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A – P CONGRUENCES ON BAER SEMIGROUPS

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In this paper a coordinatizing Baer semigroup is used to pick out an interesting sublattice of the lattice of congruence relations on a lattice with 0 and 1. These congruences are defined for any lattice with 0 and 1 and have many of the nice properties enjoyed by congruence relations on a relatively complemented lattice.

These results generalize the work of S. Maeda on Rickart (Baer) rings and are related to G. Gratzer and E. T. Schmidt's work on standard ideals.

In [7] M. F. Janowitz shows that lattice theory can be approached by means of Baer semigroups. A *Baer semigroup* is a multiplicative semigroup S with 0 and 1 in which the left and right annihilators, $L(x) = \{y \in S : yx = 0\}$ and $R(x) = \{y \in S : xy = 0\}$, of any $x \in S$ are principal left and right ideals generated by idempotents. For any Baer semigroup $S, \mathcal{L}(S) = \{L(x) : x \in S\}$ and $\mathcal{R}(S) = \{R(x) : x \in S\}$, ordered by set inclusion, are dual isomorphic lattices with 0 and 1. The Baer semigroup S is said to *coordinatize* the lattice L if $\mathcal{L}(S)$ is isomorphic to L. The basic point is Theorem 2.3, p. 1214 of [7], which states : a partially ordered set P with 0 and 1 is a lattice if and only if it can be coordinatized by a Baer semigroup.

It will be convenient to introduce the convention that S will always denote a Baer semigroup and that for any $x \in S$, x^{l} and x^{r} will denote idempotent generators of L(x) and R(x) respectively. Also the letters e, f, g, and h shall always denote idempotents of S.

Some background material is presented in §1. In §2, A - P congruences are defined and it is shown that every A - P congruence ρ on S induces a lattice congruence θ_{ρ} on $\mathscr{L}(S)$ such that $\mathscr{L}(S)/\theta_{\rho} \cong \mathscr{L}(S/\rho)$. In §3 congruences which arise in this manner are characterised as the set of all equivalence relations on $\mathscr{L}(S)$ which are compatible with a certain set of maps on $\mathscr{L}(S)$. These congruences are called compatible with S. They are standard congruences and are thus determined by their kernels.

The ideals of $\mathscr{L}(S)$ which are kernels of congruences compatible with S are characterised in §4. In §5 it is shown that a principal ideal, [(0), Se], is the kernel of a congruence compatible with S if and only if e is central in S. In §6 this is applied to complete Baer semigroups to show that, in this case, the congruence compatible with S form a Stone lattice. 1. Preliminaries. We shall let $L(M) = \{y \in S : yx = 0 \text{ for all } x \in M\}$ and $R(M) = \{y \in S : xy = 0 \text{ for all } x \in M\}$ for any set $M \subseteq S$. The following is a summary of results found on pp. 85-86 of [8].

LEMMA 1.1. Let $x, y \in S$.

(i) $xS \subseteq yS \text{ implies } L(y) \subseteq L(x)$; $Sx \subseteq Sy \text{ implies } R(y) \subseteq R(x)$.

(ii) $Sx \subseteq LR(x)$; $xS \subseteq RL(x)$.

(iii) L(x) = LRL(x); R(x) = RLR(x).

(iv) $Sx \in \mathcal{L}(S)$ if and only if Sx = LR(x); $xS \in \mathcal{R}(S)$ if and only if xS = RL(x).

(v) The mappings $eS \rightarrow L(eS)$ and $Sf \rightarrow R(Sf)$ are mutually inverse dual isomorphisms between $\mathscr{R}(S)$ and $\mathscr{L}(S)$.

(vi) Let Se, $Sf \in \mathscr{L}(S)$ and $Sh = L(ef^r)$. Then $he = (he)^2$, Se \cap Sf = She $\in \mathscr{L}(S)$, and Se \bigvee Sf = L(e^rS \cap f^rS).

