A CHARACTERIZATION OF THE LINEAR SETS SATISFYING HERZ’S CRITERION

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Let $E$ be a closed subset of $T$, the circle group, which we identify with the real numbers modulo 1. $E$ is said to satisfy Herz’s criterion (briefly, $E$ satisfies (H)), if there exists an infinite set of positive integers $N$, such that

\[(*) \quad \text{for all integers } j \text{ with } 0 \leq j < N, \text{ each of the numbers } j/N \text{ either belongs to } E \text{ or is distant by at least } 1/N \text{ from } E.\]

The main theorem proved here, is that $E$ satisfies (H) if and only if there exists a sequence of sets $F_1, F_2, \cdots$ with $E = \bigcap_{i=1}^{\infty} F_i$ and positive integers $N_1 < N_2 < \cdots$ satisfying the following properties for all $i$:

1. $N_i$ divides $N_{i+1}$ and $F_i \supset F_{i+1}$.
2. $F_i$ is a finite union of disjoint closed intervals each of whose end points is of the form $j/N_i$ for some integer $j$.
3. If for some integer $j$, $j/N_i \in F_i$, then $j/N_i \in F_{i+1}$.

The motivation for studying sets $E$ satisfying (H) is the result of Herz (c.f. [1]) that all such sets satisfy spectral synthesis, and of course that the Cantor set is an example. (See also [2], Chapter IX).

Now suppose that $E = \bigcap_{i=1}^{\infty} F_i$, with $F_i$ and $N_i$ satisfying (1)-(3) for all $i$. It is then evident that $E$ satisfies (H), since the numbers $N_i$ will satisfy (*) for all $i$. Moreover, $E$ is obtained by a sort of dissection procedure. Indeed, $F_{i+1}$ may be obtained from $F_i$ by removing from certain of the closed intervals $[j/N_i, (j+1)/N_i]$ included in $F_i$, one or more open intervals of the form

\[\left(\frac{l}{N_{i+1}}, \frac{q}{N_{i+1}}\right)\]

where $j/N_i \leq l/N_{i+1} < q/N_{i+1} \leq (j+1)/N_i$.

The “only if” part of our main result is demonstrated following the proof of Theorem 4 below. The latter result is somewhat stronger than our main theorem, and enables us to show that certain sets fail to satisfy (H) (in particular, the symmetric sets of ratio $\xi$, where $\xi$ is a rational number with $1/\xi$ unequal to an integer. (C.f. [2], pp. 13-15 for the definition of these sets).

§ 1. Preliminaries. We identify the points of $T$ with $[0,1)$, where addition and subtraction are taken modulo 1. If $x$ and $y$ belong to $T$, then the distance between them, $\rho(x, y)$, is defined to be the distance from $x-y$ to the closest integer on the real line. If $E$
is a subset of $T$, then $\rho(x, E)$ is defined as $\inf_{f \in E} \rho(x, f)$.

Throughout this paper, $E$ shall refer to a closed proper nonempty subset of $T$ and $\mathcal{N}$ shall denote the set of all positive integers $N$ satisfying $(\ast)$. (Thus if $E$ satisfies $(H)$, $\mathcal{N}$ is an infinite set (and conversely)). Every variable "$N"$, with or without sub or superscripts, refers to a member of $\mathcal{N}$, and every variable "$j"$ refers to an integer.

If $L$ and $M$ are positive integers, we write $L \mid M$ if there is an integer $q$ with $Lq = M$.

Given a set $S$, "$\sim S"$ denotes its complement.

Let $[x]$ be the greatest integer less than or equal to $x$. We remind the reader that if $U$ is a proper connected open subset of $T$, there will exist unique real numbers $a < b \leq a + 1$, such that $0 \leq b < 1$, and such that $U = \{x - [x]: a < x < b\}$. We then define the length of $U$ to be $b - a$, with the left and right end points of $U$ being $a - [a]$ and $b$ respectively.

**DEFINITION.** Let $x$ be a member of $E$ for which there exists a $j$ with $0 \leq j < N$, such that $x = j/N$.

$x$ is called $N$-initial if $(j - 1)/N \notin E$.

$x$ is called $N$-terminal if $(j + 1)/N \notin E$.

$x$ is called an $N$-end if $x$ is $N$-initial or $N$-terminal.

