

Pacific Journal of Mathematics

REPRESENTATION OF REAL NUMBERS BY GENERALIZED GEOMETRIC SERIES

EUGENE A. MAIER

REPRESENTATION OF REAL NUMBERS BY GENERALIZED GEOMETRIC SERIES

E. A. MAIER

We shall say that the series of real numbers, $\sum_{i=0}^{\infty} 1/a_i$, is a **generalized geometric series (g.g.s.)** if and only if $a_i^2 \leq a_{i+1}a_{i-1}$ for all $i \geq 1$. (Note that the series is geometric if and only if equality holds.) In this paper we investigate the representation of positive real numbers less than or equal to one by generalized geometric series of the form $\sum_{i=0}^{\infty} x^i/c_i$, where the c_i are positive integers and $x \geq 1$.

1. Preliminary results.

LEMMA 1. *If $\sum_{i=0}^{\infty} 1/a_i$ is a g.g.s. and*

$$|a_k| < |a_{k+1}|, \quad \text{then} \quad \sum_{i=k+1}^{\infty} \left| \frac{1}{a_i} \right| \leq \frac{1}{|a_{k+1}| - |a_k|}.$$

Proof. Since $|a_k/a_{k+t}| \leq |a_k/a_{k+1}|^t$ for all $t \geq 1$, we have

$$\sum_{i=k+1}^{\infty} \frac{1}{|a_i|} \leq \frac{1}{|a_k|} \sum_{i=1}^{\infty} \left| \frac{a_k}{a_{k+1}} \right|^i = \frac{1}{|a_{k+1}| - |a_k|}.$$

The following theorem readily follows from Lemma 1.

THEOREM 1. *The g.g.s. $\sum_{i=0}^{\infty} 1/a_i$ converges if and only if there exists k such that $|a_k| < |a_{k+1}|$.*

THEOREM 2. *Let $\sum_{i=0}^{\infty} 1/a_i$ be a g.g.s. with $0 < a_0 < a_1$. Let $\alpha = \sum_{i=1}^{\infty} 1/a_i$, $S_k = \sum_{i=0}^k 1/a_i$ and $t_{k+1} = a_{k+1}/a_k - 1$. Then*

- (i) *the sequence of half-open intervals $\{(S_k, S_k + 1/(a_{k+1} - a_k))\}$ is a sequence of nested intervals whose intersection is α ,*
- (ii) $t_k \leq t_{k+1} \leq 1/a_k(\alpha - S_k)$.

Proof. Since the series is a g.g.s., we have

$$\frac{1}{a_{k+1} - a_k} \geq \frac{1}{a_{k+1}} + \frac{1}{a_{k+2} - a_{k+1}}.$$

Hence the sequence of intervals in (i) above is nested. Also $a_k < a_{k+1}$ for all $k \geq 0$. Thus, using Lemma 1,

$$(1) \quad S_k < \alpha \leq S_k + \frac{1}{a_{k+1} - a_k} = S_k + \frac{1}{a_k t_{k+1}}.$$

Since $a_k/a_{k+1} \leq a_0/a_1$ for all $k \geq 0$, we have

$$0 \leq \lim_{k \rightarrow \infty} \frac{1}{a_{k+1} - a_k} \leq \lim_{k \rightarrow \infty} \frac{1}{a_k \left(\frac{a_1}{a_0} - 1 \right)} = 0,$$

and it follows from (1) that the intervals converge to α .

Inequalities (ii) are obtained from (1) and the definition of t_k .

COROLLARY. *Let $x > 0$ and let $\alpha = \sum_{i=0}^{\infty} x^i/c_i$ be a g.g.s. with $0 < c_0x < c_i$; let $S_k = \sum_{i=0}^k x^i/c_i$ and $s_{k+1} = c_{k+1}/c_k - 1$. Then*

- (i) *the sequence of half-open intervals $\{(S_k, S_k + x^{k+1}/(c_{k+1} - xc_k))\}$ is a sequence of nested intervals whose intersection is α ,*
- (ii) $s_k \leq s_{k+1} \leq x^{k+1}/(c_k(\alpha - S_k)) + (x - 1)$.

Proof. Apply the theorem with $a_k = c_k/x^k$ observing that in this case

$$t_{k+1} = \frac{c_{k+1}}{xc_k} - 1 = \frac{s_{k+1} + 1}{x} - 1.$$

3. The representation of reals. The corollary to Theorem 2 suggests an algorithm for constructing a g.g.s. of the form $\sum_{i=0}^{\infty} x^i/c_i$, where the c_i are integers and $x \geq 1$, which converges to a given positive real number $\beta \leq 1$.

