frac{1}{2}(27m^2+21m)+2} \\ = -x^2 \prod_{n=0}^{\infty} [(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})].$$

LEMMA 4. Writing $v \equiv i$ simply for $v \equiv i \pmod{3}$

$$\sum_{v \equiv 0} (\mp x^v) = \prod_{n=0}^{\infty} [(1 - x^{27n+12})(1 - x^{27n+15})(1 - x^{27n+27})] \\ \sum_{v \equiv 1} (\mp x^v) = -x \prod_{n=0}^{\infty} [(1 - x^{27n+6})(1 - x^{27n+21})(1 - x^{27n+27})]. \\ \sum_{v \equiv 2} (\mp x^v) = -x^2 \prod_{n=0}^{\infty} [(1 - x^{27n+3})(1 - x^{27n+24})(1 - x^{27n+27})].$$

Lemma 5, given below is derived from Lemma 2 after the substitution in it of the product expressions for $\sum_{v \equiv i} (\mp x^v)$ as given in

the above lemma. The following fact also is to be taken into consideration.

$$\begin{aligned}
 (12) \quad & \prod_{n=0}^{\infty} (1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27})/f(x) \\
 &= \prod_{n=0}^{\infty} [(1 - x^{27n+r})(1 - x^{27n+27-r})(1 - x^{27n+27})]/[(1-x)(1-x^2)(1-x^3)\cdots] \\
 &= \sum_{n=0}^{\infty} {}_{27}^r p(n)x^n.
 \end{aligned}$$

LEMMA 5.

$$\begin{aligned}
 U_0 &\equiv \sum_{n=0}^{\infty} {}_{12}^{27} p(n)x^n \pmod{3} \\
 U_1 &\equiv - \sum_{n=0}^{\infty} {}_6^{27} p(n-1)x^n \pmod{3} \\
 U_2 &\equiv - \sum_{n=0}^{\infty} {}_3^{27} p(n-2)x^n \pmod{3}.
 \end{aligned}$$

We require another set of congruences which are obtained from the classical result, due to Catalan [1, p. 290].

$$(13) \quad p(n-1) + 2p(n-2) - 5p(n-5) - 7p(n-7) + \cdots = \sigma(n),$$

and another result due to Glaisher [1, p. 312]

$$\begin{aligned}
 (14) \quad & p(n-1) + 2^2p(n-2) - 5^2p(n-5) - 7^2p(n-7) + \cdots \\
 &= -\frac{1}{12}[5\sigma_3(n) - (18n-1)\sigma(n)].
 \end{aligned}$$

These results can be rewritten according to our notation as

$$(15) \quad \sum_v [\mp v p(n-v)] = -\sigma(n),$$

$$(16) \quad \sum_v [\mp v^2 p(n-v)] = \frac{1}{12}[5\sigma_3(n) - (18n-1)\sigma(n)].$$

Now from (5) we have

$$\begin{aligned}
 (17) \quad u_r &= \sum_v (\mp v^r x^v)/f(x) \\
 &= \sum_v (\mp v^r x^v) \cdot \sum_{n=0}^{\infty} p(n)x^n \\
 &= \sum_{n=1}^{\infty} \left\{ \sum_v [\mp v^r p(n-v)] \right\} x^n, \quad r > 0.
 \end{aligned}$$

It is now easy to establish the validity of the following lemma from the above three relations (15), (16) and (17).

LEMMA 6.

$$u_1 = - \sum_{n=1}^{\infty} \sigma(n)x^n$$

$$u_2 = \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n .$$

The next lemma can be easily obtained by the substitution of the above values of u_1 and u_2 in (7).

LEMMA 7.

$$U_0 - 1 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 1)\sigma(n)]x^n ,$$

$$U_1 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n + 11)\sigma(n)]x^n ,$$

$$U_2 = - \frac{1}{12} \sum_{n=1}^{\infty} [5\sigma_3(n) - (18n - 13)\sigma(n)]x^n .$$

The congruences given in Lemma 8 are elementary and can be readily proved.

LEMMA 8.

$$\sigma(3n - 1) \equiv 0 \pmod{3} .$$

$$\sigma(3^i n) \equiv \sigma(n) \pmod{3} , \quad \lambda \geq 0 .$$

4. Proof of the theorems. By comparing the coefficients of like powers of x in the expressions (modulo 3) for U_i given in Lemmas 5 and 7 we obtain the following congruences for $n > 0$.

$$(18) \quad {}_{12}^{27}p(n) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n - 1)\sigma(n)] \pmod{3}$$

$$(19) \quad - {}_{6}^{27}p(n - 1) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n + 11)\sigma(n)] \pmod{3}$$

$$(20) \quad - {}_{3}^{27}p(n - 2) \equiv - \frac{1}{12} [5\sigma_3(n) - (18n - 13)\sigma(n)] \pmod{3} .$$

Remembering the well-known congruence, [4 ; 2, p. 167],

$$(21) \quad \sigma_k(n) \equiv 0 \pmod{M} \text{ for almost all } n$$

for arbitrarily fixed M and odd k , it is a straightforward matter to

deduce Theorem 1 from the above congruences.

