A NOTE ON RECURSIVELY DEFINED ORTHOGONAL POLYNOMIALS

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Let \( \{a_i\}_{i=0}^{\infty} \) and \( \{b_i\}_{i=0}^{\infty} \) be real sequences and suppose the \( b_i \)s are all positive. Define a sequence of polynomials \( \{P_i(x)\}_{i=0}^{\infty} \) as follows:

\[
P_0(x) = 1, \quad P_i(x) = (x - a_0)/b_0, \quad \text{and for } n \geq 1
\]

\[
b_n P_{n+1}(x) = (x - a_n)P_n(x) - b_{n-1}P_{n-1}(x).
\]

Favard showed that the polynomials \( \{P_i(x)\} \) are orthonormal with respect to a bounded increasing function \( \psi \) defined on \( (-\infty, +\infty) \). This note generalizes recent constructive results which deal with connections between the two sequences \( \{a_i\} \) and \( \{b_i\} \) and the spectrum of \( \psi \). (The spectrum of \( \psi \) is the set \( \Sigma(\psi) = \{\lambda: \psi(\lambda + \epsilon) - \psi(\lambda - \epsilon) > 0 \text{ for all } \epsilon > 0\}. \) It is shown that if \( b_i \to 0 \) then every limit point of the sequence \( \{a_i\} \) is in \( \Sigma(\psi) \).

2. Preliminaries. In order to use theorems from functional analysis, consider the space \( L^2(\psi) = \{f: \int_{-\infty}^{\infty} f^2 d\psi < \infty\} \). This is a Hilbert space where the inner product is given by \( (f, g) = \int_{-\infty}^{\infty} fg d\psi \) and where we identify all functions which agree on \( \Sigma(\psi) \). In [2], (p. 215), Carleman showed that the condition \( \sum 1/\sqrt{b_i} = \infty \) implies that when \( \psi \) is normalized to be continuous from the left and to have \( \psi(-\infty) = 0, \psi(+\infty) = 1 \), then it is unique. In [6], M. Riesz showed that if \( \psi \) is essentially unique then Parseval’s relation holds for the orthonormal set \( \{P_i\} \) in the space \( L^2(\psi) \). Hence the set \( \{P_i\} \) is dense in this space.

We now make the assumption that \( \lim b_i = 0 \). Combining the Carleman result and the Riesz result we see that \( \psi \) is essentially unique and the polynomials \( \{P_i\} \) are a dense set in \( L^2(\psi) \). Using this information we define an operator \( A \) on a dense subset of \( L^2(\psi) \). The domain of \( A \) is the set of all functions \( f \) which are in \( L^2(\psi) \) and for which \( xf \) is also in \( L^2(\psi) \). We take \( A \) to be the self-adjoint operator defined by \( (Af)(x) = xf(x) \). By inspection of (*) we see that for \( i = 1, 2, 3, \cdots \) we have

\[
A(P_i) = b_{i-1}P_{i-1} + a_iP_i + b_{i+1}P_{i+1}.
\]

We call \( A \) the operator associated with the sequences \( \{a_i\} \) and \( \{b_i\} \).

3. Theorems. Let \( \sigma(A) \) be the spectrum of the operator \( A \), i.e., all points \( \lambda \) where \( A - \lambda I \) does not have a bounded inverse. Then we have the following:
LEMMA. \(\sigma(A) \subset S(\psi)\).

Proof. Let \(\lambda \in \sigma(A)\). Since \(A\) is self-adjoint, \(\lambda\) is in the approximate point spectrum of \(A\). Hence there exists a sequence \(\{f_n\}\) in the domain of \(A\) satisfying \(\|f_n\| = 1, n = 1, 2, \ldots\), and \(\|(A - \lambda)f_n\| \to 0\) as \(n \to \infty\). Now by the definition of the norm in \(L^2(\psi)\) this means \(\int_{-\infty}^{\infty} f_n^2 d\psi = 1, n = 1, 2, \ldots\), and \(\int_{x+\varepsilon}^{x-\varepsilon} (x - \lambda)^2 f_n^2 d\psi \to 0\) as \(n \to \infty\). Now suppose \(\lambda \in S(\psi)\). Then there exists \(\varepsilon > 0\) such that

\[
\psi(\lambda + \varepsilon) - \psi(\lambda - \varepsilon) = 0.
\]

