STRONG CYCLIC, PARABOLIC AND CONICAL DIFFERENTIABILITY

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An arc \( A \) in the real inversive plane is strongly conformally
differentiable at a point \( p \) of \( A \) if the circle through a fixed
point different from \( p \) and two variable points of \( A \) converges
as these points tend to \( p \), and the circle through three points
of \( A \) which tend to \( p \) converges. Analogous definitions of
strong parabolic differentiability in the real affine plane, using
parabolas and three conditions, and of strong conical differenti-
ability in the real projective plane, using conics and four
conditions, may be formulated.

Relations among the strong differentiability conditions in
each of the above three geometries can be summarized as follows:
Let \( N = \text{II} \) in the inversive case; \( N = \text{II} \) or \( \text{III} \) in the
affine case; and \( N = \text{II}, \text{III} \) or \( \text{IV} \) in the projective case. Then the
\( N \)th strong differentiability condition implies the \( K \)th strong
condition for \( 0 < K < N - 1 \), but it does not imply the \( (N - 1) \)-th
strong condition. The \( (N - 1) \)-th strong condition will also
hold in each of the three geometries, however, if, in addition
to the \( N \)th strong condition, any one of the following is assumed:

(i) The limit characteristic curve of the \( N \)th strong con-
dition is nondegenerate,

(ii) The point \( p \) is an end-point of the arc \( A \),

(iii) The arc \( A \) satisfies a weaker condition \( (N - 1) \) at \( p \).

These results are related to a general principle due to O. Haupt
which states that if an arc has order \( n \) with respect to an \( n \)-parametric
family of characteristic curves, then each point of the arc is \( (n - 2) \)-
times strongly differentiable with respect to these curves. The rela-
tionship appears in the following manner.

Suppose that a curve of an \( n \)-parametric family of characteristics
is determined by \( n \) points of an arc and converges to a curve \( C \) of
the family as the \( n \) points converge to \( p \). If \( R \in C \), then no curve
of the family through a point \( R \) will meet \( A \) near \( p \) more than \( n - 1 \)
times.

2. Notation. An arc \( A \) in the real inversive, affine, or projective
plane is the continuous image of a real parameter interval. The letters
\( P, Q, \cdots \) denote points in the plane; \( p, q, \cdots \) denote points of arcs.
A neighborhood of \( p \) on \( A \) is the image of a neighborhood of the
parameter \( p \) on the parameter interval. Since the points of the arcs
which we shall consider will be in one-to-one correspondence with the
points of the parameter interval in a small neighborhood of \( p \), there
will be little danger of ambiguity if the same letters \( p, q, \cdots \) denote
both the points of the parameter interval and their images on $A$.

Whenever several points tend to $p$ on $A$, they may be assumed to be distinct unless otherwise stated.

$C$ will denote a circle, $\pi$ a parabola, and $\gamma$ a conic. The region lying to the left [right] of an oriented line $\mathcal{L}$ or a circle $C$ is denoted by $C[\mathcal{L}]^*$ and $C[C]^*$ respectively.

$\pi^*$ and $\gamma^*$ denote the interiors of the parabola $\pi$ and the conic $\gamma$ respectively, while $\pi^*$ and $\gamma^*$ denote their exteriors.

The $N$-th strong differentiability condition in the conformal, affine and projective cases will be denoted by $CN'$, $AN'$ and $PN'$ respectively.

3. Conditions I and I'. Let $A$ be an arc in the real inversive plane.

**CONDITION CI'.** There exists a point $R \neq p$ such that if the parameters $s$ and $t$ are sufficiently close to the parameter $p$, $s \neq t$, then the circle $C(s, t, R)$ through the points $s$, $t$ and $R$ exists. It converges if $s$ and $t$ tend to $p$.

The special case of Condition CI' in which $s = p$ is called **Condition CI**; cf. [7, 5]. The limit tangent circle of CI is denoted by $C(p^*, R)$. If Condition CI or CI' holds for one point $R \neq p$, then it holds for every point $Q \neq p$. Condition CI' implies

$$\lim_{s,t \to p} C(s, t, Q) = C(p^*, R) \text{ for all } R \neq p.$$ 

If we designate $R$ as the infinite point, then Conditions CI and CI' become the **Affine or Projective Conditions I and I'** respectively.

**CONDITION AI' = CONDITION PI'.** If the parameters $s$ and $t$ are sufficiently close to the parameter $p$, $s \neq t$, then the straight line $\mathcal{L}(s, t)$ through the points $s$ and $t$ exists. It converges if $s$ and $t$ tend to $p$.

**CONDITION AI = Condition PI** is the special case of this condition when $s = p$.

The following properties are well known.

**LEMMA 1.** A convex arc always satisfies Conditions AI and AI' at an endpoint; cf. [2, 2.7].

**LEMMA 2.** A convex arc which satisfies Condition AI at an interior point $p$ also satisfies AI' at $p$; cf. [2, 3.1].

**LEMMA 3.** If an arc which has no cusp at $p$, satisfies Condition AI at $p$, and AI' one-sidedly at $p$, then it also satisfies Condition AI' at $p$. 
4. Strong cyclic differentiability.

CONDITION CII. The arc $A$ satisfies CI at $p$ and

$$C(p^2) = \lim_{s \to p} C(p^2, s)$$

exists.

CONDITION CII'. $\lim_{t,u,v \to p} C(t, u, v)$ exists.

The following example shows that CII' does not imply CI or CI'.

The arc $A$ defined by $x = t^2$, $y = t^3$, $0 \leq t \leq 1/2$; $x = -t^2$, $y = t^3$, $-1/2 \leq t < 0$, satisfies CII' at the origin $p$, and

$$\lim_{t,u,v \to p} C(t, u, v) = p.$$ 

If $R \neq p$, $\lim_{t,u,v \to p} C(t, u, R)$ exists and touches the $x$-axis at the origin when $t > 0$, $u > 0$. This limit also exists when $t < 0$, $u < 0$, but it then touches the $y$-axis at the origin. Hence Conditions CI and CI' do not hold at $p$.