(vii) Let $eS, fS \in \mathscr{R}(S)$ and $gS = R(f^{l}e)$. Then $eg = (eg)^{2}$, $eS \cap fS = egS \in \mathscr{R}(S)$, and $eS \bigvee fS = R(Se^{l} \cap Sf^{l})$.

Note that the meet operation in $\mathscr{L}(S)$ and $\mathscr{R}(S)$ is set intersection and that the trivial ideals, S and (0), are the largest and smallest elements of both $\mathscr{L}(S)$ and $\mathscr{R}(S)$.

We shall be interested in a class of isotone maps introduced by Croisot in [2].

DEFINITION 1.2. Let P be a partially ordered set. An isotone map ϕ of P into itself is called *residuated* if there exists an isotone map ϕ^+ of P into P such that for any $p \in P$, $p\phi^+\phi \leq p \leq p\phi\phi^+$. In this case ϕ^+ is called a *residual* map.

Clearly ϕ^+ is uniquely determined by ϕ and conversely. The pair (ϕ, ϕ^+) sets up a Galois connection between P and its dual. Thus we can combine results from [2], [3], and [11] to get.

LEMMA 1.3. Let P be a partially ordered set and ϕ and ψ maps of P into itself.

(i) If ϕ and ψ are residuated then $\phi\psi$ is residuated and $(\phi\psi)^+ = \psi^+\phi^+$.

(ii) If ϕ is residuated then $\phi = \phi \phi^+ \phi$ and $\phi^+ = \phi^+ \phi \phi^+$.

(iii) Let ϕ be residuated and $\{x_{\alpha}\}$ be any family of elements of P. If $\bigvee_{\alpha} x_{\alpha}$ exists then $\bigvee_{\alpha} (x_{\alpha}\phi)$ exists and $\bigvee_{\alpha} (x_{\alpha}\phi) = (\bigvee_{\alpha} x_{\alpha})\phi$. Dually if $\bigwedge_{\alpha} x_{\alpha}$ exists then $\bigwedge_{\alpha} (x_{\alpha}\phi^{+})$ exists and $\bigwedge_{\alpha} (x_{\alpha}\phi^{+}) = (\bigwedge_{\alpha} x_{\alpha})\phi^{+}$.

(iv) A necessary and sufficient condition that ϕ be residuated is that for any $x \in L$, $\{z : z\phi \leq x\}$ has a largest element x^* . In this case ϕ^+ is given by $x\phi^+ = x^*$. According to Lemma 1.3 (i) the set of residuated maps forms a semigroup for any partially ordered set P. We shall denote the semigroup of residuated maps on P by S(P). In [7], Theorem 2.3, p. 1214, it is shown that P is a lattice if and only if S(P) is a Baer semigroup. In this case S(P) coordinatizes P.

In [8], pp. 93, 94, it is shown that any Baer semigroup S can be represented as a semigroup of residuated maps on $\mathcal{L}(S)$. We shall be interested in the maps introduced to achieve this.

LEMMA 1.4. For any $x \in S$ define $\phi_x : \mathscr{L}(S) \to \mathscr{L}(S)$ by $Se\phi_x = LR(ex)$.

(i) ϕ_x is residuated with residual ϕ_x^+ given by $Se\phi_x^+ = L(xe^r)$.

(ii) If LR(y) = Se then $Se\phi_x = LR(yx)$.

(iii) Let $S_0 = \{\phi_x : x \in S\}$. Then S_0 is a Baer semigroup which coordinatizes $\mathscr{L}(S)$.

(iv) The map $x \rightarrow \phi_x$ is a homomorphism, with kernel {0}, of S into S_0 .

We shall now develop an unpublished result due to D. J. Foulis and M. F. Janowitz.

DEFINITION 1.5. A semigroup S is a complete Baer semigroup if for any subset M of S there exist idempotents e, f such that L(M) = Seand R(M) = fS.