Thus \(\psi\) has no mass in the interval \([\lambda - \varepsilon, \lambda + \varepsilon]\), and we have

\[
\int_{x-\varepsilon}^{x+\varepsilon} f_n^2 d\psi + \int_{x+\varepsilon}^{x+\varepsilon} f_n^2 d\psi = 1, \quad n = 1, 2, \ldots,
\]

and

\[
\int_{x-\varepsilon}^{x+\varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{x+\varepsilon}^{x+\varepsilon} (x - \lambda)^2 f_n^2 d\psi \to 0 \quad \text{as} \quad n \to \infty.
\]

But these are contradictory since

\[
\int_{x-\varepsilon}^{x+\varepsilon} (x - \lambda)^2 f_n^2 d\psi + \int_{x+\varepsilon}^{x+\varepsilon} (x - \lambda)^2 f_n^2 d\psi \geq \varepsilon^2 \left[ \int_{x-\varepsilon}^{x+\varepsilon} f_n^2 d\psi + \int_{x+\varepsilon}^{x+\varepsilon} f_n^2 d\psi \right] = \varepsilon^2.
\]

This completes the proof.

We are now ready for our result about \(S(\psi)\). It is motivated by the results in [5] where we constructed \(\psi\) in the case where \(b_i \to 0\) and \(\{a_i\}\) has only a finite number of limit points.

THEOREM. Let the sequence of polynomials \(\{P_i\}\) be recursively defined by (*) and assume \(b_i > 0\) for each \(i\) and \(b_i \to 0\). Then each limit point of the sequence \(\{a_i\}\) is a point of the spectrum of the associated distribution function \(\psi\).

Proof. From the above lemma it suffices to show that each limit point of the sequence \(\{a_i\}\) is in \(\sigma(A)\). Thus let \(\lambda\) be a limit point of \(\{a_i\}\) and suppose \(\{a_{i(n)}\}\) is a subsequence converging to \(\lambda\). Next let \(f_n(x) = P_{i(n)}(x), n = 1, 2, 3, \ldots\). By the defining relation (*) and by the definition of \(A\), we have

\[
\|(A - \lambda)f_n\|^2 = \|(x - \lambda)P_{i(n)}\|^2
\]

\[
= \int_{x-\varepsilon}^{x+\varepsilon} \left( (b_{i(n)} - 1)P_{i(n)-1} + (a_{i(n)} - \lambda)P_{i(n)} + b_{i(n)}P_{i(n)+1} \right)^2 d\psi
\]

\[
= b_{i(n)-1}^2 + (a_{i(n)} - \lambda)^2 + b_{i(n)}^2.
\]
Now $b_i \to 0$ and $a_{i(n)} \to \lambda$, so we see $\| (A - \lambda) f_n \|^2 \to 0$ as $n \to \infty$. Moreover $\| f_n \| = \| P_{i(n)} \| = 1$, so $\lambda \in \sigma(A)$ and the proof is complete.

**Remark.** If we choose the $a_i$'s to be dense in the real line, for example any enumeration of the rationals, then for every set of $b_i$'s satisfying $b_i \to 0$ we have $S(\psi) = (-\infty, +\infty)$.

**Conjecture.** The converse of the above theorem does not hold since in [5] our construction exhibited points of $S(\psi)$ which were not limit points of $\{a_i\}$. However each limit point of $S(\psi)$ was a limit point of $\{a_i\}$. So it seems reasonable to conjecture that when $b_i \to 0$, $\lambda$ is a limit point of $S(\psi)$ if and only if $\lambda$ is a limit point of $\{a_i\}$.

**References**


Received January 12, 1968.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsuusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.
Pacific Journal of Mathematics
Vol. 28, No. 3 May, 1969

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