To establish Theorem 2 we obtain by a process of addition or subtraction of (18), (19) and (20) in pairs the following.

$$(22) \quad -\frac{27}{12}p(n) - \frac{27}{6}p(n-1) \equiv \frac{27}{12}p(n) + \frac{27}{3}p(n-2) \\ \equiv \frac{27}{6}p(n-1) - \frac{27}{3}p(n-2) \equiv \sigma(n) \pmod{3}.$$

Now writing $3n+2$ for n in (22) and making use of the first relation of Lemma 8 we obtain the theorem immediately.

To derive a generalization from (22) we write $3^\lambda n$ for n in it and make use of the last congruence of Lemma 8 to obtain,

$$(23) \quad -\frac{27}{12}p(3^\lambda n) - \frac{27}{6}p(3^\lambda n-1) \equiv \frac{27}{12}p(3^\lambda n) + \frac{27}{3}p(3^\lambda n-2) \\ \equiv \frac{27}{6}p(3^\lambda n-1) - \frac{27}{3}p(3^\lambda n-2) \\ \equiv \sigma(n) \pmod{3}.$$

We need write $3n-1$ for n in (23) and use the first congruence of Lemma 8 to arrive at the more general Theorem 3.

THEOREM 3. *With respect to the modulus 3*

$$-\frac{27}{12}p(3^{\lambda+1}n - 3^\lambda) \equiv \frac{27}{6}p(3^{\lambda+1}n - 3^\lambda - 1) \equiv \frac{27}{3}p(3^{\lambda+1}n - 3^\lambda - 2).$$

Finally, it might be of interest to note that the three restricted partition functions $\frac{27}{r}p(n)$, $r = 3, 6$ and 12 , are connected by the identical relation,

$$(24) \quad \frac{27}{12}p(n) = \frac{27}{6}p(n-1) + \frac{27}{3}p(n-2), \quad n > 0.$$

This is seen to be true by a joint consideration of (6), Lemma 4, and (12). The first relation gives

$$(25) \quad \sum_{i=0}^2 \sum_{v=i}^2 (\mp x^v)/f(x) = 1.$$

We substitute the values of $\sum_{v=i}^2 (\mp x^v)$ in the product form as given in Lemma 4, and then make use of (12) in order to express the left hand side of (25) as a power series in x whose coefficients are simple linear functions of the restricted partition functions. Now (24) is obtained directly by equating to zero the coefficient of x^n , $n > 0$.

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Jon F. Carlson, <i>Automorphisms of groups of similitudes over F_3</i>	485
W. Wistar (William) Comfort, Neil Hindman and Stelios A. Negrepointis, <i>F'-spaces and their product with P-spaces</i>	489
Archie Gail Gibson, <i>Triples of operator-valued functions related to the unit circle</i>	503
David Saul Gillman, <i>Free curves in E^3</i>	533
E. A. Heard and James Howard Wells, <i>An interpolation problem for subalgebras of H^∞</i>	543
Albert Emerson Hurd, <i>A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems</i>	555
E. W. Johnson and J. P. Lediaev, <i>Representable distributive Noether lattices</i>	561
David G. Kendall, <i>Incidence matrices, interval graphs and seriation in archeology</i>	565
Robert Leroy Kruse, <i>On the join of subnormal elements in a lattice</i>	571
D. B. Lahiri, <i>Some restricted partition functions; Congruences modulo 3</i>	575
Norman D. Lane and Kamla Devi Singh, <i>Strong cyclic, parabolic and conical differentiability</i>	583
William Franklin Lucas, <i>Games with unique solutions that are nonconvex</i>	599
Eugene A. Maier, <i>Representation of real numbers by generalized geometric series</i>	603
Daniel Paul Maki, <i>A note on recursively defined orthogonal polynomials</i>	611
Mark Mandelker, <i>F'-spaces and z-embedded subspaces</i>	615
James R. McLaughlin and Justin Jesse Price, <i>Comparison of Haar series with gaps with trigonometric series</i>	623
Ernest A. Michael and A. H. Stone, <i>Quotients of the space of irrationals</i>	629
William H. Mills and Neal Zierler, <i>On a conjecture of Golomb</i>	635
J. N. Pandey, <i>An extension of Haimo's form of Hankel convolutions</i>	641
Terence John Reed, <i>On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles</i>	653
Haskell Paul Rosenthal, <i>A characterization of the linear sets satisfying Herz's criterion</i>	663
George Thomas Sallee, <i>The maximal set of constant width in a lattice</i>	669
I. H. Sheth, <i>On normaloid operators</i>	675
James D. Stasheff, <i>Torsion in BBSO</i>	677
Billy Joe Thorne, <i>$A - P$ congruences on Baer semigroups</i>	681
Robert Breckenridge Warfield, Jr., <i>Purity and algebraic compactness for modules</i>	699
Joseph Zaks, <i>On minimal complexes</i>	721