**Theorem 1.** Let $A$ satisfy Condition CII' at $p$. Then $A$ will satisfy CI at $p$ if any one of the following also holds.

(i) $C = \lim_{t,u,v \to p} C(t, u, v) \neq p$;

(ii) $p$ is an end-point of $A$;

(iii) $A$ satisfies CI at $p$.

**Proof.** Assume that $A$ satisfies CII' at $p$.

(i) Assume, for the moment, that $C \neq p$. Choose a point $S$ on $C(t, u, v)$ such that $S$ does not converge to $p$ as $t, u, v$ converge on $A$ to $p$.

Let $R$ be a given point, $R \in C$, and let $D$ be any accumulation circle of the $C(t, u, R)$. Then the angle between $C(t, u, R)$ and $C(t, u, S) = C(t, u, v)$ may be defined as the amplitude of the cross ratio of $t, R, u, S$. Now this angle tends to $0$ as $t, u, v$ tend to $p$. Hence $D$ is the unique circle through $R$ which is tangent to $C$ at $p$. Thus $A$ satisfies CI at $p$.

Next, preparing for (ii) and (iii), we remove the restriction $C \neq p$. Since $R \in C$, we can choose a neighborhood $B$ of $p$ on $A$ such that $R \in C(t, u, v)$ for every choice of $t, u, v$ on $B$. This implies that $C(t, u, R)$ meets $B$ only at $t$ and $u$. If we designate $R$ as the infinite point, this means that $B$ is convex.

(ii) If $p$ is an end-point of $B$, then Lemma 1 implies that $B$ satisfies AI' at $p$.

(iii) If $A$ satisfies CI at $p$, then $C(p^2, R)$ exists and, with $R = \infty$, it is the tangent line of $B$ at $p$. Thus $B$ satisfies AI at $p$. By
Lemma 2, $B$ also satisfies $A\Gamma'$ at $p$.

Thus in both of the cases (ii) and (iii), $C(t,u,R)$ converges as $t$ and $u$ converge on $A$ to $p$. Hence $A$ satisfies Condition $C\Gamma'$ at $p$.

5. Strong parabolic differentiability.

5.1. Let $A$ be an arc in the real affine plane. If $A\ I$ holds at $p$ we shall denote the family of nondegenerate parabolas which touch the tangent line $\mathcal{T}$ of $A$ at $p$ by $\tau$ and its compactification by $\overline{\tau}$. If $Q$ and $R$ are distinct points which lie on the same side of $\mathcal{T}$ and $p, Q, R$ are not collinear, then the quadrangle $p, s, Q, R$ will be convex when $s$ is close to $p$ on $A$. Hence there will be two parabolas through these four points. When $s$ tends to $p$, any limit parabola will be one of the two parabolas of $\overline{\tau}$ which pass through $Q$ and $R$. We call it a tangent parabola of $A$ at $p$; cf. [1, 3].

CONDITION AII. Let $A$ satisfy $A\ I$ at $p$ and let the points of $A - \{p\}$ lie in one of the closed half-planes bounded by the tangent $\mathcal{T}$, say in $\mathcal{T}_+ \cup \mathcal{T}_-$. If $R \in \mathcal{T}_+$, then the two tangent parabolas of $A$ at $p$ through $R$ and $t$ converge when $t$ tends to $p$; cf. [1, 4.1].

The limit osculating parabolas of $A$ at $p$ through $R$ are denoted by $\pi_i(p',R)$, $i = 1, 2$. The family of all the osculating parabolas of $A$ at $p$ is denoted by $\sigma$. The set $\sigma$ is one of the following three subsets of the family $\overline{\tau}$.

Type 1. $\sigma$ is a one-parameter family of parabolas of $\tau$ each of which intersects a given member at $p$ and at exactly one other point.

Type 2. $\sigma$ consists of all the double rays of $\overline{\tau}$ with the common vertex $p$ which lie in $\mathcal{T}_+ \cup \mathcal{T}_-$.

Type 3. $\sigma$ consists of all the pairs of parallel lines of $\overline{\tau}$ which lie in $\mathcal{T}_+ \cup \mathcal{T}_-$; cf. [1, 4.4].

5.2. Let $A$ be a convex arc. If $p$ is an interior point of $A$ let $\mathcal{T}$ and $\mathcal{T}'$ be the one-sided tangents of $A$ at $p$. Let $A - \{p\} \in \mathcal{T}_+ \cap \mathcal{T}_-$. Choose a point $R \in \mathcal{T}_+ \cap \mathcal{T}_-$. If $p$ is an end-point of $A$, let $A - \{p\}$ and $R$ lie in $\mathcal{T}_+$. Let $t, u, v$, be mutually distinct points which lie sufficiently close to $p$ on $A$. Then there will be two parabolas through $t, u, v, R$. We denote them by $\pi_i(t,u,v,R)$ and $\pi_2(t,u,v,R)$ cf. [2, 2.8].

5.3. CONDITION AII'. There exists a point $R \neq p$ with the following properties. If $t, u, v$ are mutually distinct and lie sufficiently close to $p$, then the parabolas $\pi_i(t,u,v,R)$ exist, $i = 1, 2$. They converges as $t, u, v$ converge to $p$. 
We note that AIΓ does not imply AΓ. For example the arc \( A \) which was discussed in § 4 also satisfies AIΓ at \( p \). In particular, if \( R \) lies in the first quadrant, then both of parabolas \( \pi_i(t, u, v, R) \) tend to the double ray through \( R \) with the origin \( p \) as the initial point as \( t, u, v \) tend to \( p \). This arc, however, does not satisfy AI or AI' at \( p \).

