GAMES WITH UNIQUE SOLUTIONS THAT ARE NONCONVEX

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In 1944 von Neumann and Morgenstern introduced a theory of solutions (stable sets) for n-person games in characteristic function form. This paper describes an eight-person game in their model which has a unique solution that is nonconvex. Former results in solution theory had not indicated that the set of all solutions for a game should be of this nature.

First, the essential definitions for an n-person game will be stated. Then, a particular eight-person game is described. Finally, there is a brief discussion on how to construct additional games with unique and nonconvex solutions.

The author [2] has subsequently used some variations of the techniques described in this paper to find a ten-person game which has no solution; thus providing a counterexample to the conjecture that every n-person game has a solution in the sense of von Neumann and Morgenstern.

2. Definitions. An n-person game is a pair \((N, v)\) where \(N = \{1, 2, \ldots, n\}\) and \(v\) is a real valued characteristic function on \(2^N\), that is, \(v\) assigns the real number \(v(S)\) to each subset \(S\) of \(N\) and \(v(\emptyset) = 0\). The set of all imputations is

\[
A = \{x : \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N\}
\]

where \(x = (x_1, x_2, \ldots, x_n)\) is a vector with real components. If \(x\) and \(y\) are in \(A\) and \(S\) is a nonempty subset of \(N\), then \(x \text{ dom}_S y\) means \(\sum_{i \in S} x_i \leq v(S)\) and \(x_i > y_i\) for all \(i \in S\). For \(B \subset A\) let \(\text{Dom}_S B = \{y \in A : \text{there exists } x \in B \text{ such that } x \text{ dom}_S y\}\) and let \(\text{Dom} B = \bigcup_{S \subset N} \text{Dom}_S B\). A subset \(K\) of \(A\) is a solution if \(K \cap \text{Dom} K = \emptyset\) and \(K \cup \text{Dom} K = A\). The core of a game is

\[
C = \{x \in A : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N\}.
\]

The core consists of those imputations which are maximal with respect to all of the relations \(\text{dom}_S\), and hence it is contained in every solution.

3. Example. Consider the game \((N, v)\) where \(N = \{1, 2, 3, 4, 5, 6, 7, 8\}\) and where \(v\) is given by: \(v(N) = 4, v(\{1, 4, 6, 7\}) = 2, v(\{1, 2\}) = 3\).
\[ v(\{3, 4\}) = v(\{5, 6\}) = v(\{7, 8\}) = 1, \text{ and } v(S) = 0 \text{ for all other } S \subseteq N. \]

For this game

\[ A = \left\{ x; \sum_{i \in N} x_i = 4 \text{ and } x_i \geq 0 \text{ for all } i \in N \right\} \]

and

\[ C = \{ x \in A; x_1 + x_2 = x_3 + x_4 = x_5 = x_7 + x_8 = 1 \]
\[ \text{and } x_1 + x_4 + x_6 + x_7 \geq 2 \} . \]

FIG. 1. Traces in \( H \) of \( L, C \) and \( K-C \)
Also define the four-dimensional hypercube

\[ H = \{ x \in A : x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1 \} \] .

Three traces of \( H \) as well as its 16 vertices are pictured in Fig. 1. The unique solution for this game is

\[ K = C \cup F_1 \cup F_4 \cup F_6 \cup F_7 \]

where the cube \( F_i \) is the face of \( H \) given by

\[ F_i = H \cap \{ x : x_i = 1 \} \quad i = 1, 4, 6, 7 . \]

Each \( F_i - C \) is a tetrahedron with one face meeting \( C \). In the three traces of \( H \) illustrated in Fig. 1, the traces of \( C \) are shown in heavy solid lines and the traces of the \( F_i - C \) are shown in heavy broken lines.

The proof that \( K \) is the unique solution follows readily from two observations. First, \( K \) is just those imputations in \( H \) which are maximal in \( H \) with respect to the relation \( \text{dom} \{1, 4, 6, 7\} \). Second, the closed line segment \( L \) joining the imputations \((0, 1, 0, 1, 0, 1, 0, 1)\) and \((1, 0, 1, 0, 1, 0, 0, 1)\) has the properties \( L \subset C \) and \( \bigcup_S \text{Dom}_S L = A - H \) when \( S = \{1, 2\}, \{3, 4\}, \{5, 6\}, \) and \( \{7, 8\} \).

To see that \( K \) is nonconvex, note the lower trace

\[ F_8 = H \cap \{ x : x_8 = 1 \} \]

in Fig. 1. The heavy lines (solid and broken) in this trace show \( K \cap F_8 \), which is clearly not convex. For example, the imputation

\[ \frac{1}{3} (1, 2, 2, 1, 2, 1, 0, 3) = \frac{1}{3} (0, 1, 1, 0, 1, 0, 1, 1) \\
+ \frac{1}{3} (0, 1, 0, 1, 1, 0, 0, 1) + \frac{1}{3} (1, 0, 1, 0, 1, 0, 0, 1) \]

is a linear combination of points in \( K \), but it is not itself in \( K \).

4. Remarks. The original von Neumann-Morgenstern theory [3] assumed that the characteristic function of a game is superadditive, that is, \( v(S_1 \cup S_2) \geq v(S_1) + v(S_2) \) whenever \( S_1 \) and \( S_2 \subset N \) and \( S_1 \cap S_2 = \emptyset \). Using the method of Gillies [1, p. 68] this example can be made into a game with a superadditive characteristic function without changing \( A, C, \) or the unique solution \( K \).

The essential idea in the example above is that \( \bigcup_S \text{Dom}_S L = A - H \) where \( S = \{1, 2\}, \{3, 4\}, \{5, 6\}, \) and \( \{7, 8\} \). One can generalize this relation in various ways to obtain many games in other dimensions which have a similar property. He can then introduce into these games additional \( S \subset N \) with \( v(S) > 0 \), but in such a way as to maintain the corresponding \( L \) as a subset of the core. As a result he will
obtain large classes of interesting solutions, many of which are unique and nonconvex.

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