In proving Lemma 2.3 of [7] the crucial observation was [7] Lemma 2.1, p. 1213, where it is shown that for any lattice L and any $a \in L$ there are idempotent residuated maps θ_a and ψ_a given by:

$$x heta_a=egin{cases} x&x\leq a\ a& ext{otherwise} \end{cases} \qquad x\psi_a=egin{cases} 0&x\leq a\ xigvee a& ext{otherwise}. \end{cases}$$

THEOREM 1.6. Let P be a partially ordered set with 0 and 1. Then the following conditions are equivalent.

- (i) P is a complete lattice.
- (ii) S(P) is a complete Baer semigroup.
- (iii) P can be coordinatized by a complete Baer semigroup.

Proof. (i) \Rightarrow (ii) Let P be a complete lattice and $M \subseteq S(P)$ with $m = \bigvee \{1\phi : \phi \in M\}$ and $n = \bigwedge \{0\phi^+ : \phi \in M\}$. It is easily verified that $L(M) = S(P)\theta_n$ and $R(M) = \psi_m S(P)$.

(ii) \Rightarrow (iii) follows from [7], Theorem 2.3.

(iii) \Rightarrow (i) Let S be a complete Baer semigroup coordinatizing P and $\mathscr{P}(S)$ the complete lattice of all subsets of S. Define α and β

mapping $\mathscr{P}(S)$ into $\mathscr{P}(S)$ by $M\alpha = L(M)$ and $M\beta = R(M)$. Clearly (α, β) sets up a Galois connection of $\mathscr{P}(S)$ with itself. Since S is a complete Baer semigroup $\mathscr{L}(S)$ is the set of Galois closed objects of (α, β) . Thus $\mathscr{L}(S)$ is a complete lattice.

We conclude this section with some relatively well known facts about lattice congruences. An equivalence relation Θ on a lattice is a *lattice congruence* if $a\Theta b$ and $c\Theta d$ imply $(a \bigvee c)\Theta(b \bigvee d)$ and $(a \bigwedge c)\Theta(b \bigwedge d)$. We shall sometimes write $a \equiv b(\Theta)$ in place of $a\Theta b$. With respect to the order $\Theta \leq \Theta'$ if and only if $a\Theta b$ implies $a\Theta'b$, the set of all lattice congruences on a lattice L is a complete lattice, denoted by $\Theta(L)$, with meet and join given as follows:

THEOREM 1.7. Let L be a lattice and Γ a subset of $\Theta(L)$. (i) $a \equiv b(\bigwedge \Gamma)$ if and only if $a\gamma b$ for all $\gamma \in \Gamma$. (ii) $a \equiv b(\bigvee \Gamma)$ if and only if there exist finite sequences a_0, a_1, \dots, a_n of elements of L and $\gamma_1, \dots, \gamma_n$ of elements of Γ , such that $a = a_0, a_n = b$, and $a_{i-1} \gamma_i a_i$ for $i = 1, \dots, n$.

The largest element ι of $\Theta(L)$ is given by $a\iota b$ for all $a, b \in L$ and the smallest element ω is given by $a\omega b$ if and only if a = b.

In [4] it is shown that $\Theta(L)$ is distributive. In fact we have:

THEOREM 1.8. Let L be a lattice. The O(L) is a distributive lattice such that for any family $\{\Theta_{\alpha}\} \subseteq O(L)$

$$(\bigvee_{\alpha} \Theta_{\alpha}) \bigwedge \Psi = \bigvee_{\alpha} (\Theta_{\alpha} \bigwedge \Psi)$$

for any $\Psi \in \Theta(L)$.

Thus by Theorem 15, p. 147, of [1] we have:

THEOREM 1.9. For any lattice $L, \Theta(L)$ is pseudo-complemented.

Finally we mention that if $\Theta \in \Theta(L)$ then $a\Theta b$ if and only if $x\Theta y$ for all $x, y \in [a \land b, a \lor b]$.