If AIΓ holds, then \( \pi_i(t, u, v, R) \) always exist when \( t, u, v \) are sufficiently close to \( p \). Hence a sufficiently small neighborhood of \( p \) on \( A \) is convex.

We also note the following extension of AIΓ.

**Lemma 4.** Let \( A \) satisfy AIΓ at \( p \). Then there exists a point \( R \neq p \) such that

\[
\lim_{t, u, v \to p} \pi_i(t, u, v, Q) \text{ exists, } i = 1, 2.
\]

A proof is given in [2, 3.3].

**Lemma 5.** AI' and AI imply AIΓ.

*Proof.* Let \( p \) be an interior point of \( A \). Since \( A \) is convex, the \( \pi_i(t, u, v, R) \) are defined when \( t, u, v \) are close to \( p \) and \( R \) and \( A - \{ p \} \) will lie on the same side of the tangent \( \mathcal{T} \) of \( A \) at \( p \). Hence the \( \pi_i(p^2, v, R) \) are defined if \( v \) is close to \( p \). We can approximate each \( \pi_i(p^2, v, R) \) by a \( \pi_i(p, u, v, R) \) with \( u \) between \( p \) and \( v \). Since AI holds at \( p \), \( \lim_{u, v \to p} \pi_i(p, u, v, R) \) exists. Hence \( \lim_{u, v \to p} \pi_i(p^2, v, R) = \lim_{u, v \to p} \pi_i(p, u, v, R) \) also exists. By [1, Theorem 3], AIΓ also holds at \( p \).

We wish to prove

**Theorem 2.** Let \( A \) satisfy AIΓ at \( p \). Then \( A \) will also satisfy AI at \( p \) if any one of the following conditions also holds.

(i) \( \pi_i = \lim_{u, v \to p} \pi_i(t, u, v, R) \) is not a double ray through \( R \) with the vertex \( p \).

(ii) \( p \) is an end-point of \( A \).

(iii) \( A \) satisfies AI at \( p \).

*Proof.* Let \( A \) satisfy AIΓ at \( p \). Then there exists a neighborhood of \( p \) on \( A \) which is convex.

Parts (ii) and (iii) now follow directly from Lemmas 1 and 2 respectively.

We now prepare for (i). If \( p \) is an interior point of \( A \), then \( A \) will satisfy Condition I one-sidedly at \( p \). Since \( A \) satisfies AI and AI' one-sidedly at \( p \), it will also satisfy AIΓ one-sidedly there; cf. Lemmas 5. As a special case of AIΓ, \( \lim \pi_i(p^2, v, R) = \pi_i \) and hence \( \pi_i \) will be a nondegenerate parabola (Type 1), or a double ray with the vertex \( p \) (Type 2), or a pair of parallel lines (Type 3).

(i) Assume that \( \pi_i \) is not a double ray. Thus \( \pi_i \) is nondegenerate.
or a pair of parallel lines. In either case, both of the one-sided tangents of $A$ at $p$ will be tangents of $\pi_i$ at $p$. Hence these one-sided tangents will coincide. By Lemma 2, $A$ will satisfy $A_1'$ at $p$.

**Remark.** The implications (ii), (iv) and (vi) in [2, 3.2] are false and (iii) should read: $A - \{p\}$ and $R$ will lie on the same side of either of the one-sided tangents of $A$.

5.4. Let $A$ be a convex arc and let $t, u, v, w$ be mutually distinct points which lie sufficiently close to $p$ on $A$ in that order. Then there will be two parabolas through these four points. We denote by $\pi_i(t, u, v, w)$ that parabola such that the order of the points $t, u, v, w$ on the parabola is the same as their order on $A$. $\pi_i(t, u, v, w)$ denotes the other parabola through these points; cf [2, 2.8]. We shall be concerned mainly with $\pi_i$'s from now on. It will be convenient to write $\pi(t, u, v, w)$ for $\pi_i(t, u, v, w)$ and refer to it as the principal parabola through $t, u, v, w$. As a limiting case, $\pi_i(p^3, t)$ will be denoted by $\pi(p^3, t)$.

**Condition AIII.** $A$ satisfies Condition AII at $p$ and $\lim_{t \to p} \pi(p; t)$ exists.

The limit superosculating parabola is denoted by $\pi(p')$. If $p$ is of Type 1, it is either nondegenerate, or a double ray on $X$ with the vertex $p$. If $p$ is an end-point of $A$, $p$ is of Type 1b [Type 1c] if this double ray is equal [opposite] to the limit of the double ray through $t$ with the vertex $p$ as $t$ tends to $p$ on $A$.

**Condition AIII'.** Suppose that the points $t, u, v, w$ are mutually distinct and lie on $A$ in that order. If they are sufficiently close to $p$, the parabola $\pi(t, u, v, w)$ exists. It converges as these four points converge to $p$.

Condition AIII' does not, in general, imply AII'. Consider, for example, the arc $A = A \cup p_i \cup A'_i$, where $p$ is the origin, $A_i$ is given by

$$x = s^2, \quad y = s^4 - s^5, \quad 0 < s < \delta \quad \text{(Type 1b)}$$

and $A'_i$ is given by

$$x = -s^3/2, \quad y = s^4 + s^5, \quad 0 < s < \delta \quad \text{(Type 1c)}.$$ 

Here $\delta$ is sufficiently small. The arc $A$ satisfies $A_1'$ at $p$ and the tangent $\tau$ at $p$ lies along the $x$-axis. Condition AII, however, is not satisfied, since the osculating parabolas of $A_i$ at $p$ are given by $(y - \lambda x)^2 = \lambda^2 y$, while those of $A'_i$ at $p$ are given by $(y - \lambda x)^2 = \lambda^2 y/4$. 


