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**A CHARACTERIZATION OF THE LINEAR SETS SATISFYING
HERZ'S CRITERION**

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Let E be a closed subset of T , the circle group, which we identify with the real numbers modulo 1. E is said to satisfy Herz's criterion (briefly, E satisfies (H)), if there exists an infinite set of positive integers N , such that

(*) for all integers j with $0 \leq j < N$, each of the numbers j/N either belongs to E or is distant by at least $1/N$ from E .

The main theorem proved here, is that E satisfies (H) if and only if there exists a sequence of sets F_1, F_2, \dots with $E = \bigcap_{i=1}^{\infty} F_i$ and positive integers $N_1 < N_2 < \dots$ satisfying the following properties for all i :

- (1) N_i divides N_{i+1} and $F_i \supset F_{i+1}$.
- (2) F_i is a finite union of disjoint closed intervals each of whose end points is of the form j/N_i for some integer j .
- (3) If for some integer j , $j/N_i \in F_i$, then $j/N_i \in F_{i+1}$.

The motivation for studying sets E satisfying (H) is the result of Herz (c.f. [1]) that all such sets satisfy spectral synthesis, and of course that the Cantor set is an example. (See also [2], Chapter IX).

Now suppose that $E = \bigcap_{i=1}^{\infty} F_i$, with F_i and N_i satisfying (1)–(3) for all i . It is then evident that E satisfies (H) , since the numbers N_i will satisfy (*) for all i . Moreover, E is obtained by a sort of dissection procedure. Indeed, F_{i+1} may be obtained from F_i by removing from certain of the closed intervals $[j/N_i, (j+1)/N_i]$ included in F_i , one or more open intervals of the form

$$\left(\frac{l}{N_{i+1}}, \frac{q}{N_{i+1}} \right)$$

where $j/N_i \leq l/N_{i+1} < q/N_{i+1} \leq (j+1)/N_i$.

The "only if" part of our main result is demonstrated following the proof of Theorem 4 below. The latter result is somewhat stronger than our main theorem, and enables us to show that certain sets fail to satisfy (H) (in particular, the symmetric sets of ratio ξ , where ξ is a rational number with $1/\xi$ unequal to an integer. (C.f. [2], pp. 13–15 for the definition of these sets).

§1. *Preliminaries.* We identify the points of T with $[0, 1)$, where addition and subtraction are taken modulo 1. If x and y belong to T , then the distance between them, $\rho(x, y)$, is defined to be the distance from $x - y$ to the closest integer on the real line. If E

is a subset of T , then $\rho(x, E)$ is defined as $\inf_{f \in E} \rho(x, f)$.

Throughout this paper, E shall refer to a closed proper nonempty subset of T and \mathcal{N} shall denote the set of all positive integers N satisfying (*). (Thus if E satisfies (H), \mathcal{N} is an infinite set (and conversely)). Every variable " N ", with or without sub or superscripts, refers to a member of \mathcal{N} , and every variable " j " refers to an integer.

If L and M are positive integers, we write $L \mid M$ if there is an integer q with $Lq = M$.

Given a set S , " $\sim S$ " denotes its complement.

Let $[x]$ be the greatest integer less than or equal to x . We remind the reader that if U is a proper connected open subset of T , there will exist unique real numbers $a < b \leq a + 1$, such that $0 \leq b < 1$, and such that $U = \{x - [x]: a < x < b\}$. We then define the length of U to be $b - a$, with the left and right end points of U being $a - [a]$ and b respectively.

DEFINITION. Let x be a member of E for which there exists a j with $0 \leq j < N$, such that $x = j/N$.

x is called N -initial if $(j - 1)/N \notin E$.

x is called N -terminal if $(j + 1)/N \notin E$.

x is called an N -end if x is N -initial or N -terminal.

We note that if x is N -initial (N -terminal) then x is a right (left) end point of a component of $\sim E$ of length at least $2/N$. Indeed, if x is N -initial, we may close a j so that $x - (1/N) = j/N$, and $j/N \in E$. Hence the open interval $((j/N) - (1/N), (j/N) + (1/N))$ cannot contain any points of E , and of course $x = (j + 1)/N$ belongs to E .

