

# Pacific Journal of Mathematics

**TORSION IN BBSO**

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**The cohomology of BBSO, the classifying space for the stable Grassmanian BSO, is shown to have torsion of order precisely  $2^r$  for each natural number  $r$ . Moreover, the elements of order  $2^r$  appear in a pattern of striking simplicity.**

Many of the stable Lie groups and homogeneous spaces have torsion at most of order 2 [1, 3, 5]. There is one such space, however, with interesting torsion of higher order. This is  $BBSO = SU/\text{Spin}$  which is of interest in connection with Bott periodicity and in connection with the J-homomorphism [4, 7]. By the notation  $SU/\text{Spin}$  we mean that  $BBSO$  can be regarded as the fibre of  $B \text{ Spin} \rightarrow BSU$  or that, up to homotopy, there is a fibration

$$SU \rightarrow BBSO \rightarrow B \text{ Spin}$$

induced from the universal  $SU$  bundle by  $B \text{ Spin} \rightarrow BSU$ . The mod 2 cohomology  $H^*(BBSO; Z_2)$  has been computed by Clough [4]. The purpose of this paper is to compute enough of  $H^*(BBSO; Z)$  to obtain the mod 2 Bockstein spectral sequence [2] of  $BBSO$ .

Given a ring  $R$ , we shall denote by  $R[x_i \mid i \in I]$  the polynomial ring on generators  $x_i$  indexed by elements of a set  $I$ . The set  $I$  will often be described by an equation or inequality in which case  $i$  is to be understood to be a natural number. Similarly  $E(x_i \mid i \in I)$  will denote the exterior algebra on generators  $x_i$ . In this case, we will need only  $R = Z_2$ .

Let us recall the results on  $B \text{ Spin}$  as given by Thomas [6] and on  $BBSO$  as given by Clough [4].

$$H^*(B \text{ Spin}; Z_2) \approx Z_2[w_i \mid i \neq 2^j + 1]$$

where  $w_i$  is (the image of) the Stiefel-Whitney class  $w_i$ .

$$H^*(B \text{ Spin}; Z) \approx Z[Q_i \mid i > 0] \oplus \hat{T}$$

where  $2\hat{T} = 0$  and  $Q_i \in H^{4i}$ .

$$H^*(BBSO; Z_2) \approx E(e_i \mid i \geq 3)$$

where  $e_i \in H^i$  and is the image of  $w_i$  if  $i \neq 2^j + 1$  while  $e_{2^j+1}$  maps to an indecomposable element in  $H^*(SU; Z_2)$ .

Now let  ${}_{\beta}E_r$  denote the mod 2 Bockstein spectral sequence of  $BBSO$  [2]. In particular,  ${}_{\beta}E_2 = \text{Ker } Sq^1/\text{Im } Sq^1$ . Now  $Sq^1 w_{2i} = w_{2i+1}$  in  $BSO$  and  $Sq^1 w_{2i+1} = 0$  while  $Sq^1 e_{2^j} = 0$  in  $B \text{ Spin}$ . We will see that

$e_{2^j+1}$  can be chosen to have  $Sq^1 e_{2^j+1} = 0$  except for  $Sq^1 e_3 = e_4$ . Thus

$${}_{\beta}E_2 = E(e_3 e_4, e_{2^2+i}, v_{4i+1} \mid i > 0)$$

where  $v_{4i+1} = e_{2i} e_{2i+1}$  except  $v_{2^j+1} = e_{2^j+1}; j > 1$ .

**THEOREM 1.**

$${}_{\beta}E_r \approx E(e_3 e_4 \cdots e_{2^r}, e_{2^r+i}, v_{4i+1} \mid i > 0)$$

and  $d_r(e_3 \cdots e_{2^r}) = e_{2^r+1}$  modulo decomposable elements.

To prove Theorem 1, we will exhibit torsion of order  $2^r$  for all  $r$ .