We note that if $x$ is $N$-initial ($N$-terminal) then $x$ is a right (left) end point of a component of $\sim E$ of length at least $2/N$. Indeed, if $x$ is $N$-initial, we may close a $j$ so that $x = (1/N) = j/N$, and $j/N \notin E$. Hence the open interval $((j/N) - (1/N), (j/N) + (1/N))$ cannot contain any points of $E$, and of course $x = (j + 1)/N$ belongs to $E$.

2. Our first result shows that if $E$ satisfies $(H)$, then the boundary points of components of $\sim E$ must be rational numbers.

**LEMMA 1.** Let $U$ be a component of $\sim E$, of length $l$. Then if $N > 1/l$, the end points of $U$ are $N$-ends.

**Proof.** Let $x$ be the left end point of $U$. Then $x \in E$. Suppose it were false that $x = j/N$ for some $j$. There would then exist a $0 \leq j < N$ such that $x \in (j/N, (j + 1)/N)$. Since $(1/N) < l$, we would have that $((i + 1)/N \in U)$, so $(j + 1)/N \notin E$. But

$$\rho\left(\frac{j + 1}{N}, E\right) \leq \rho\left(\frac{j + 1}{N}, x\right) < \frac{1}{N},$$

a contradiction. Thus, there exists a $j$, $0 \leq j < N$, with $x = j/N$. But then $(j + 1)/N \in E$, since the length of $(j/N, (j + 1)/N)$ is $1/N < l$, hence $(j + 1)/N \in U$. Thus, $x$ is $N$-terminal. The proof that the
right end point of $U$ is $N$-initial is similar.

Our next task is to define certain sets that are finite unions of disjoint closed intervals, that approximate $E$. First, we note that if $x$ is $N$-initial, then $x$ is associated with a unique $N$-terminal number (possibly equal to $x$), as follows: let $k$ be the smallest integer $l$, with $0 \leq l < N$, such that $x + (l + 1)/N \in E$. (Note that $l = N - 2$ is such an integer.) Then $x + k/N$ is the uniquely determined $N$-terminal number.

We define $I_x = [x, x + (k/N)]$ and $E_N = \bigcup \{I_x: x \text{ is } N\text{-initial}\}$. If there do not exist any $N$-ends, set $E_N = T$. Let $l_1$ be the maximum of the lengths of components of $\sim E$.

Then if $N > 1/l_1$, there will exist $N$-ends by Lemma 1 and hence $E_N$ will be a proper subset of $T$. Of course, $I_x \cap I_{x'} = \emptyset$ for $x$ and $x'$ different $N$-ends; so $E_N$ is a disjoint union of intervals with end points all of the form $j/N$.

**Lemma 2.** For all $N$ and $N'$, $N' < N$ implies $E_N \subset E_{N'}$.

*Proof.* Let $N' < N$ be fixed, and let $x$ be a fixed $N$-initial number. It follows directly from the definitions that $E \subset E_{N'}$; thus since $x \in E$, there is a (unique) $N'$-end $y$, such that $x \in I_y$, where $I_y = [y, z]$, with $z$ the unique $N'$-terminal number associated with $y$.

Now choose an integer $l$ with $0 \leq l < N$ such that

$$z \in \left[\frac{l}{N}, \frac{l + 1}{N}\right].$$

Then $(l + 1)/N \notin E$, since $(l + 1)/N \in (z, z + 1/N)$. Thus we must have that $z = l/N$, or else $\rho(l/N, E) \leq \rho(l/N, z) < 1/N$. Hence $z$ is $N$-terminal, and so it follows from the definition of $I_x$ that $I_x \subset I_y$.

Thus $E_N \subset \bigcup \{I_y: y \text{ is } N'-\text{initial}\} = E_{N'}$.

Our last lemma enables us to obtain certain canonical members of $N$ crucial for the proof of Theorem 4 (whose proof also shows that the number $N/d$ below equals $q_i$, where $l_{i+1} \leq \frac{1}{N} < l_i$ and $q_i, l_i$ are defined directly preceding the statement of Theorem 4).

**Lemma 3.** Let $S_N = \{0 \leq j < N: j/N \text{ is an } N\text{-end}\}$.

Let $d$ be a positive integer such that $d \mid N$ and $d \mid j$ for all $j \in S_N$. Then $(N/d) \in \mathcal{N}$.  