To obtain such a series let $\{s_0, s_1, s_2 \dots\}$ be any sequence of positive integers such that $s_0 = [1/\beta]$ and, for $k \geq 0$,

$$\max \left\{ \frac{x^{k+1}}{c_k(\beta - S_k)} - 1, s_k - 1 \right\} < s_{k+1} \leq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1)$$

where $c_k = \prod_{j=0}^k (s_j + 1)$ and $S_k = \sum_{j=0}^k x^j/c_j$.

Such a sequence of integers exists since

$$\begin{aligned} c_k(\beta - S_k) &= c_{k-1}(s_k + 1)(\beta - S_{k-1}) - x^k \\ &\leq c_{k-1} \left(\frac{x^k}{c_{k-1}(\beta - S_{k-1})} + x \right) (\beta - S_{k-1}) - x^k = xc_{k-1}(\beta - S_{k-1}) \end{aligned}$$

and hence

$$s_k \leq \frac{x^k}{c_{k-1}(\beta - S_{k-1})} + (x - 1) \leq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1).$$

The resulting series, $\sum_{i=0}^{\infty} x^i/c_i$ where $c_i = \prod_{j=0}^i (s_j + 1)$, is a g.g.s. since $s_{k+1} \geq s_k$. Also since $\beta \leq 1$,

$$c_1 = (s_1 + 1)c_0 > \frac{c_0x}{c_0\beta - 1} = \frac{c_0x}{\left(\left[\frac{1}{\beta}\right] + 1\right)\beta - 1} \geq xc_0 .$$

Thus the series satisfies the hypotheses of the corollary to Theorem 2. Now from the manner in which the sequence $\{s_k\}$ has been obtained, we have

$$\frac{x^{k+1}}{c_k(\beta - S_k)} - 1 < s_{k+1} \leq \frac{x^{k+1}}{c_k(\beta - S_k)} + (x - 1)$$

and thus

$$\begin{aligned} S_{k+1} &= S_k + \frac{x^{k+1}}{c_k(s_{k+1} + 1)} < \beta \leq S_k + \frac{x^{k+1}}{c_k(s_{k+1} + 1 - x)} \\ &= S_k + \frac{x^{k+1}}{c_{k+1} - xc_k} . \end{aligned}$$

Therefore, by (i) of the corollary, $\beta = \sum_{i=0}^{\infty} x^i/c_i$.

If $x \neq 1$, the sequence $\{s_k\}$ obtained by the above process is not unique. For example, if $\beta = 1$ and $x = 2$, we have $s_0 = 1$,

$$\begin{aligned} \max \left\{ \frac{x}{c_0\beta - 1} - 1, s_0 - 1 \right\} \\ = \max \{1, 0\} = 1 \quad \text{and} \quad \frac{x}{c_0 - 1} + x - 1 = 3 . \end{aligned}$$

Thus there are two possible values for s_1 . To obtain uniqueness, we must further restrict the s_k . One restriction that leads to a unique representation is to require that $s_k \geq xs_{k-1}$. We now turn our attention to series which satisfy this condition.

THEOREM 3. *Let $s > 0$, $x \geq 1$. For $k \geq 0$, let $s_k = x^k s$, $c_k = \prod_{i=0}^k (s_i + 1)$. Then $\sum_{i=0}^{\infty} x^i/c_i = 1/s$; that is*

$$\frac{1}{s} = \frac{1}{s + 1} + \frac{x^2}{(s + 1)(xs + 1)} + \frac{x^2}{(s + 1)(xs + 1)(x^2s + 1)} + \dots .$$

Proof. Let $S_k = \sum_{i=0}^k x_i/c_i$. We shall show by induction that $S_k + 1/c_k s = 1/s$ for all $k \geq 0$. For $k = 0$, we have

$$s_0 + \frac{1}{c_0 s} = \frac{1}{c_0} \left(1 + \frac{1}{s}\right) = \frac{1}{s + 1} \left(\frac{s + 1}{s}\right) = \frac{1}{s} .$$

If $S_k + 1/c_k s = 1/s$, then

$$\begin{aligned}
 S_{k+1} + \frac{1}{c_{k+1}s} &= S_k + \frac{x^{k+1}}{c_{k+1}} + \frac{1}{c_{k+1}s} \\
 &= \frac{1}{s} - \frac{1}{c_k s} + \frac{1}{c_k} \left(\frac{sx^{k+1} + 1}{s(sx^{k+1} + 1)} \right) = \frac{1}{s}.
 \end{aligned}$$

It also follows by induction that $c_k > (s + 1)^k$ and hence, since $s + 1 > 1$, $\lim_{k \rightarrow \infty} 1/c_k s = 0$. Thus $\lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} (1/s + 1/c_k s) = 1/s$.