**THEOREM 2.** *In  $H^*(BBSO; Z)$ , we have*

$$2^r Q_{2^r} \neq 0 \text{ and } 2^{r+1} Q_{2^r} = 0.$$

$H^*(BBSO; Z_2)$ . We recall some of Clough's observations on  $H^*(BBSO; Z_2)$ . We know  $H^*(SU; Z_2) = E(y_i \mid i > 1)$  where  $y_i \in H^{2i+1}$  transgresses universally to the mod 2 reduction of the Chern class  $c_i$  and hence to the image of  $w_i^2$  in  $B$  Spin. Thus  $w_i^2 = 0$  in  $BBSO$  for  $i \neq 2^j + 1$  and  $y_{2^j}$  is the restriction of a class  $e_{2^j+1+1}$ . In particular since  $Sq^{2^j}(w_{2^j-1+1})^2 = (w_{2^j+1})^2$  we can take  $e_{2^j+1}$  to be  $Sq^{2^j-1} Sq^{2^j-2} \cdots Sq^1 Sq^2 e_3$ . The class  $e_3$  is uniquely determined ( $H^3(BBSO; Z_2) \approx Z_2$ ) and this definition of  $e_{2^j+1}$  implies  $Sq^1 e_{2^j+1+1} = (e_{2^j+1})^2 = 0$  if  $e_3^2 = 0$ . The only alternative to  $e_3^2 = 0$  is  $e_3^2 = e_6$ ; there is no other class in this dimension. Since  $Sq^1 w_6 = w_7$  in  $B$  Spin and  $w_6, w_7$  map to  $e_6, e_7$ , we have  $Sq^1 e_6 = e_7$  but  $Sq^1(e_3)^2 = 0$ ; therefore  $e_3^2$  must be zero.

$H^*(BBSO, Z)$ . Consider  $BBSO$  as the fibre of  $B$  Spin  $\rightarrow BSU$ . The latter map factors:  $B$  Spin  $\xrightarrow{\pi} BSO \rightarrow BSU$ . Recall that

$$H^*(BSU; Z) = Z[c_i \mid i > 1] \text{ and } H^*(BSO; Z) = Z[P_i] \oplus T$$

where  $T$  is the torsion ideal,  $2T = 0$ ,  $c_{2i+1}$  maps into  $T$  and  $c_{2i}$  maps to  $P_i$ . To determine  $\text{Im}(H^*(B$  Spin)) in  $H^*(BBSO)$ , we need to know  $\pi^*[P_i]$  in  $H^*(B$  Spin).

**THEOREM 3** (Thomas [6]). *If  $i$  is not a power of 2,  $\pi^*P_i = Q_i$ . If  $j = 2^r, r > 0, \pi^*P_{2^j} = 2Q_{2^j} + Q_j^2 - \pi^*\Phi_{2^j}$ .  $\pi^*P_1 = 2Q_1$ .*

**LEMMA.**  $\pi^*\Phi_{2^j}$  maps into  $\text{Im } T \subset H^*(BBSO)$ .

*Proof.*  $H^*(BSO; Z)$  maps onto  $\text{Im } T$  in  $H^*(BBSO)$  since  $H^*(BSU)$  maps onto the  $Z[P_i]$  part.

Since  $\pi^*P_j$  goes to zero in  $BBSO$ , we have in  $H^*(BBSO; Z)$

$$2Q_{2^j} = -Q_j^2 + t \quad \text{where } 2t = 0 \quad \text{and } j = 2^r .$$

$$2Q_1 = 0 .$$

By iteration we find

$$2^{r+1}Q_{2^r} = \pm 2Q_{2^r}Q_{2^{r-1}} \cdots Q_2(Q_1)^2 = 0 .$$

To determine the order of  $Q_{2^i}$  in *BBSO*, consider  $\Gamma(u \mid 2u = 0)$ , a divided polynomial algebra on a single generator  $u$  of dimension 4 and order 2; i.e., additively  $\Gamma$  has generators  $\gamma_i(u)$  in dimension  $4i$  and the multiplication table is  $\gamma_i(u)\gamma_j(u) = (i, j)\gamma_{i+j}(u)$  where  $(i, j)$  is the binomial coefficient  $\{(i + j)!/i!j!\}$ .

In particular  $i!\gamma_i(u) = u^i$ .

We construct a map  $f$  from  $\text{Im}(H^*(B \text{ Spin}; Z) \rightarrow H^*(BBSO; Z))$  to  $\Gamma$  by mapping  $\hat{T}$  to zero,  $Q_i$  to zero for  $i \neq 2^j$  and  $Q_{2^j}$  to  $-\gamma_2(f(Q_{2^{j-1}}))$  with  $f(Q_1) = u$ . Since  $2Q_{2^j} = -Q_{2^{j-1}}^2 + \pi^*\Phi_{2^j}$ , and  $\Phi_{2^j}$  goes into  $\text{Im } \hat{T}$  in *BBSO*, the map  $f$  is well defined. Since for any  $x$ , the order of  $\gamma_2(x)$  is twice the order of  $x$ , we have

$$\text{ord } f(Q_{2^j}) = 2 \text{ ord } f(Q_{2^{j-1}}) = 2^j \text{ ord } f(Q_1) = 2^{j+1} .$$

Thus the order of  $Q_{2^j}$  is at least  $2^{j+1}$  and that  $2^{j+1}Q_{2^i}$  is in fact zero we have already seen.