*Proof.* We may and shall assume that $d > 1$. Put $M = N/d$, and let $l$ be an integer with $0 \leq l < M$, such that $l/M \notin E$. It remains
for us to show that \( \rho(l/M, E) \geq 1/M \). If this is not the case, then either \(((l - 1)/M, l/M)\) or \(\{l/M, (l + 1)/M\}\) contains a point of \(E\). Suppose the first possibility; then
\[
\left( \frac{l - 1}{M}, \frac{l}{M} \right) = \left( \frac{d(l - 1)}{N}, \frac{dl}{N} \right)
\]
contains an \(N\)-end.

Indeed there is, in the first place, an integer \(r\), \(d(l - 1) < r < dl\), such that \(r/N \in E\). For if
\[
x \in \left( \frac{d(l - 1)}{N}, \frac{dl}{N} \right)
\]
begins to \(E\), we can certainly find such an \(r\) with \(\rho(x, r/N) < 1/N\). Then \(r/N \in E\) since \(N \in \mathcal{N}\) is always assumed. Now let \(k\) be the least integer greater than or equal to \(r\) such that \((k + 1)/N \notin E\). Evidently \(k \leq dl - 1\) since \(l/M = dl/N \in E\), and \(k/N\) is an \(N\)-end.

Hence there is a \(j \in S_N\) such that \(k/N = j/N \pmod{1}\). Since \(d \mid N\) and \(d \mid j\), it follows that \(d \mid k\). But \(d(l - 1) < k < dl\), hence
\[
l - 1 < \frac{k}{d} < l,
\]
a contradiction.

The argument for the case when \(((l/M), (l + 1)/M)\) contains a point of \(E\), is practically identical to this.

The next result implies our main theorem, and is useful in determining if a given set fails \((H)\). We shall need the following assumptions and notation:

Assume that \(\sim E\) has infinitely many components, all with rational end points.

Let \(l_1, l_2, \ldots\) be an enumeration of their lengths, with \(l_i > l_{i+1} > 0\) for all \(i\). Evidently \(\sum_{i=1}^{\infty} l_i \leq 1\), so \(l_i \to 0\) as \(i \to \infty\).

Let \(U_i\) be the union of all the components of \(\sim E\) of lengths greater than or equal to \(l_i\), \(K_i\) the set of end points of these components, and \(q_i\) the least common multiple of the denominators of the members of \(K_i\), expressed in the lowest form.

**Theorem 4.** If \(E\) satisfies \((H)\), then for infinitely many integers \(i\), the following three conditions must hold simultaneously:

(a) \(l_{i+1} \leq \frac{1}{q_i}\).
(b) \(2l_{i+1} < l_i\).
(c) For each integer \(j\) with \(0 \leq j < q_i\), if \(j/q_i \in E\), then \(j/q_i \in U_i\).
REMARK. If $E$ is a set for which condition (c) holds for infinitely many $i$, then $E$ satisfies (H). Indeed, the boundary points of $U_i$ are all of the form $j/q_i$; thus if $i$ satisfies (c), $N = q_i$ satisfies ($\ast$). Moreover, $\{q_i : i \text{satisfies (c)}\}$ will then be an infinite set. Indeed, $(1/q_i) \leq l_i$ for all $i$. Thus fixing $i$, if we choose $k > i$ such that $l_k < (1/q_i)$, we have that $(1/q_k) < (1/q_i)$, so there are at most finitely many $j$'s such that $q_j = q_i$.

Proof of Theorem 4. Assume that $E$ satisfies (H), and fix $N \in \mathcal{N}$ with $N > 1/l_i$.

Then there is a unique $i$ such that $l_{i+1} \leq (1/N) < l_i$. By Lemma 1, each member of $K_i$ is an $N$-end. Letting $E_N$ be as defined before the proof of Lemma 2, we thus have $U_i \subset \sim E_N$. Moreover, every component of $\sim E_N$ is a component of $\sim E$, of length greater than or equal to $2/N$, by the definition of $E_N$. Thus, every component of $\sim E_N$ is of length greater than $l_{i+1}$, whence $\sim E_N \subset U_i$, and every $N$-end is a member of $K_i$, since it is an end point of a component of $\sim E$ of length greater than or equal to $l_i$. Thus $E_N = \sim U_i$ and the set of $N$-ends equals $K_i$. So every element in $K_i$ is of the form $j/N$, whence $q_i | N$, so $q_i \leq N$, and thus (a) follows. Since $2/N$ is less than or equal to the lengths of all the components of $\sim E_N = U_i$, it follows that $2/N \leq l_i$, whence (b) holds. Finally, it follows from the definition of $q_i$, that if $d$ is the greatest common divisor of $S_N \cup \{N\}$, then $q_i = N/d$ (where $S_N$ is defined in Lemma 3). Thus by Lemma 3, $q_i \in \mathcal{N}$, whence since $q_i \leq N$, $E_{q_i} \supset E_N$ by Lemma 2. So suppose that $j/q_i \in E$. Then

$$\frac{j}{q_i} \in E_{q_i},$$

by the latter's definition, so $j/q_i \notin E_N$, whence $j/q_i \in U_i$, so (c) holds.