THEOREM 4. *Let $x \geq 1$. Let $\{s_0, s_1, s_2, \dots\}$ be a sequence of positive integers such that $s_k \geq xs_{k-1}$ for all $k \geq 0$ and let $c_k = \prod_{i=0}^k (s_i + 1)$. Then $\sum_{i=0}^{\infty} x^i/c_i$ is a convergent g.g.s. Furthermore if $\alpha = \sum_{i=0}^{\infty} x^i/c_i$ and $S_k = \sum_{i=0}^k x_i/c_i$ then*

- (i) *the sequence of half-open intervals $\{(S_k, S_k + x^{k+1}/(c_{k+1} - c_k))\}$ is a sequence of nested intervals whose intersection is α ,*
- (ii) *$s_{k+1} = [x^{k+1}/c_k(\alpha - S_k)]$ for all $k \geq 0$, $s_0 = [1/\alpha]$,*
- (iii) *if x is rational, then α is rational if and only if $sk = xs_{k-1}$ for all k sufficiently large.*

Proof. Since $c_i/c_{i-1} = s_i + 1 \leq s_{i+1} + 1 = c_{i+1}/c_i$, it follows that

$$\left(\frac{c_i}{x^i} \right)^2 \leq \left(\frac{c_{i-1}}{x^{i-1}} \right) \left(\frac{c_{i+1}}{x^{i+1}} \right)$$

and hence $\sum_{i=0}^{\infty} x^i/c_i$ is a g.g.s. The series converges by Theorem 1 since $c_0 < c_1/x$.

To establish (i), we first observe that $s_{k+j+1} \geq x^j s_{k+1}$ for all $j \geq 0$. Thus, using Theorem 3, we have

$$\begin{aligned}
 S_k < \alpha &= S_k + \sum_{i=0}^{\infty} \frac{x^{k+j+1}}{c_{k+j+1}} \\
 &= S_k + \frac{x^{k+1}}{c_k} \left(\frac{1}{s_{k+1} + 1} + \frac{x}{(s_{k+1} + 1)(s_{k+2} + 1)} \right. \\
 &\quad \left. + \frac{x^2}{(s_{k+1} + 1)(s_{k+2} + 1)(s_{k+3} + 1)} + \dots \right) \\
 &\leq S_k + \frac{x^{k+1}}{c_k} \left(\frac{1}{s_{k+1} + 1} + \frac{x}{(s_{k+1} + 1)(xs_{k+1} + 1)} \right. \\
 &\quad \left. + \frac{x^2}{(s_{k+1} + 1)(xs_{k+1} + 1)(x^2s_{k+1} + 1)} + \dots \right) \\
 &= S_k + \frac{x^{k+1}}{c_k} \cdot \frac{1}{s_{k+1}} = S_k + \frac{x^{k+1}}{c_{k+1} - c_k}.
 \end{aligned}$$

Furthermore, since $s_{k+2} - xs_{k+1} \geq 0$, we have

$$\begin{aligned} \frac{x^{k+1}}{c_k s_{k+1}} - \frac{x^{k+2}}{c_{k+1} s_{k+2}} &= \frac{x^{k+1}}{c_{k+1}} \left(\frac{s^{k+1} + 1}{s_{k+1}} - \frac{x}{s_{k+2}} \right) \\ &= \frac{x^{k+1}}{c_{k+1}} \left(1 + \frac{1}{s_{k+1}} - \frac{x}{s_{k+2}} \right) \geq \frac{x^{k+1}}{c_{k+1}}. \end{aligned}$$

Thus

$$S_{k+1} + \frac{x^{k+2}}{c_{k+1} s_{k+2}} = S_k + \frac{x^{k+1}}{c_{k+1}} + \frac{x^{k+2}}{c_{k+1} s_{k+2}} \leq S_k + \frac{x^{k+1}}{c_k s_{k+1}}$$

and the sequence of intervals in (i) is nested. Since

$$S_k + \frac{x^{k+1}}{c_{k+1} - c_k} \leq S_k + \frac{x^{k+1}}{c_{k+1} - x c_k},$$

by part (i) of the corollary to Theorem 2, the intersection of the intervals is α .