Thus we have  $2^r$  torsion for each  $r$ . From the exact cohomology sequence derived from  $0 \rightarrow Z \xrightarrow{2^r} Z \rightarrow Z_{2^r} \rightarrow 0$ , we see that  $Q_{2^{r-1}} = \beta_{2^r}^\infty x_r$  for some class  $x_r \in H^*(BBSO; Z_{2^r})$ , where  $\beta_{2^r}^\infty$  is the connecting homomorphism  $H^*( ; Z_{2^r}) \rightarrow H^{*+1}( ; Z)$ .

LEMMA.  $(\beta_{2^r}^\infty x_r)_2 = d_r(x_r)_2$  where  $( )_2$  means reduction mod 2.

*Proof.* Recall how  $d_r$  is defined:  $d_r(x) = (\beta_2^\infty(x)/2^{r-1})_2$ . From the commutativity of the diagram

$$\begin{array}{ccccc} Z & \xrightarrow{2^r} & Z & \longrightarrow & Z_{2^r} \\ \downarrow 2^{r-1} & & \parallel & & \downarrow \\ Z & \xrightarrow{2} & Z & \longrightarrow & Z_2 \end{array}$$

it follows that  $\beta_2^\infty = 2^{r-1}\beta_{2^r}^\infty$ . In particular,  $d_r(x_r)_2 = (Q_{2^{r-1}})_2$ . According to Thomas,  $(Q_{2^{r-1}})_2 = \pi^*(w_{2^{r+1}} + \psi_{2^{r+1}})$  where  $\psi_{2^{r+1}}$  is decomposable. In particular,  $(Q_1)_2 = W_4$ .

We prove Theorem 2 by induction. Since

$$Sq^1 w_{2^i} = w_{2^{i+1}} \quad \text{and} \quad Sq^1 w_{2^{i+1}} = 0 ,$$

we know  $Sq^1 e_{2^i} = e_{2^{i+1}}$  and  $Sq^1 e_{2^{i+1}} = 0$  unless  $i = 2^j$ . Since we have chosen  $e_{2^{j+1}} = Sq^{2^j-1} \cdots Sq^2 e_3$ , we have  $Sq^1 e_{2^{j+1}} = (e_{2^{j-1+1}})^2 = 0$  for

$j \geq 2$ . For  $j = 1$ , we have  $Sq^1 e_3 = e_4$  because  $e_4 = (Q_1)_2$  which is in the image of  $Sq^1$  since  $2Q_1 = 0$ .

Thus

$$\begin{aligned} \beta E_2 &= \text{Ker } Sq^1 / \text{Im } Sq^1 \\ &= E(e_3 e_4) \otimes E(e_{2i} e_{2i+1} \mid 2 < i \neq 2^j) \otimes E(e_{2^j+1}, e_{2^{j+1}} \mid j \geq 2). \end{aligned}$$

Since  $d_2(x_2)_2 = (Q_2)_2 = e_3$ , we must have  $x_2 = e_3 e_4$ .

In general  $d_r(x_r)_{2^j} = (Q_{2^r-1})_{2^j} = e_{2^r+1}$  modulo decomposables. Now consider  $H^*(BBSO; Q)$ . Since  $H^*(BSO; Q) = Q[P_i]$  with the usual diagonal  $m^*(P_i) = \sum_{j+h=i} P_j \otimes P_h$ , we have  $H^*(BBSO; Q) = E(R_i)$  where  $\dim R_i \in H^{4j+1}$ . Thus  $\beta E_\infty = E(S_{4i+1})$  and the only possibility is

$$\begin{aligned} S_{4i+1} &= e_{2i} e_{2i+1} \quad i \neq 2^j, \\ S_{2^i+1} &= e_{2^i+1} \end{aligned}$$

modulo terms decomposable in terms of the  $S_{4i+1}$ . This leaves  $e_3 e_4 \cdots e_{2^r}$  as the only possibility for  $x_r$ , i.e.,  $d_r(e_3 e_4 \cdots e_{2^r}) = e_{2^r+1}$  mod decomposables as claimed.

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