2. Our first result shows that if E satisfies (H), then the boundary points of components of $\sim E$ must be rational numbers.

LEMMA 1. Let U be a component of $\sim E$, of length l . Then if $N > 1/l$, the end points of U are N -ends.

Proof. Let x be the left end point of U . Then $x \in E$. Suppose it were false that $x = j/N$ for some j . There would then exist a $0 \leq j < N$ such that $x \in (j/N, (j + 1)/N)$. Since $(1/N) < l$, we would have that $((j + 1)/N \in U)$, so $(j + 1)/N \notin E$. But

$$\rho\left(\frac{j + 1}{N}, E\right) \leq \rho\left(\frac{j + 1}{N}, x\right) < \frac{1}{N},$$

a contradiction. Thus, there exists a j , $0 \leq j < N$, with $x = j/N$. But then $(j + 1)/N \notin E$, since the length of $(j/N, (j + 1)/N)$ is $1/N < l$, hence $(j + 1)/N \in U$. Thus, x is N -terminal. The proof that the

right end point of U is N -initial is similar.

Our next task is to define certain sets that are finite unions of disjoint closed intervals, that approximate E . First, we note that if x is N -initial, then x is associated with a unique N -terminal number (possibly equal to x), as follows: let k be the smallest integer l , with $0 \leq l < N$, such that $x + (l + 1)/N \notin E$. (Note that $l = N - 2$ is such an integer.) Then $x + k/N$ is the uniquely determined N -terminal number.

We define $I_x = [x, x + (k/N)]$ and $E_N = \bigcup \{I_x: x \text{ is } N\text{-initial}\}$. If there do not exist any N -ends, set $E_N = T$. Let l_1 be the maximum of the lengths of components of $\sim E$.

Then if $N > 1/l_1$, there will exist N -ends by Lemma 1 and hence E_N will be a proper subset of T . Of course, $I_x \cap I_{x'} = \phi$ for x and x' different N -ends; so E_N is a disjoint union of intervals with end points all of the form j/N .

LEMMA 2. For all N and N' , $N' < N$ implies $E_N \subset E_{N'}$.

Proof. Let $N' < N$ be fixed, and let x be a fixed N -initial number. It follows directly from the definitions that $E \subset E_{N'}$; thus since $x \in E$, there is a (unique) N' -end y , such that $x \in I'_y$, where $I'_y = [y, z]$, with z the unique N' -terminal number associated with y .

Now choose an integer l with $0 \leq l < N$ such that

$$z \in \left[\frac{l}{N}, \frac{l + 1}{N} \right).$$

Then $(l + 1)/N \in E$, since $(l + 1)/N \in (z, z + 1/N)$. Thus we must have that $z = l/N$, or else $\rho(l/N, E) \leq \rho(l/N, z) < 1/N$. Hence z is N -terminal, and so it follows from the definition of I_x that $I_x \subset I'_y$.

Thus $E_N \subset \bigcup \{I'_y: y \text{ is } N'\text{-initial}\} = E_{N'}$.

Our last lemma enables us to obtain certain canonical members of N crucial for the proof of Theorem 4 (whose proof also shows that the number N/d below equals q_i , where $l_{i+1} \leq \frac{1}{N} < l_i$ and q_i, l_i are defined directly preceding the statement of Theorem 4).

LEMMA 3. Let $S_N = \{0 \leq j < N: j/N \text{ is an } N\text{-end}\}$.

Let d be a positive integer such that $d \mid N$ and $d \mid j$ for all $j \in S_N$. Then $(N/d) \in \mathcal{N}$.