To establish (ii), we have from (i) that

$$S_k + \frac{x^{k+1}}{c_k(s_{k+1} + 1)} = S_{k+1} < \alpha \leq S_k + \frac{x^{k+1}}{c_k s_{k+1}}.$$

Solving these inequalities for s_{k+1} , we have

$$(2) \quad \frac{x^{k+1}}{c_k(\alpha - S_k)} - 1 < s_{k+1} \leq \frac{x^{k+1}}{c_k(\alpha - S_k)}.$$

Also, since $s_k \geq x^k s_0$, using Theorem 3, we have

$$\begin{aligned} \frac{1}{s_0 + 1} = S_0 < \alpha &\leq \frac{1}{s_0 + 1} + \frac{1}{(s_0 + 1)(x s_0 + 1)} \\ &+ \frac{x^2}{(s_0 + 1)(x s_0 + 1)(x^2 s_0 + 1)} + \dots = \frac{1}{s_0} \end{aligned}$$

and hence $s_0 = \lceil 1/\alpha \rceil$.

We turn now to the proof of (iii). Suppose $s_k = x s_{k-1}$ for all $k > k_0$. Then $s_{k_0+j} = x^j s_{k_0}$ for all $j \geq 0$. Thus, again using Theorem 3, we have

$$\begin{aligned} \alpha &= S_{k_0-1} + \frac{x^{k_0}}{c_{k_0-1}} \left(\frac{1}{s_{k_0+1}} \right. \\ &\quad \left. + \frac{x}{(s_{k_0+1})(x s_{k_0} + 1)} + \frac{x^2}{(s_{k_0+1})(x s_{k_0+1})(x^2 s_{k_0+1})} + \dots \right) \\ &= S_{k_0-1} + \frac{x^{k_0}}{c_{k_0-1}} \cdot \frac{1}{s_{k_0}} \end{aligned}$$

which is rational if x is rational.

Conversely, suppose α is rational. From (2) we have

$$\frac{x^{k+1}}{s_{k+1} + 1} < C_k(\alpha - S_k) \leq \frac{x^{k+1}}{s_{k+1}}.$$

Thus

$$\begin{aligned} 0 < c_{k+1}(\alpha - S_{k+1}) &= c_{k+1}\left(\alpha - S_k - \frac{x^{k+1}}{c_{k+1}}\right) \\ (3) \qquad &= c_k(s_{k+1} + 1)(\alpha - S_k) - x^{k+1} \\ &\leq c_k\left(\frac{x^{k+1}}{c_k(\alpha - S_k)} + 1\right)(\alpha - S_k) - x^{k+1} = c_k(\alpha - S_k). \end{aligned}$$

Hence, if $\alpha = p/q$, for all k we have

$$0 < c_{k+1}(p - S_{k+1}q) \leq c_k(p - S_kq).$$

Therefore, noting that $c_k S_k$ is an integer, $\{c_k(p - S_kq)\}$ is a nonincreasing sequence of positive integers and thus for k sufficiently large, say $k > k_0$, the terms of the sequence become constant. Hence, for $k > k_0$, $c_{k+1}(\alpha - S_{k+1}) = c_k(\alpha - S_k)$ and thus equality must hold in (3). Therefore $s_{k+1} = x^{k+1}/c_k(\alpha - S_k)$ for $k > k_0$ and, for k sufficiently large,

$$s_{k+1} = \frac{x^{k+1}}{c_k(\alpha - S_k)} = x \cdot \frac{x^k}{c_{k-1}(\alpha - S_{k-1})} = x s_k.$$

THEOREM 5. *Let $0 < \beta \leq 1$ and let x be a positive integer. Then there exists a unique sequence of positive integers $\{s_0, s_1, s_2, \dots\}$ such that $s_k \geq x s_{k-1}$ for $k \geq 1$ and $\beta = \sum_{i=0}^{\infty} x^i/c_i$ where $c_i = \prod_{j=0}^i (s_j + 1)$.*