By Lemma 5, All' is not satisfied either. It may be verified that
\( \pi(t, u, v, w) \) always converges to the double ray on \( \mathbb{X} \) given by \( y = 0, x \geq 0 \), as the distinct points \( t, u, v, w \) converge on \( A \) to \( p \) in any manner. Thus \( A \) satisfies All' at \( p \).

**Lemma 6.** All' and All imply All and

\[
\pi(p) = \lim_{w \to p} \pi(p, w) = \lim_{t, u, v, w \to p} \pi(t, u, v, w).
\]

**Proof.** Any accumulation parabola of the \( \pi(p, w) \) as \( w \) tends to \( p \), can be approximated in turn by a \( \pi(p, v, w) \), a \( \pi(p, u, v, w) \) and a \( \pi(t, u, v, w) \).

5.5. We wish to prove

**Theorem 3.** Let \( A \) satisfy All' at \( p \). Then \( A \) will satisfy All at \( p \). It will also satisfy All at \( p \) if, in addition to All', any of the following conditions holds.

(i) \( \pi = \lim_{t, u, v, w \to p} \pi(t, u, v, w) \) is nondegenerate or the double line on \( \mathbb{X} \).

(ii) \( p \) is an end-point of \( A \).

(iii) \( A \) satisfies All at \( p \).

**Proof.** Assume that \( A \) satisfies All' at \( p \). Since \( \pi(t, u, v, w) \) exists when \( t, u, v, w \) are close to \( p \), we may assume that \( A \) is convex.

(i) Let \( \pi \) be nondegenerate. Then the tangent of \( \pi \) at \( p \) will be the tangent \( Z \) of \( A \) at \( p \); cf. [1, 2.12]. Thus \( A \) satisfies All at \( p \). Using the convexity of both \( A \) and \( \pi \), we readily verify that \( A - \{p\} \) and \( \pi - \{p\} \) will lie on the same side of \( Z \).

Let \( R \) be a point on \( \pi \), \( R \neq p \). Let \( Q \in \pi(t, u, v, w) \) and let \( Q \) converge to \( R \) as \( t, u, v, w \) tend to \( p \). Then \( \pi(t, u, v, w) \) is one of the \( \pi_i(t, u, v, Q) \), say \( \pi(t, u, v, Q) \), and \( \lim_{t, u, v, w \to p} \pi(t, u, v, Q) = \pi \). Since \( A \) is convex, and \( A \) and \( R \) lie on the same side of \( \mathbb{X} \), \( \pi_i(t, u, v, R) \) will exist when \( t, u, v \) are sufficiently close to \( p \). Any accumulation parabola \( \pi_i \) of the \( \pi_i(t, u, v, R) \) will meet \( \pi \) at \( p \) with a multiplicity \( \geq 3 \), and will also pass through \( R \). Hence either \( \pi_1 \) or \( \pi_2 \) will coincide with \( \pi \); cf. [2, 3.3]. Thus \( A \) satisfies All at \( p \) if \( \pi \) is nondegenerate. We shall deal with the case where \( \pi \) is the double line on \( \mathbb{X} \) after considering case (ii).

(ii) Let \( p \) be an end-point of \( A \). By Lemma 1, \( A \) satisfies All' and All at \( p \). Again \( \mathbb{X} \) denotes the tangent of \( A \) at \( p \). Let \( R \) be a point which lies on the same side of \( \mathbb{X} \) as \( A - \{p\} \) and let \( R \in \pi \). Then there exists a neighborhood \( B \) of \( p \) on \( A \) such that each of the parabolas \( \pi_i(t, u, v, R) \) does not meet \( B \) elsewhere. Let \( \pi(p, t, R) \) denote one of the \( \pi_i(p, t, R) \), \( i = 1, 2 \). As a limiting case, as \( t \) moves continuously and monotonically...
cally on $B$, $\pi(p^2, t, R)$ moves continuously and monotonically in one of the families of tangent parabolas of $B$ at $p$ through $R$. Hence $\lim_{t \to p} \pi(p^2, t, R)$ exists. Thus $A$ satisfies All at $p$.

Let $p, u, v, w$ lie on $B$ in that order. Then $\pi(p, u, v, R)$ lies between $\pi(p^2, v, R)$ and $\pi(p, v, w, R)$ in one of the families of parabolas through $p, v, w$, for each choice of $u$ on $B$ between $p$ and $w$. Letting $u$ and $v$ converge to $p$, one obtains that any accumulation parabola of the $\pi(p, u, v, R)$ lies between $\pi(p^2, R)$ and $\pi(p^2, w, R)$ for each choice of $w$ on $B$. Letting $w$ tend to $p$, we obtain that $\lim_{u, v \to p} \pi(p, u, v, R) = \pi(p^2, R)$.

By using a similar argument, one now readily verifies that

$$\lim_{t, u, v \to p} \pi(t, u, v, R) = \pi(p^2, R).$$

Thus, $A$ satisfies All' in case (ii) also.

Next, we deal with the case where $p$ is an interior point of $A$ and $\pi$ is degenerate. From the above, each of the subarcs into which $A$ is decomposed by $p$ will satisfy All and All' at $p$. By Lemma 6 AllI. By the same Lemma $\pi = \pi(p^2)$ with respect to each of these subarcs. It follows that $\pi$ will have to be a double ray (Types 1b, 1c or 2), or a double line (Type 3), on each of the one-sided tangents of $A$ at $p$. Hence, these one-sided tangents coincide and $A$ satisfies AllI at $p$. Since $A$ is convex, however, it also satisfies All' at $p$. Thus AllI' always implies All'.