Proof. We may and shall assume that $d > 1$. Put $M = N/d$, and let l be an integer with $0 \leq l < M$, such that $l/M \notin E$. It remains

for us to show that $\rho(l/M, E) \geq 1/M$. If this is not the case, then either $\{(l-1)/M, l/M\}$ or $\{l/M, (l+1)/M\}$ contains a point of E . Suppose the first possibility; then

$$\left(\frac{l-1}{M}, \frac{l}{M}\right) = \left(\frac{d(l-1)}{N}, \frac{dl}{N}\right)$$

contains an N -end.

Indeed there is, in the first place, an integer r , $d(l-1) < r < dl$, such that $r/N \in E$. For if

$$x \in \left(\frac{d(l-1)}{N}, \frac{dl}{N}\right)$$

belongs to E , we can certainly find such an r with $\rho(x, r/N) < 1/N$. Then $r/N \in E$ since $N \in \mathcal{N}$ is always assumed. Now let k be the least integer greater than or equal to r such that $(k+1)/N \notin E$. Evidently $k \leq dl-1$ since $l/M = dl/N \notin E$, and k/N is an N -end.

Hence there is a $j \in S_N$ such that $k/N = j/N \pmod{1}$. Since $d|N$ and $d|j$, it follows that $d|k$. But $d(l-1) < k < dl$, hence

$$l-1 < \frac{k}{d} < l,$$

a contradiction.

The argument for the case when $\{(l/M), (l+1)/M\}$ contains a point of E , is practically identical to this.

The next result implies our main theorem, and is useful in determining if a given set fails (H). We shall need the following assumptions and notation:

Assume that $\sim E$ has infinitely many components, all with rational end points.

Let l_1, l_2, \dots be an enumeration of their lengths, with $l_i > l_{i+1} > 0$ for all i . Evidently $\sum_{i=1}^{\infty} l_i \leq 1$, so $l_i \rightarrow 0$ as $i \rightarrow \infty$.

Let U_i be the union of all the components of $\sim E$ of lengths greater than or equal to l_i , K_i the set of end points of these components, and q_i the least common multiple of the denominators of the members of K_i , expressed in the lowest form.

THEOREM 4. *If E satisfies (H), then for infinitely many integers i , the following three conditions must hold simultaneously:*

- (a) $l_{i+1} \leq \frac{1}{q_i}$.
- (b) $2l_{i+1} < l_i$.
- (c) For each integer j with $0 \leq j < q_i$, if $j/q_i \in E$, then $j/q_i \in U_i$.

REMARK. If E is a set for which condition (c) holds for infinitely many i , then E satisfies (H). Indeed, the boundary points of U_i are all of the form j/q_i ; thus if i satisfies (c), $N = q_i$ satisfies (*). Moreover, $\{q_i : i \text{ satisfies (c)}\}$ will then be an infinite set. Indeed, $(1/q_i) \leq l_i$ for all i . Thus fixing i , if we choose $k > i$ such that $l_k < (1/q_i)$, we have that $(1/q_k) < (1/q_i)$, so there are at most finitely many j 's such that $q_j = q_i$.

Proof of Theorem 4. Assume that E satisfies (H), and fix $N \in \mathcal{N}$ with $N > 1/l_1$.

Then there is a unique i such that $l_{i+1} \leq (1/N) < l_i$. By Lemma 1, each member of K_i is an N -end. Letting E_N be as defined before the proof of Lemma 2, we thus have $U_i \subset \sim E_N$. Moreover, every component of $\sim E_N$ is a component of $\sim E$, of length greater than or equal to $2/N$, by the definition of E_N . Thus, every component of $\sim E_N$ is of length greater than l_{i+1} , whence $\sim E_N \subset U_i$, and every N -end is a member of K_i , since it is an end point of a component of $\sim E$ of length greater than or equal to l_i . Thus $E_N = \sim U_i$ and the set of N -ends equals K_i . So every element in K_i is of the form j/N , whence $q_i | N$, so $q_i \leq N$, and thus (a) follows. Since $2/N$ is less than or equal to the lengths of all the components of $\sim E_N = U_i$, it follows that $2/N \leq l_i$, whence (b) holds. Finally, it follows from the definition of q_i , that if d is the greatest common divisor of $S_N \cup \{N\}$, then $q_i = N/d$ (where S_N is defined in Lemma 3). Thus by Lemma 3, $q_i \in \mathcal{N}$, whence since $q_i \leq N$, $E_{q_i} \supset E_N$ by Lemma 2. So suppose that $j/q_i \in E$. Then

$$\frac{j}{q_i} \notin E_{q_i}$$

by the latter's definition, so $j/q_i \in E_N$, whence $j/q_i \in U_i$, so (c) holds.