2. A - P congruences. In [10] S. Maeda defines annihilator preserving homomorphisms for rings. We shall take the same definition for semigroups with 0.

DEFINITION 2.1. A homomorphism ϕ of a semigroup S with 0 is called an annihilator preserving (A - P) homomorphism if for any $x \in S$, $R(x)\phi = R(x\phi) \cap S\phi$ and $L(x)\phi = L(x\phi) \cap S\phi$. A congruence relation ρ on a semigroup S is called an A - P congruence if the natural homomorphism induced by ρ is an A - P homomorphism.

For any congruence ρ on a semigroup S and any $x \in S$ let x/ρ denote the equivalence class of S/ρ containing x. Similarly for any set $A \subseteq S$, let $A/\rho = \{x/\rho \in S/\rho : x \in A\}$. If S has a 0 then $R(x)/\rho \subseteq$ $R(x/\rho)$ and $L(x)/\rho \subseteq L(x/\rho)$. Thus a congruence ρ is an A - P congruence if and only if $R(x/\rho) \subseteq R(x)/\rho$ and $L(x/\rho) \subseteq L(x)/\rho$. Note that we are using L and R to denote the left and right annihilators both in S and in S/ρ .

THEOREM 2.2. Let ρ be an A - P congruence on a semigroup S. If e and f are idempotents of S such that Se = L(x) and fS = R(y)for some $x, y \in S$, then $(S/\rho)(e/\rho) = L(x/\rho)$ and $(f/\rho)(S/\rho) = R(y/\rho)$. Thus if S is a Baer semigroup so is S/ρ .

Proof. Since ρ is an A - P congruence $L(x/\rho) = L(x)/\rho$. Thus L(x) = Se gives $L(x/\rho) = L(x)/\rho = (Se)/\rho = (S/\rho)(e/\rho)$. Similarly R(x) = fS gives $R(x/\rho) = (f/\rho)(S/\rho)$.

We now use an A - P congruence ρ on S to induce a homomorphism of $\mathscr{L}(S)$ onto $\mathscr{L}(S|\rho)$.

THEOREM 2.3. Let ρ be an A-P congruence on S. Then $\theta_{\rho}: \mathscr{L}(S) \to \mathscr{L}(S/\rho)$ by $L(x)\theta_{\rho} = L(x/\rho)$ is a lattice homomorphism of $\mathscr{L}(S)$ onto $\mathscr{L}(S/\rho)$.

Proof. Let Se, $Sf \in \mathcal{L}(S)$ and note that, by Theorem 2.2,

$$Se heta_
ho=(S/
ho)(e/
ho) \quad ext{and} \quad Sf heta_
ho=(S/
ho)(f/
ho) \; .$$

Clearly θ_{ρ} is well defined since if L(x) = L(y) then

$$L(x/\rho) = L(x)/\rho = L(y)/\rho = L(y/\rho)$$

By Lemma 1.1 (vi), $She = Se \cap Sf$ where $Sh = L(ef^r)$. Applying Theorem 2.2 gives $(f^r/\rho)(S/\rho) = R(f/\rho)$ and $(S/\rho)(h/\rho) = L((e/\rho)(f^r/\rho))$. Thus applying Lemma 1.1 (vi) to S/ρ yields

$$(S/\rho)(e/\rho) \cap (S/\rho)(f/\rho) = (S/\rho)(h/\rho)(e/\rho) = (S/\rho)(he/\rho) .$$

Therefore, $Se\theta_{\rho} \cap Sf\theta_{\rho} = (Se \cap Sf)\theta_{\rho}$. By a dual argument $\theta_{\rho}^* \mathscr{R}(S) \to \mathscr{R}(S/\rho)$ by $R(x)\theta_{\rho}^* = R(x/\rho)$ is also a meet homomorphism.

