

# Pacific Journal of Mathematics

**GAMES WITH UNIQUE SOLUTIONS THAT ARE NONCONVEX**

WILLIAM FRANKLIN LUCAS

## GAMES WITH UNIQUE SOLUTIONS THAT ARE NONCONVEX

W. F. LUCAS

**In 1944 von Neumann and Morgenstern introduced a theory of solutions (stable sets) for  $n$ -person games in characteristic function form. This paper describes an eight-person game in their model which has a unique solution that is nonconvex. Former results in solution theory had not indicated that the set of all solutions for a game should be of this nature.**

First, the essential definitions for an  $n$ -person game will be stated. Then, a particular eight-person game is described. Finally, there is a brief discussion on how to construct additional games with unique and nonconvex solutions.

The author [2] has subsequently used some variations of the techniques described in this paper to find a ten-person game which has no solution; thus providing a counterexample to the conjecture that every  $n$ -person game has a solution in the sense of von Neumann and Morgenstern.

**2. Definitions.** An  $n$ -person game is a pair  $(N, v)$  where  $N = \{1, 2, \dots, n\}$  and  $v$  is a real valued characteristic function on  $2^N$ , that is,  $v$  assigns the real number  $v(S)$  to each subset  $S$  of  $N$  and  $v(\varnothing) = 0$ . The set of all *imputations* is

$$A = \left\{ x: \sum_{i \in N} x_i = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for all } i \in N \right\}$$

where  $x = (x_1, x_2, \dots, x_n)$  is a vector with real components. If  $x$  and  $y$  are in  $A$  and  $S$  is a nonempty subset of  $N$ , then  $x \text{ dom}_S y$  means  $\sum_{i \in S} x_i \leq v(S)$  and  $x_i > y_i$  for all  $i \in S$ . For  $B \subset A$  let  $\text{Dom}_S B = \{y \in A: \text{there exists } x \in B \text{ such that } x \text{ dom}_S y\}$  and let  $\text{Dom } B = \bigcup_{S \subset N} \text{Dom}_S B$ . A subset  $K$  of  $A$  is a *solution* if  $K \cap \text{Dom } K = \varnothing$  and  $K \cup \text{Dom } K = A$ . The *core* of a game is

$$C = \left\{ x \in A: \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N \right\}.$$

The core consists of those imputations which are maximal with respect to all of the relations  $\text{dom}_S$ , and hence it is contained in every solution.

**3. Example.** Consider the game  $(N, v)$  where  $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$  and where  $v$  is given by:  $v(N) = 4$ ,  $v(\{1, 4, 6, 7\}) = 2$ ,  $v(\{1, 2\}) =$

$v(\{3, 4\}) = v(\{5, 6\}) = v(\{7, 8\}) = 1$ , and  $v(S) = 0$  for all other  $S \subset N$ . For this game

$$A = \left\{ x: \sum_{i \in N} x_i = 4 \text{ and } x_i \geq 0 \text{ for all } i \in N \right\}$$

and

$$C = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1 \text{ and } x_1 + x_4 + x_6 + x_7 \geq 2\}.$$

