

# Pacific Journal of Mathematics

**A NOTE ON RECURSIVELY DEFINED ORTHOGONAL  
POLYNOMIALS**

DANIEL PAUL MAKI

## A NOTE ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS

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**Let  $\{a_i\}_{i=0}^\infty$  and  $\{b_i\}_{i=0}^\infty$  be real sequences and suppose the  $b_i$ 's are all positive. Define a sequence of polynomials  $\{P_i(x)\}_{i=0}^\infty$  as follows:  $P_0(x) = 1$ ,  $P_1(x) = (x - a_0)/b_0$ , and for  $n \geq 1$**

$$(*) \quad b_n P_{n+1}(x) = (x - a_n)P_n(x) - b_{n-1}P_{n-1}(x).$$

**Favard showed that the polynomials  $\{P_i(x)\}$  are orthonormal with respect to a bounded increasing function  $\psi$  defined on  $(-\infty, +\infty)$ . This note generalizes recent constructive results which deal with connections between the two sequences  $\{a_i\}$  and  $\{b_i\}$  and the spectrum of  $\psi$ . (The spectrum of  $\psi$  is the set  $S(\psi) = \{\lambda : \psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) > 0 \text{ for all } \varepsilon > 0\}$ .) It is shown that if  $b_i \rightarrow 0$  then every limit point of the sequence  $\{a_i\}$  is in  $S(\psi)$ .**

2. Preliminaries. In order to use theorems from functional analysis, consider the space  $\mathcal{L}^2(\psi) = \{f : \int_{-\infty}^{+\infty} f^2 d\psi < \infty\}$ . This is a Hilbert space where the inner product is given by  $(f, g) = \int f g d\psi$  and where we identify all functions which agree on  $S(\psi)$ . In [2], (p. 215), Carleman showed that the condition  $\sum 1/\sqrt{b_i} = \infty$  implies that when  $\psi$  is normalized to be continuous from the left and to have  $\psi(-\infty) = 0$ ,  $\psi(+\infty) = 1$ , then it is unique. In [6], M. Riesz showed that if  $\psi$  is essentially unique then Parseval's relation holds for the orthonormal set  $\{P_i\}$  in the space  $\mathcal{L}^2(\psi)$ . Hence the set  $\{P_i\}$  is dense in this space.

We now make the assumption that  $\lim b_i = 0$ . Combining the Carleman result and the Riesz result we see that  $\psi$  is essentially unique and the polynomials  $\{P_i\}$  are a dense set in  $\mathcal{L}^2(\psi)$ . Using this information we define an operator  $A$  on a dense subset of  $\mathcal{L}^2(\psi)$ . The domain of  $A$  is the set of all functions  $f$  which are in  $\mathcal{L}^2(\psi)$  and for which  $xf$  is also in  $\mathcal{L}^2(\psi)$ . We take  $A$  to be the self-adjoint operator defined by  $(Af)(x) = xf(x)$ . By inspection of (\*) we see that for  $i = 1, 2, 3, \dots$  we have

$$(**) \quad A(P_i) = b_{i-1}P_{i-1} + a_i P_i + b_i P_{i+1}.$$

We call  $A$  the operator associated with the sequences  $\{a_i\}$  and  $\{b_i\}$ .

3. Theorems. Let  $\sigma(A)$  be the spectrum of the operator  $A$ , i.e., all points  $\lambda$  where  $A - \lambda I$  does not have a bounded inverse. Then we have the following:

LEMMA.  $\sigma(A) \subset S(\psi)$ .