By Lemma 1.1 (vi) Se \bigvee Sf = L(R(e) \cap R(f)). Let $gS = R(e) \cap R(f)$ so that Se \bigvee Sf = Sg^l. Since θ_{ρ}^* is a meet homomorphism,

$$R(e/
ho)\cap R(f/
ho)=(g/
ho)(S/
ho)$$
 .

Noting that $L(g/\rho) = (S/\rho)(g^{i}/\rho)$ and applying Lemma 1.1 (vi) to S/ρ gives

$$(S/\rho)(e/\rho) \bigvee (S/\rho)(f/\rho) = (S/\rho)(g^{i}/\rho)$$
.

Thus $Se\theta_{\rho} \bigvee Sf\theta_{\rho} = (Se \bigvee Sf)\theta_{\rho}$ and θ_{ρ} is a lattice homomorphism. Clearly θ_{ρ} is onto.

For any A - P congruence ρ on S let Θ_{ρ} denote the lattice congruence $\theta_{\rho} \circ \theta_{\rho}^{-1}$ induced on $\mathscr{L}(S)$ by θ_{ρ} .

COROLLARY 2.4. $\mathscr{L}(S)/\Theta_{\rho} \cong \mathscr{L}(S/\rho).$

3. Compatible congruences. In this section we shall characterise lattice congruence which are induced by an A - P congruence on a coordinatizing Baer semigroup in the manner given in Theorem 2.3. Since $L \cong \mathscr{L}(S)$ for any Baer semigroup S coordinatizing L, we shall lose no generality by considering only lattices of the form $\mathscr{L}(S)$.

The residuated maps $\phi_x, x \in S$, defined in Lemma 1.4, play a central role in the theory of Bear semigroups. We shall be interested in equivalence relations on $\mathscr{L}(S)$ which are compatible with ϕ_x and ϕ_x^+ , considered as unary operations on $\mathscr{L}(S)$.

DEFINITION 3.1. An equivalence relation E on $\mathcal{L}(S)$ is called *compatible with* S if for any $x \in S$,

$$SeESf \Longrightarrow (Se\phi_x)E(Sf\phi_x)$$
 and $(Se\phi_x^+)E(Sf\phi_x^+)$.

By [7] Lemma 3.1 and 3.2, pp. 1214–1215, $Se \cap Sf = Se \cap S\phi_f = Se\phi_f^+\phi_f$. Dually $Se \bigvee Sf = Se\phi_{fr}\phi_{fr}^+$. Thus we have :

LEMMA 3.3. Any equivalence relation compatible with S is a lattice congruence.

We now consider an A - P congruence ρ on S and Θ_{ρ} , the lattice congruence induced on $\mathscr{L}(S)$ by ρ as in Theorem 2.3.

THEOREM 3.4. Let ρ be an A - P congruence on S. Then Θ_{ρ} is compatible with S.

Proof. Since $Se\theta_{\rho}Sf$ if and only if $(S/\rho)(e/\rho) = (S/\rho)(f/\rho)$, $Se\theta_{\rho}Sf$ implies $(e/\rho) = (e/\rho)(f/\rho)$ and $(f/\rho) = (f/\rho)(e/\rho)$. Note that for any $y \in S$, $LR(y)\theta_{\rho} = LR(y/\rho)$. If $Se\theta_{\rho}Sf$ we have

$$(Se\phi_x) heta_
ho = LR((ex)/
ho) = LR((e/
ho)(x/
ho)) \ = LR((e/
ho)(f/
ho)(x/
ho)) \subseteq LR((f/
ho)(x/
ho)) = (Sf\phi_x) heta_
ho \;.$$

By symmetry $(Sf\phi_x)\theta_{\rho} = (Se\phi_x)\theta_{\rho}$ ie. $Se\phi_x\theta_{\rho}Sf\phi_x$.