From the above, we observe that if $\pi$ is the double line on $B$, then the one-sided osculating parabolas will be pairs of parallel lines one of which is $T$ and All will be satisfied automatically at $p$. This reduces our discussion to the case (iii).

(iii) Let $A$ satisfy All and AllI' at $p$. Then $A$ also satisfies AI and AllI there. Thus the families $\tau$ and $\sigma$ of tangent and osculating parabolas of $A$ at $p$ are defined and so is the superosculating parabola $\pi(p^2)$. Let $R$ be a point on the same side of $B$ as $A - \{p\}$; $R \in \pi(p^2)$. We may assume that $p$ is an interior point of $A$. Put $A = B \cup p \cup B'$. It is sufficient to prove that

$$\lim_{t, u, v \to p} \pi(t, u, v, R) = \pi(p^2, R).$$

where $\pi(t, u, v, R)$ is one of the $\pi_i(t, u, v, R)$.

The proof is similar to that given in (ii). There exists a neighborhood $M = N \cup p \cup N'$ of $p$ on $A$ such that the parabola $\pi(t, u, v, R)$ does not meet $A$ elsewhere if $t, u, v$ lie on $M$. Let $s, t \in N'$; $u, v \in N$. Then $\pi(t, u, v, R)$ will lie between $\pi(s, u, v, R)$ and $\pi(p, u, v, R)$ in one of the families of parabolas through $u, v, R$ for each choice of $t$ between $p$ and $s$ on $N'$. Letting $t, u, v$ tend to $p$, we obtain that
any accumulation parabola of the \( \pi(t, u, v, R) \) will lie between \( \pi(s, p^2, R) \) and \( \pi(p^3, R) \) for each choice of \( s \) on \( N' \). Letting \( s \) tend to \( p \), we obtain that \( \lim \pi(t, u, v, R) = \pi(p^3, R) \).

5.6. The method used to prove parts (ii) and (iii) of Theorem 3 can be generalized and will be needed later on.

Let \( A \) be a plane arc of order \( n \) with respect to a \( n \)-parameter family of characteristic curves \( K \). Let \( p, t_1, \ldots, t_n, t_{n+1} \) be distinct points of \( A \) in that order and let \( K(t_1, \ldots, t_n) \) denote the unique curve of \( K \) determined by \( t_1, \ldots, t_n \). Assume that

\[
K(p^{n-1}, t_n) = \lim_{t_1, \ldots, t_{n-1} \to p} K(t_1, \ldots, t_{n-1}, t_n)
\]

exists and

\[
K(p^n) = \lim_{t_n \to p} K(p^{n-1}, t_n) = \lim_{t_2, \ldots, t_n \to p} K(p, t_2, \ldots, t_n)
\]

exists and is nondegenerate.

We wish to show that

\[
\lim_{t_1, \ldots, t_n \to p} K(Kt_1, \ldots, t_n) = K(p^n).
\]

**Proof.** \( K(t_1, \ldots, t_n) \) will 'lie between' \( K(p, t_1, \ldots, t_n) \) and \( K(t_2, \ldots, t_{n+1}) \) in the one-parameter subfamily of \( K \) through the points \( t_2, \ldots, t_n \). Hence if \( t_1, \ldots, t_n \) tend to \( p \), any accumulation curve of the \( K(t_1, \ldots, t_n) \) will lie between \( K(p^n) \) and \( K(p^{n-1}, t_{n+1}) \) for each choice of \( t_{n+1} \) on \( A \).

Letting \( t_{n+1} \) tend to \( p \) we obtain the desired result.


6.1. Let \( A \) be an arc in the real projective plane. If \( A \) satisfies PI then \( \tau \) will denote the family of nondegenerate tangent conics of \( A \) at \( p \). The closure \( \overline{\tau} \) of the tangent conics is discussed in [3, 3.1].

**CONDITION PII.** Let \( A \) satisfy PI at \( p \) and let \( Q \) and \( R \) be any fixed points, \( Q \in \mathcal{X}, R \in \mathcal{X}; p, Q, R \) not collinear. If \( s \) is close to \( p \), \( s \neq p \), the unique tangent conic \( \gamma(\overline{\tau}, s, Q, R) \) or \( \gamma(p^3, s, Q, R) \) of \( A \) at \( p \) through \( Q, R \) and \( s \) converges as \( s \) tends to \( p \); cf. [3, 5.1].

The limiting *osculating conic* of \( A \) at \( p \) through \( Q \) and \( R \) is denoted by \( \gamma(\sigma; Q, R) \) or \( \gamma(p^3, Q, R) \). The family of all the osculating conics of \( A \) at \( p \) is denoted by \( \sigma \).

If PII holds, then \( \sigma \) is one of the following three subsets of \( \overline{\tau} \):

**Type 1.** \( \sigma \) consists of all the conics of \( \tau \) which have at least three-point contact at \( p \) with any particular member of \( \sigma \);

**Type 2.** \( \sigma \) consists of the pairs of distinct lines through \( p \), both
of them different from $\mathcal{X}$;

Type 3. $\sigma$ consists of the pairs of lines one of which is $\mathcal{X}$ while the other does not pass through $p$; cf. [3, Th. 5].

Any accumulation line of the lines $2(u, v)$ as $u$ and $v$ tend to $p$ on $A$ is called a general tangent of $A$ at $p$.

CONDITION PIΓ'. There exist two distinct points $Q$ and $R$, which are not collinear with $p$ and which do not lie on a general tangent of $A$ at $p$, with the following properties. If $t, u, v$ are mutually distinct and lie sufficiently close to $p$ on $A$, then the conic $\gamma(t, u, v, Q, R)$ exists. It converges as $t, u, v$ converge to $p$.

We note that the above condition is weaker than the corresponding condition in [4, 3.2], which also assumes PI.