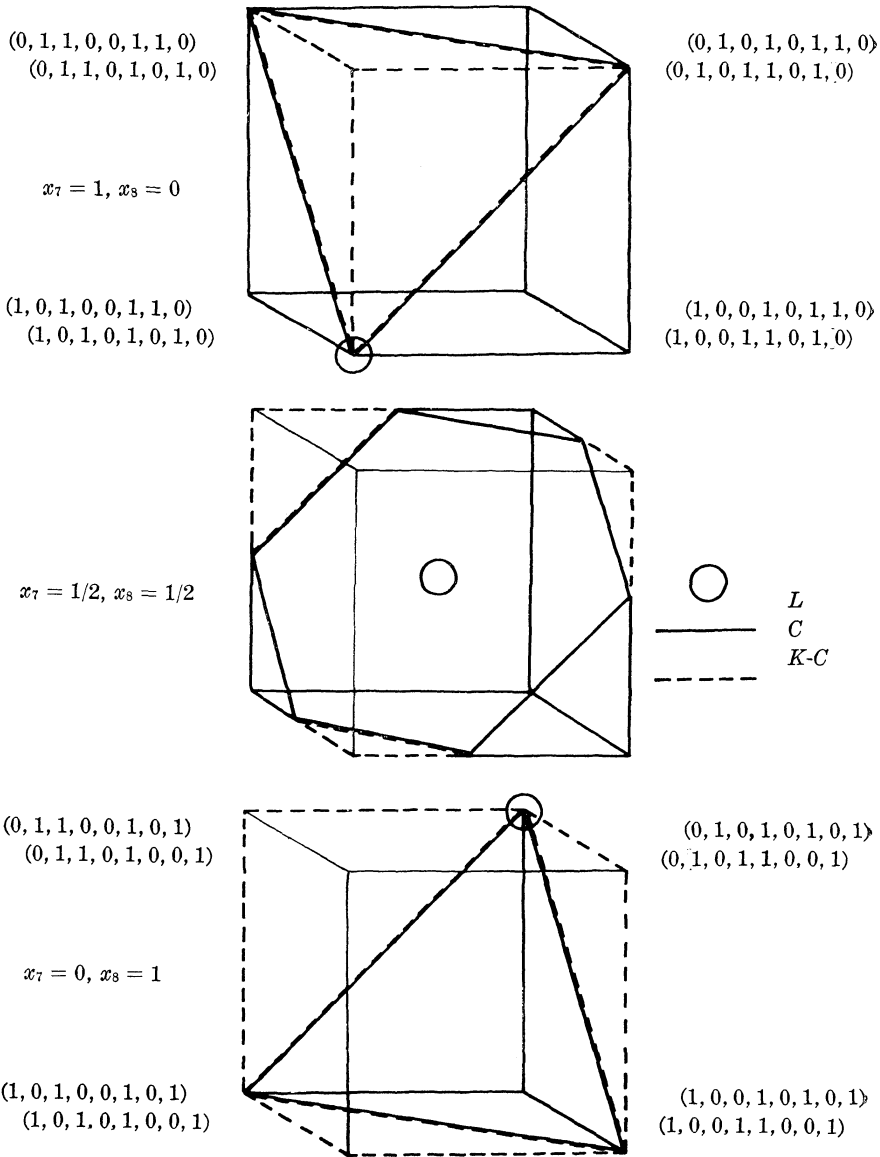


FIG. 1. Traces in  $H$  of  $L, C$  and  $K-C$

Also define the four-dimensional hypercube

$$H = \{x \in A: x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = 1\}.$$

Three traces of  $H$  as well as its 16 vertices are pictured in Fig. 1.

The unique solution for this game is

$$K = C \cup F_1 \cup F_4 \cup F_6 \cup F_7$$

where the cube  $F_i$  is the face of  $H$  given by

$$F_i = H \cap \{x: x_i = 1\} \quad i = 1, 4, 6, 7.$$

Each  $F_i - C$  is a tetrahedron with one face meeting  $C$ . In the three traces of  $H$  illustrated in Fig. 1, the traces of  $C$  are shown in heavy solid lines and the traces of the  $F_i - C$  are shown in heavy broken lines.

The proof that  $K$  is the unique solution follows readily from two observations. First,  $K$  is just those imputations in  $H$  which are maximal in  $H$  with respect to the relation  $\text{dom}_{\{1,4,6,7\}}$ . Second, the closed line segment  $L$  joining the imputations  $(0, 1, 0, 1, 0, 1, 0, 1)$  and  $(1, 0, 1, 0, 1, 0, 1, 0)$  has the properties  $L \subset C$  and  $\bigcup_S \text{Dom}_S L = A - H$  when  $S = \{1, 2\}, \{3, 4\}, \{5, 6\}$ , and  $\{7, 8\}$ .

To see that  $K$  is nonconvex, note the lower trace

$$F_8 = H \cap \{x: x_8 = 1\}$$

in Fig. 1. The heavy lines (solid and broken) in this trace show  $K \cap F_8$ , which is clearly not convex. For example, the imputation

$$\begin{aligned} \frac{1}{3}(1, 2, 2, 1, 2, 1, 0, 3) &= \frac{1}{3}(0, 1, 1, 0, 0, 1, 0, 1) \\ &+ \frac{1}{3}(0, 1, 0, 1, 1, 0, 0, 1) + \frac{1}{3}(1, 0, 1, 0, 1, 0, 0, 1) \end{aligned}$$

is a linear combination of points in  $K$ , but it is not itself in  $K$ .