Finally since \mathcal{N} is infinite, there must be infinitely many i 's for which there exists an $N \in \mathcal{N}$ with $l_{i+1} \leq 1/N < l_i$, and consequently for which (a), (b), and (c) all hold.

Proof of the main theorem. Let E satisfy (H), and assume first that $\sim E$ has infinitely many components. Then by Lemma 1, the end points of these components are all rational numbers, so Theorem 4 is applicable; thus condition (c) of that result holds for infinitely many integers i . Now fixing i for which (c) holds, if $N > q_i$, then $q_i | N$; indeed, since $q_i \geq 1/l_i$, we obtain by Lemma 1 that every element of K_i is an N -end, and thus expressible in the form j/N . Moreover, since the boundary points of U_i are all of the form j/q_i , we obtain that $q_i \in \mathcal{N}$.

Thus simply let j_1, j_2, \dots be an enumeration of a subset of the

i 's satisfying (c), such that $q_{i_r} < q_{i_{r'}}$ for all $r < r'$. Then if we put $F_i = \sim U_{j_i}$ and $N_i = q_{j_i}$ for all i , $E = \bigcap_{i=1}^{\infty} F_i$ and (1)–(3) are satisfied for all i . We have also established that *when E satisfies (H) and its complement, has infinitely many components then there exist $N_1 < N_2 < \dots$ such that for all i and N , if $N \geq N_i$ then $N_i | N$.*

Now if E satisfies (H) and $\sim E$ has only finitely many components, then by Lemma 1, the boundary points of E are all rational numbers. Let M be the least common multiple of the denominators of these numbers expressed in the lowest form; then setting $N_i = 2^{i-1}M$ and $F_i = E$ for all i , it is easily verified that (1)–(3) hold. We remark finally that if $\sim E$ has finitely many components with rational boundary points, then E satisfies (H), and in fact letting M be as above, then for all $L \geq M$, $L \in \mathcal{N}$ if and only if $M | L$. (Thus the statement ending the preceding paragraph fails for E 's such that $\sim E$ has finitely many components.)

We wish finally to give some examples of sets which fail to satisfy (H). If ξ is a real number with $0 < \xi < 1/2$, S_ξ , the symmetric set of ratio ξ , consists of all numbers x in T such that

$$x = (1 - \xi) \sum_{j=0}^{\infty} \varepsilon_j \xi^j$$

where $\varepsilon_j = 0$ or 1 , all j . (See pages 13–15 of [2].)

Now ξ is an end point of a component of $\sim S_\xi$, namely $(\xi, 1 - \xi)$.

Hence if ξ is irrational, then S_ξ fails (H) by Lemma 1. If $\xi = 1/L$ for some integer L , then it is well known that S_ξ satisfies (H). We shall show that if $\xi = p/q$, where p and q are relatively prime integers with $p > 1$, then S_ξ fails (H).

Defining l_i and q_i for $E = S_\xi$, we have that $l_i = (1 - 2\xi)\xi^{i-1}$ and $q_i = q^i$ for $i = 1, 2, \dots$. (It follows from page 14 of [2] that all the end points of components of U_i are of the form l/q^i for some integer l ; but p^i/q^i is such an end point, and p^i and q^i are relatively prime.) Now if $l_{i+1} \leq 1/q_i$, then $(1 - 2(p/q))(p/q)^i \leq 1/q_i$, or $p^i \leq q/(q - 2p)$; thus condition (a) of Theorem 4 will be violated for all i sufficiently large.

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