Condition PIΓ' does not imply PI or PI' at $p$. The example used in § 4 also satisfies PIΓ at $p$ and

$$\lim_{t, u, v \to p} \gamma(t, u, v, Q, R) = 2(p, Q) \cup 2(p, R).$$

This arc, however, does not satisfy Condition PI.

We shall prove some of the assertions made in [4, 3.2] in the following.

**Theorem 4.** Let $A$ satisfy PIΓ at $p$. Then $A$ will satisfy PI if any one of the following also holds.

(i) $\gamma = \lim_{t, u, v \to p} \gamma(t, u, v, Q, R)$ is not a pair of lines through $p$,

(ii) $p$ is an end-point of $A$,

(iii) $A$ satisfies PI at $p$.

**Proof.** Let $A$ satisfy PIΓ at $p$.

(i) Assume that $\gamma$ is nondegenerate or a pair of lines one of which $2(Q, R)$. Then $\gamma$ has a unique tangent $\mathcal{X}$ at $p$. Thus $\mathcal{X} = \lim 2(U, V)$ where $U$ and $V$ lie on $\gamma(t, u, v, Q, R)$ and $U$ and $V$ tend to $p$ as $t, u, v$ tend to $p$. In particular, $\mathcal{X} = \lim_{u, v \to p} 2(u, v)$. Thus $A$ satisfies PIΓ at $p$.

(ii) Let $p$ be an end-point of $A$. Choose a point $S$ such that no three of $p, Q, R, S$ are collinear and $S$ does not lie on $\gamma$. Then there exists a neighborhood $B$ of $p$ on $A$ such that no conic through $Q, R$ and $S$ will meet $B$ more than twice. It follows that $\gamma(p, t, Q, R, S)$ varies continuously and monotonically in the pencil of conics through $p, Q, R, S$ as $t$ moves continuously and monotonically on $B$ to $p$. Hence $\gamma_0 = \lim_{t \to p} \gamma(p, t, Q, R, S)$ exists. Now $\gamma_0$ is not a pair of lines through $p$, and hence $\gamma_0$ has a tangent $\mathcal{X}$ at $p$. As in (i), $\lim_{t \to p} 2(p, t) = \mathcal{X}$. Thus $A$ satisfies PI at $p$. It will be shown in (iii) below that PI and PIΓ' imply PIΓ.
(iii) Assume that $A$ satisfies PI and PII at $p$. Then $A$ also satisfies PII there. Let $\mathcal{X}$ be the tangent of $A$ at $p$. Choose points $Q, R$ such that $p, Q, R$ are not collinear and $Q$ and $R$ do not lie on $\mathcal{X}$. Then $\gamma(p, Q, R)$ exists.

Next, choose $S \in \gamma(p^2 Q, R) \cup \mathcal{X}$ and such that no three of the points $p, Q, R, S$ are collinear. Then there exists a neighborhood $B$ of $p$ such that no conic through $Q, R$ and $S$ meets $B$ more than twice.

Let $t, u, v$ lie on $B$ with $u$ between $p$ and $t$. Then $\gamma(u, v, Q, R, S)$ will lie between $\gamma(t, v, Q, R, S)$ and $\gamma(p, v, Q, R, S)$ in the family of conics through $v, Q, R, S$. By applying §5.6 we obtain that $\lim_{u \to p} \gamma(u, v, Q, R, S) = \gamma(p, v, Q, R, S)$. By our choice of $S$, $\gamma(p^3, Q, R, S)$ is nondegenerate. Hence it has a tangent at $p$ and this tangent is $\mathcal{X}$. Hence $\lim_{u \to p} \mathcal{X}(u, v) = \mathcal{X}$ as before.

6.2. The next condition is:

CONDITION PIII. $A$ satisfies PII at $p$ and if $Q \in \mathcal{X}$, then $\gamma(p^3, s, Q)$ converges as $s$ tends to $p$ on $A$.

The limit superosculating conic of $A$ at $p$ through $Q$ is denoted by $\gamma(p^4, Q)$. If $p$ is Type 1, then the family $\rho$ of all the superosculating conics of $A$ at $p$ is one of the following subsets of $\sigma$.

Type 1a. $\rho$ is a subfamily of $\sigma$ consisting of all those conics of $\sigma$ which have four-point contact at $p$ with a particular conic of $\sigma$.

Type 1b. $\rho$ consists of all pairs of lines through $p$, one of which is $\mathcal{X}$; cf. [3, 6].

In Types 2 and 3, PIII is satisfied automatically.

CONDITION PIII'. There exists a point $R$, which does not lie on a general tangent of $A$ at $p$, with to following properties. If $t, u, v, w$ are mutually distinct and lie sufficiently close to $p$, the conic $\gamma(t, u, v, w, R)$ is uniquely defined. It converges as $t, u, v, w$ converge to $p$.

The above condition is weaker than that in [4, 3.4] which assumes PII'.

REMARK. If $A$ satisfies PIII' at $p$ and $\lim \gamma(t, u, v, w, R) = \gamma$, then $\lim_{t \to p} \gamma(t, u, v, w, Q) = \gamma$; for any accumulation conic of the $\gamma(t, u, v, w, Q)$ will have four-point contact with $\gamma$ at $p$ and it will also pass through $R$. Similarly, if $S \neq p$ and there is a sequence of points $Q$ tending to $S$ such that $\gamma(t, u, v, w, Q)$ converges, then $\gamma(t, u, v, w, S)$ converges to the same limit; thus PIII' holds.
We observe that $\text{PIII}'$ does not imply $\text{PI}'$ or even $\text{PII}$, as the example $A = A_4 \cup p \cup A'_y$ in § 5.4 shows. The arc $A$ satisfies $\text{PIII}'$ at $p$. In particular, if $R$ does not lie on the $x$-axis, then $\lim_{t, u, v, w \rightarrow p} \gamma(t, u, w, R) = \mathcal{X}(p, R) \cup \mathcal{X}$. The osculating conics of $A_4$ and of $A'_y$ at $p$ are given by

$$a(x^2 - y) + bxy + cy^2 = 0 \quad \text{and} \quad a(x^2 - y/4) + bxy + cy^2 = 0,$$

respectively. Hence $A$ does not satisfy $\text{PII}$ at $p$.