**4. Remarks.** The original von Neumann-Morgenstern theory [3] assumed that the characteristic function of a game is superadditive, that is,  $v(S_1 \cup S_2) \geq v(S_1) + v(S_2)$  whenever  $S_1$  and  $S_2 \subset N$  and  $S_1 \cap S_2 = \varphi$ . Using the method of Gillies [1, p. 68] this example can be made into a game with a superadditive characteristic function without changing  $A, C$ , or the unique solution  $K$ .

The essential idea in the example above is that  $\bigcup_S \text{Dom}_S L = A - H$  where  $S = \{1, 2\}, \{3, 4\}, \{5, 6\}$ , and  $\{7, 8\}$ . One can generalize this relation in various ways to obtain many games in other dimensions which have a similar property. He can then introduce into these games additional  $S \subset N$  with  $v(S) > 0$ , but in such a way as to maintain the corresponding  $L$  as a subset of the core. As a result he will

obtain large classes of interesting solutions, many of which are unique and nonconvex.

#### REFERENCES

1. D. B. Gillies, *Solutions to general non-zero-sum games*, Annals of Mathematics Studies, No. 40, A. W. Tucker and R. D. Luce (editors), Princeton University Press, Princeton, 1959.
2. W. F. Lucas, *A Game with no solution*, RAND Memorandum RM-5518-PR, The RAND Corporation, Santa Monica, November 1967.
3. J. von Neumann and O. Morgenstern, *Theory of games and economic behavior*, Princeton University Press, Princeton, (1944).

Received June 26, 1967.

THE RAND CORPORATION  
SANTA MONICA, CALIFORNIA

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN  
Stanford University  
Stanford, California

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. R. PHELPS  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

# Pacific Journal of Mathematics

Vol. 28, No. 3

May, 1969

Jon F. Carlson, <i>Automorphisms of groups of similitudes over <math>F_3</math></i> .....	485
W. Wistar (William) Comfort, Neil Hindman and Stelios A. Negrepointis, <i><math>F'</math>-spaces and their product with <math>P</math>-spaces</i> .....	489
Archie Gail Gibson, <i>Triples of operator-valued functions related to the unit circle</i> .....	503
David Saul Gillman, <i>Free curves in <math>E^3</math></i> .....	533
E. A. Heard and James Howard Wells, <i>An interpolation problem for subalgebras of <math>H^\infty</math></i> .....	543
Albert Emerson Hurd, <i>A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems</i> .....	555
E. W. Johnson and J. P. Lediaev, <i>Representable distributive Noether lattices</i> .....	561
David G. Kendall, <i>Incidence matrices, interval graphs and seriation in archeology</i> .....	565
Robert Leroy Kruse, <i>On the join of subnormal elements in a lattice</i> .....	571
D. B. Lahiri, <i>Some restricted partition functions; Congruences modulo 3</i> ....	575
Norman D. Lane and Kamla Devi Singh, <i>Strong cyclic, parabolic and conical differentiability</i> .....	583
William Franklin Lucas, <i>Games with unique solutions that are nonconvex</i> .....	599
Eugene A. Maier, <i>Representation of real numbers by generalized geometric series</i> .....	603
Daniel Paul Maki, <i>A note on recursively defined orthogonal polynomials</i> ....	611
Mark Mandelker, <i><math>F'</math>-spaces and <math>z</math>-embedded subspaces</i> .....	615
James R. McLaughlin and Justin Jesse Price, <i>Comparison of Haar series with gaps with trigonometric series</i> .....	623
Ernest A. Michael and A. H. Stone, <i>Quotients of the space of irrationals</i> ....	629
William H. Mills and Neal Zierler, <i>On a conjecture of Golomb</i> .....	635
J. N. Pandey, <i>An extension of Haimo's form of Hankel convolutions</i> .....	641
Terence John Reed, <i>On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles</i> .....	653
Haskell Paul Rosenthal, <i>A characterization of the linear sets satisfying Herz's criterion</i> .....	663
George Thomas Sallee, <i>The maximal set of constant width in a lattice</i> .....	669
I. H. Sheth, <i>On normaloid operators</i> .....	675
James D. Stasheff, <i>Torsion in BBSO</i> .....	677
Billy Joe Thorne, <i><math>A - P</math> congruences on Baer semigroups</i> .....	681
Robert Breckenridge Warfield, Jr., <i>Purity and algebraic compactness for modules</i> .....	699
Joseph Zaks, <i>On minimal complexes</i> .....	721