Proof. Define

$$s_0 = \left[\frac{1}{\beta} \right], \quad c_0 = s_0 + 1, \quad S_0 = \frac{1}{c_0}$$

and, for $k > 0$,

$$s_{k+1} = \left[\frac{x^{k+1}}{c_k(\beta - S_k)} \right], \quad c_{k+1} = (s_{k+1} + 1)c_k, \quad S_{k+1} = S_k + \frac{x^{k+1}}{c_{k+1}}.$$

Then, in the same manner as inequality (3) was obtained, we have

$$c_{k+1}(\beta - S_{k+1}) \leq c_k(\beta - S_k).$$

Thus

$$\begin{aligned}
 s_{k+1} &> \frac{x^{k+1}}{c_k(\beta - S_k)} - 1 \geq \frac{x^{k+1}}{c_{k-1}(\beta - S_{k-1})} - 1 \\
 &= x \left(\frac{x^k}{c_{k-1}(\beta - S_{k-1})} \right) - 1 \geq x s_k - 1.
 \end{aligned}$$

Since s_{k+1} and $x s_k - 1$ are integers, it follows that $s_{k+1} \geq x s_k$. Also from the definition of s_{k+1} ,

$$\frac{x^{k+1}}{c_k(\beta - S_k)} - 1 < s_{k+1} \leq \frac{x^{k+1}}{c_k(\beta - S_k)}.$$

Therefore

$$S_k < S_{k+1} = S_k + \frac{x^{k+1}}{c_{k+1}} < \beta \leq S_k + \frac{x^{k+1}}{c_k(\beta - S_k)}.$$

Thus from Theorem 4 (i), $\beta = \sum_{i=0}^{\infty} x^i/c_i$. The uniqueness of the sequence $\{s_k\}$ follows from Theorem 4 (ii).

Received March 20, 1968.

UNIVERSITY OF OREGON AND
PACIFIC LUTHERAN UNIVERSITY

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Jon F. Carlson, <i>Automorphisms of groups of similitudes over F_3</i>	485
W. Wistar (William) Comfort, Neil Hindman and Stelios A. Negrepointis, <i>F'-spaces and their product with P-spaces</i>	489
Archie Gail Gibson, <i>Triples of operator-valued functions related to the unit circle</i>	503
David Saul Gillman, <i>Free curves in E^3</i>	533
E. A. Heard and James Howard Wells, <i>An interpolation problem for subalgebras of H^∞</i>	543
Albert Emerson Hurd, <i>A uniqueness theorem for weak solutions of symmetric quasilinear hyperbolic systems</i>	555
E. W. Johnson and J. P. Lediaev, <i>Representable distributive Noether lattices</i>	561
David G. Kendall, <i>Incidence matrices, interval graphs and seriation in archeology</i>	565
Robert Leroy Kruse, <i>On the join of subnormal elements in a lattice</i>	571
D. B. Lahiri, <i>Some restricted partition functions; Congruences modulo 3</i>	575
Norman D. Lane and Kamla Devi Singh, <i>Strong cyclic, parabolic and conical differentiability</i>	583
William Franklin Lucas, <i>Games with unique solutions that are nonconvex</i>	599
Eugene A. Maier, <i>Representation of real numbers by generalized geometric series</i>	603
Daniel Paul Maki, <i>A note on recursively defined orthogonal polynomials</i>	611
Mark Mandelker, <i>F'-spaces and z-embedded subspaces</i>	615
James R. McLaughlin and Justin Jesse Price, <i>Comparison of Haar series with gaps with trigonometric series</i>	623
Ernest A. Michael and A. H. Stone, <i>Quotients of the space of irrationals</i>	629
William H. Mills and Neal Zierler, <i>On a conjecture of Golomb</i>	635
J. N. Pandey, <i>An extension of Haimo's form of Hankel convolutions</i>	641
Terence John Reed, <i>On the boundary correspondence of quasiconformal mappings of domains bounded by quasicircles</i>	653
Haskell Paul Rosenthal, <i>A characterization of the linear sets satisfying Herz's criterion</i>	663
George Thomas Sallee, <i>The maximal set of constant width in a lattice</i>	669
I. H. Sheth, <i>On normaloid operators</i>	675
James D. Stasheff, <i>Torsion in BBSO</i>	677
Billy Joe Thorne, <i>$A - P$ congruences on Baer semigroups</i>	681
Robert Breckenridge Warfield, Jr., <i>Purity and algebraic compactness for modules</i>	699
Joseph Zaks, <i>On minimal complexes</i>	721