We wish to prove

**Theorem 5.** Let $A$ satisfy $\text{PIII}'$ at $p$. Then $A$ will satisfy $\text{PI}'$ at $p$. It will also satisfy $\text{PII}'$ at $p$ if, in addition, any one of the following holds.

(i) $\gamma = \lim_{t, u, v, w \rightarrow p} \gamma(t, u, v, w, R)$ is nondegenerate,

(ii) $p$ is an end-point of $A$.

(iii) $A$ satisfies $\text{PII}$ at $p$.

**Proof.** Let $A$ satisfy $\text{PIII}'$ at $p$.

(i) Assume that $\gamma$ is nondegenerate. Let $Q \in \gamma$, $Q \neq p$, $Q \neq R$. Then any accumulation conic $\gamma_o$ of the $\gamma(t, u, v, w, R)$ will have three-point contact with $\gamma$ at $p$ and will pass through $Q$ and $R$. Hence $\gamma = \gamma_o$. Thus $A$ satisfies $\text{PII}'$ at $p$. Since

$$\lim_{t, u, v, w \rightarrow p} \gamma(t, u, v, Q, R) = \gamma$$

is nondegenerate, $A$ also satisfies $\text{PI}$, $\text{PI}'$ and $\text{PII}$ at $p$.

(ii) Let $p$ be an end-point of $A$. Choose a point $Q$ not on a general tangent of $A$ at $p$ such that $Q \in \gamma$ and $p, Q, R$ are not collinear. Then there exists a neighborhood $B$ of $p$ on $A$ such that no conic through $Q$ and $R$ meets $B$ more than three times. By applying §5.6 one verifies that $B$ satisfies $\text{PI}$ and $\text{PII}$ at $p$. Then using the technique in [4, 5,2], one shows that $A$ satisfies $\text{PII}'$ at $p$. By §6.1, $A$ also satisfies $\text{PI}'$ there.

We now verify that $\text{PIII}'$ implies $\text{PI}'$ when $\gamma$ is degenerate and $p$ is an interior point of $A$. Let $A = B \cup p \cup B'$. Then $B$ and $B'$ both satisfy $\text{PIII}'$ at $p$. By (ii), they also satisfy $\text{PI}'$ and $\text{PII}'$. Let $\mathcal{X}$ and $\mathcal{X}'$ be the tangents of $B$ and $B'$ respectively at $p$.

Since $\gamma$ is degenerate, it must be $\mathcal{X} \cup \mathcal{X}(p, R)$ when associated with $B$ and $\mathcal{X}' \cup \mathcal{X}(p, R)$ when associated with $B'$. Hence $\mathcal{X} = \mathcal{X}'$. Thus $A$ satisfies $\text{PI}$ at $p$.

Since $\gamma(t, u, v, w, R)$ always converges as $t, u, v, w$ converge on $A$ to $p$, we readily verify that $A$ has no cusp at $p$. By Lemma 3, $A$ satisfies $\text{PI}'$ at $p$.

(iii) Assume that $A$ satisfies $\text{PII}$ and $\text{PIII}'$ at $p$. Then $A$ satisfies $\text{PIII}$ there. From the above, $A = B \cup p \cup B'$ also satisfies $\text{PI}'$ at $p$ and
both $B$ and $B'$ will satisfy PII' there. Let $\mathcal{X}$ be the tangent of $A$ at $p$. Let $Q$ and $R$ be points such that $Q \in \mathcal{X}$, $R \in \mathcal{X}$; $p, Q, R$ are not collinear, and $Q \in \gamma(p^3, R)$.

We wish to prove that $\gamma(t, u, v, Q, R)$ tends to $\gamma(p^3, Q, R)$ if $t, u, v$ tend to $p$. We may assume that $t, p, u, v$ lie on $A$ in that order with $t$ on $B'$ and $u, v$ on $B$.

Since PIII holds, a conic through $R$ and four points of $A$ tends to $\gamma(p^3, R)$ as these four points tend to $p$ on $A$. Hence there exists a neighborhood $M$ of $p$ on $A$ such that no conic through $Q$ and $R$ meets $M$ more than three times. The proof then follows the method of § 5.6.

6.3. Conical conditions IV and IV'.

CONDITION PIV. The arc $A$ satisfies PIII and the superosculating conic $\gamma(p^3, s)$ converges as $s$ tends to $p$.

The limit ultraosculating conic of $A$ at $p$ is denoted by $\gamma(p^3)$. It is nondegenerate (Type 1a(i)), or the point conic $p$ (Type 1a(ii)), or the double line on $\mathcal{X}$ (Type 1a(iii)); cf. [3, 7].

In the remaining cases, Types 1b, 2 and 3, PIV is satisfied automatically and $\gamma(p^3)$ is the double line on $\mathcal{X}$.

CONDITION PIV'. $\gamma(s, t, u, v, w)$ is uniquely defined and converges as the mutually distinct points $s, t, u, v, w$ converge on $A$ to $p$.

The above condition is weaker than the corresponding one in [4, 3.6] which also assumes PIII'.