Finally since $\mathcal{N}$ is infinite, there must be infinitely many $i$'s for which there exists an $N \in \mathcal{N}$ with $l_{i+1} \leq 1/N < l_i$, and consequently for which (a), (b), and (c) all hold.

Proof of the main theorem. Let $E$ satisfy (H), and assume first that $\sim E$ has infinitely many components. Then by Lemma 1, the end points of these components are all rational numbers, so Theorem 4 is applicable; thus condition (c) of that result holds for infinitely many integers $i$. Now fixing $i$ for which (c) holds, if $N > q_i$, then $q_i | N$; indeed, since $q_i \geq 1/l_i$, we obtain by Lemma 1 that every element of $K_i$ is an $N$-end, and thus expressible in the form $j/N$. Moreover, since the boundary points of $U_i$ are all of the form $j/q_i$, we obtain that $q_i \in \mathcal{N}$.

Thus simply let $j_1, j_2, \cdots$ be an enumeration of a subset of the
i's satisfying (c), such that \( q_{ir} < q_{ir'} \) for all \( r < r' \). Then if we put \( F_i = \sim U_i \) and \( N_i = q_{ji} \) for all \( i \), \( E = \bigcap_{i=1}^\infty F_i \) and (1)-(3) are satisfied for all \( i \). We have also established that when \( E \) satisfies (H) and its complement, has infinitely many components then there exist \( N_i < N_2 < \cdots \) such that for all \( i \) and \( N \), if \( N \geq N_i \) then \( N_i \mid N \).

Now if \( E \) satisfies (H) and \( \sim E \) has only finitely many components, then by Lemma 1, the boundary points of \( E \) are all rational numbers. Let \( M \) be the least common multiple of the denominators of these numbers expressed in the lowest form; then setting \( N_i = 2^{i-1}M \) and \( F_i = E \) for all \( i \), it is easily verified that (1)-(3) hold. We remark finally that if \( \sim E \) has finitely many components with rational boundary points, then \( E \) satisfies (H), and in fact letting \( M \) be as above, then for all \( L \geq M \), \( L \in \mathcal{N} \) if and only if \( M \mid L \). (Thus the statement ending the preceding paragraph fails for \( E \)'s such that \( \sim E \) has finitely many components.)

We wish to give some examples of sets which fail to satisfy (H). If \( \xi \) is a real number with \( 0 < \xi < 1/2 \), \( S_\xi \), the symmetric set of ratio \( \xi \), consists of all numbers \( x \) in \( T \) such that

\[
x = (1 - \xi) \sum_{j=0}^{\infty} \varepsilon_j \xi^j
\]

where \( \varepsilon_j = 0 \) or 1, all \( j \). (See pages 13-15 of [2].)

Now \( \xi \) is an end point of a component of \( \sim S_\xi \), namely \( (\xi, 1 - \xi) \).

Hence if \( \xi \) is irrational, then \( S_\xi \) fails (H) by Lemma 1. If \( \xi = 1/L \) for some integer \( L \), then it is well known that \( S_\xi \) satisfies (H). We shall show that if \( \xi = p/q \), where \( p \) and \( q \) are relatively prime integers with \( p > 1 \), then \( S_\xi \) fails (H).

Defining \( l_i \) and \( q_i \) for \( E = S_\xi \), we have that \( l_i = (1 - 2\xi)^i \) and \( q_i = q^i \) for \( i = 1, 2, \ldots \). (It follows from page 14 of [2] that all the end points of components of \( U_i \) are of the form \( l/q^i \) for some integer \( l \); but \( p^i/q^i \) is such an end point, and \( p^i \) and \( q^i \) are relatively prime.) Now if \( l_{i+1} \leq 1/q_i \), then \( (1 - 2(p/q))(p/q)^i \leq 1/q_i \), or \( p^i \leq q/(q - 2p) \); thus condition (a) of Theorem 4 will be violated for all \( i \) sufficiently large.

**References**


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