*Proof.* Let  $\lambda \in \sigma(A)$ . Since  $A$  is self-adjoint,  $\lambda$  is in the approximate point spectrum of  $A$ . Hence there exists a sequence  $\{f_n\}$  in the domain of  $A$  satisfying  $\|f_n\| = 1$ ,  $n = 1, 2, \dots$ , and  $\|(A - \lambda)f_n\| \rightarrow 0$  as  $n \rightarrow \infty$ . Now by the definition of the norm in  $\mathcal{L}^2(\psi)$  this means  $\int_{-\infty}^{+\infty} f_n^2 d\psi = 1$ ,  $n = 1, 2, \dots$ , and  $\int_{-\infty}^{+\infty} (x - \lambda)^2 f_n^2 d\psi \rightarrow 0$  as  $n \rightarrow \infty$ . Now suppose  $\lambda \notin S(\psi)$ . Then there exists  $\varepsilon > 0$  such that

$$\psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) = 0.$$

Thus  $\psi$  has no mass in the interval  $[\lambda - \varepsilon, \lambda + \varepsilon]$ , and we have

$$\int_{-\infty}^{\lambda - \varepsilon} f_n^2 d\psi + \int_{\lambda + \varepsilon}^{+\infty} f_n^2 d\psi = 1, \quad n = 1, 2, \dots,$$

and

$$\int_{+\infty}^{\lambda - \varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{\lambda + \varepsilon}^{+\infty} (x - \lambda)^2 f_n^2 d\psi \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But these are contradictory since

$$\begin{aligned} & \int_{-\infty}^{\lambda - \varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{\lambda + \varepsilon}^{+\infty} (x - \lambda)^2 f_n^2 d\psi \\ & \geq \varepsilon^2 \left[ \int_{-\infty}^{\lambda - \varepsilon} f_n^2 d\psi + \int_{\lambda + \varepsilon}^{+\infty} f_n^2 d\psi \right] = \varepsilon^2. \end{aligned}$$

This completes the proof.

We are now ready for our result about  $S(\psi)$ . It is motivated by the results in [5] where we constructed  $\psi$  in the case where  $b_i \rightarrow 0$  and  $\{a_i\}$  has only a finite number of limit points.

**THEOREM.** *Let the sequence of polynomials  $\{P_i\}_0^\infty$  be recursively defined by (\*) and assume  $b_i > 0$  for each  $i$  and  $b_i \rightarrow 0$ . Then each limit point of the sequence  $\{a_i\}$  is a point of the spectrum of the associated distribution function  $\psi$ .*

*Proof.* From the above lemma it suffices to show that each limit point of the sequence  $\{a_i\}$  is in  $\sigma(A)$ . Thus let  $\lambda$  be a limit point of  $\{a_i\}$  and suppose  $\{a_{i(n)}\}$  is a subsequence converging to  $\lambda$ . Next let  $f_n(x) = P_{i(n)}(x)$ ,  $n = 1, 2, 3, \dots$ . By the defining relation (\*) and by the definition of  $A$ , we have

$$\begin{aligned} \|(A - \lambda)f_n\|^2 &= \|(x - \lambda)P_{i(n)}\|^2 \\ &= \int_{-\infty}^{+\infty} (b_{i(n)-1}P_{i(n)-1} + (a_{i(n)} - \lambda)P_{i(n)} + b_{i(n)}P_{i(n)+1})^2 d\psi \\ &= b_{i(n)-1}^2 + (a_{i(n)} - \lambda)^2 + b_{i(n)}^2. \end{aligned}$$

Now  $b_i \rightarrow 0$  and  $a_{i(n)} \rightarrow \lambda$ , so we see  $\|(A - \lambda)f_n\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover  $\|f_n\| = \|P_{i(n)}\| = 1$ , so  $\lambda \in \sigma(A)$  and the proof is complete.

REMARK. If we choose the  $a_i$ 's to be dense in the real line, for example any enumeration of the rationals, then for every set of  $b_i$ 's satisfying  $b_i \rightarrow 0$  we have  $S(\psi) = (-\infty, +\infty)$ .

CONJECTURE. The converse of the above theorem does not hold since in [5] our construction exhibited points of  $S(\psi)$  which were not limit points of  $\{a_i\}$ . However each limit point of  $S(\psi)$  was a limit point of  $\{a_i\}$ . So it seems reasonable to conjecture that when  $b_i \rightarrow 0$ ,  $\lambda$  is a limit point of  $S(\psi)$  if and only if  $\lambda$  is a limit point of  $\{a_i\}$ .

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