We note that PIV' does not imply PIII'. To show this, let $A = A'_2 \cup p \cup A'_5$, where $p$ is the origin, $A'_5$ is given by $x = s^2, y = s^4 + s^7$, $0 < s < \delta$; and $A'_2$ is given by $x = -s^2 - s^5, y = s^4 + s^7$, $0 < s < \delta$; $\delta$ is sufficiently small. Here $p$ is of Type 1a(ii) with respect to both $A'_5$ and $A'_2$. It can be verified that $A$ satisfies PIV' at $p$. The families of superosculating conics of $A'_5$ and $A'_2$ are distinct, however, and hence PIII and PIII' do not hold there.

A similar counterexample exists for Type 1a(iii).

We wish prove

**Theorem 6.** Let $A$ satisfy PIV' at $p$. Then $A$ will satisfy PI' and PII' there. $A$ will also satisfy PIII' at $p$ if any one of the following holds.

(i) $\lim \gamma(s, t, u, v, w)$ is nondegenerate.
(ii) $p$ is an end-point of $A$.
(iii) $A$ satisfies PIII at $p$.

**Proof.** Let $A$ satisfy PIV' at $p$.

(i) Assume that $\gamma$ is nondegenerate. Let $R \in \gamma, R \neq p$, and let
$Q \in \gamma(s, t, u, v, w), Q \rightarrow R$. Then $\lim \gamma(t, u, v, w, Q) = \lim \gamma(s, t, u, v, w) = \gamma$. By the Remark in § 6.2, $A$ satisfies PII at $p$. By Theorem 5, $A$ also satisfies PII' and PI at $p$.

(ii) Let $p$ be an end-point of $A$. Choose a point $R \in \gamma$. Then there exists a neighborhood $B$ of $p$ on $A$ such that no conic through $R$ meets $B$ more than four times. By applying § 5.6, one can show that $B$ satisfies PI, PII'. We may assume $R \notin \mathcal{X}$ and again apply 5.6 to obtain PII and PIII at $p$. Then using the technique in [4, § 5], one proves that $B$ also satisfies PII' and PIII' at $p$.

(iii) Assume that $A$ satisfies PIII and PIV at $p$. Thus $A$ also satisfies PI, PII and PIV there. Thus $7 = 7(p)$.

We may assume that $p$ is an interior point of $A$; thus $A = B \cup B'$, say. By (ii), $B$ and $B'$ will satisfy PII', PII' and PIII' at $p$.

Let $R \in \mathcal{X} \cup \gamma(p)$. Following § 5.6 we can show that $\gamma(t', t, u, v, R)$ and $\gamma(u', t', t, u, R)$ both tend to $\gamma(p, R)$ as $t, u, v$ tend to $p$ on $B$ and $t', u'$ tend to $p$ on $B'$. The details will be omitted. This proves PIII'.

Next, we show that $A$ satisfies PII' at $p$, assuming only PIV'. From now on we may take $\gamma = \lim \gamma(s, t, u, v, w)$ to be degenerate and $p$ to be an interior point of $A$. If $A = B \cup p \cup B'$, then both $B$ and $B'$ will satisfy PII' at $p$. Hence $\gamma = \gamma(p)$ is the point $p$ or a double line. There exists a neighborhood $M = N \cup p \cup N'$ of $p$ on $A$ which is convex; otherwise a double segment, for example, would be among the accumulation conics through five points of $A$.

Case 1. $\gamma(p)$ is a double line. Then $\gamma(p)$ lies on both of the one-sided tangents $\mathcal{X}$ and $\mathcal{X}'$ of $A$ at $p$. Hence these tangents coincide. By Lemma 2, $A$ will satisfy PII' at $p$.

Case 2. $\gamma(p) = p$; thus, $p$ is of Type 1a(ii) with respect to both $B$ and $B'$. Since $M$ is convex, $M$ will not cross $\mathcal{X}$ or $\mathcal{X}'$ at $p$. Suppose that $\mathcal{X} \neq \mathcal{X}'$. Choose a point $R \in \mathcal{X} \cup \mathcal{X}'$ such that $R$ and $M - \{p\}$ lie in the same region bounded by $\mathcal{X}$ and $\mathcal{X}'$. If $t' \in N'$, then the $\gamma(p, t', R)$ of $B$ do not meet $M$ elsewhere. It is known that $\gamma(p, t', R)$ and $B$ touch $\mathcal{X}$ on the same side of $\mathcal{X}$. Hence the arc of $N'$ between $p$ and $t'$ will lie inside $\gamma(p, t', R)$; cf. [3, 3.3]. It follows that as $t'$ tends to $p$, $\gamma(p, t', R)$ will tend to the pair of lines $\mathcal{X}(p, R) \cup \mathcal{X}$, which intersects $A$ at $p$; cf. [3, 5 .11]. Hence the end-points of $M$ will lie on opposite sides of $\gamma(p, t', R)$ if $t'$ is sufficiently close to $p$ and $\gamma(p, t', R)$ will meet $M$ with an odd multiplicity. Since $M$ already meets $\gamma(p, t', R)$ four times, it will meet $M$ at least five times. This is a contradiction. It follows that $\mathcal{X} = \mathcal{X}'$ and by Lemma 2, $A$ satisfies PII' at $p$.

Finally, we show that PIV' implies PII'. Let $R$ be a point, $R \in \mathcal{X} \cup \gamma$ and again assume that no conic through $R$ meets a suitable neighborhood $M_i$ of $p$ on $A$ more than four times. Let $d \in M_i$ and choose a
neighborhood $M$ of $p$ on $M$, such that $d \notin M$. Then no conic through $R$ and $d$ will meet $M$ more than three times. We may also assume that $\mathcal{L}(p, R) \cup \mathcal{Z}$ does not meet $M$ outside $p$. The proof then follows the method of [4, 6.3].

The authors wish to thank Peter Scherk for reading the manuscript and making many valuable suggestions.

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Received November 28, 1967.

McMaster University,
University of California, Los Angeles, and
Lucknow University
Pacific Journal of Mathematics
Vol. 28, No. 3       May, 1969

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