Now $R(e/\rho) = R((e/\rho)(f/\rho)) \supseteq R(f/\rho) = R((f/\rho)(e/\rho)) \supseteq R(e/\rho)$. Thus $(e^r/\rho)(S/\rho) = (f^r/\rho)(S/\rho)$. But

$$(Se\phi_x^+) heta_
ho = L(xe^r) heta_
ho = L((xe^r)/
ho) = L((x/
ho)(e^r/
ho))$$

and similarly $(Sf\phi_x^+)\theta_\rho = L((x/\rho)(f^r/\rho))$. Clearly $y/\rho \in L((x/\rho)(e^r/\rho))$ if and only if $(y/\rho)(x/\rho) \in L((e^r/\rho)(S/\rho)) = L((f^r/\rho)(S/\rho))$. Thus we have $(Se\phi_x^+)\theta_\rho = (Sf\phi_x^+)\theta_\rho$. Therefore, $Se\phi_x^+\theta_\rho Sf\phi_x^+$ and θ_ρ is compatible with S.

By the following theorem every congruence compatible with S is determined by its kernel in a very nice way.

THEOREM 3.5. Let Θ be a congruence compatible with S. Then the following are equivalent.

(i) $Se\Theta Sf$.

(ii) $Se\phi_{fr} \bigvee Sf\phi_{er} \in ker \Theta$.

(iii) There is an $Sg \in ker \Theta$ such that $Se \bigvee Sf = Se \bigvee Sg = Sf \bigvee Sg$.

Proof (i) \Rightarrow (ii) Since θ is compatible with S, Se9Sf gives $Se\phi_{jr}\Theta Sf\phi_{jr} = (0)$, i.e., $Se\phi_{jr} \in \ker \theta$. By symmetry $Sf\phi_{er} \in \ker \theta$ so we have (ii).

(ii) \Rightarrow (iii) Let $Sg = Se\phi_{fr} \bigvee Sf\phi_{e^r} \in \ker \Theta$ and claim $Se \bigvee Sg = Sf \bigvee Sg$, i.e.,

 $LR(e) \bigvee LR(ef^r) \bigvee LR(fe^r) = LR(f) \bigvee LR(ef^r) \lor LR(fe^r)$.

By Lemma 1.1 (v) this is equivalent to

$$R(e) \cap R(ef^r) \cap R(fe^r) = R(f) \cap R(ef^r) \cap R(fe^r)$$
.

Let $x \in R(e) \cap R(ef^r) \cap R(fe^r)$. Then $x = e^r x$ and $fx = fe^r x = 0$ so $x \in R(f) \cap R(ef^r) \cap R(fe^r)$. By symmetry

$$R(e) \cap R(ef^r) \cap R(fe^r) = R(f) \cap R(ef^r) \cap R(fe^r)$$

so we have $Se \bigvee Sg = Sf \bigvee Sg$. To show that $Se \bigvee Sf = Se \bigvee Sg = Sf \bigvee Sg$ we need only show that $Sg \subseteq Se \bigvee Sf$. This is equivalent to $R(e) \cap R(f) \subseteq R(ef^r) \cap R(fe^r)$. But if $x \in R(e) \cap R(f)$ then $x = e^r x = f^r x$, so $ef^r x = ex = 0$ and $fe^r x = fx = 0$, i.e., $x \in R(ef^r) \cap R(fe^r)$.

(iii) \Rightarrow (i) If $Se \lor Sg = Sf \lor Sg$ and $Sg\Theta(0)$ then $Se\Theta Se \lor Sg = Sf \lor Sg\Theta Sf$.

A congruence Θ on a lattice is called *standard* if there is an ideal S such that $a\Theta b$ if and only if $a \lor b = (a \land b) \lor s$ for some $s \in S$.

COROLLARY 3.6. Any congruence Θ compatible with S is a standard congruence.

Proof. Since Θ is a lattice congruence $Se\Theta Sf$ if and only if $(Se \bigvee Sf)\Theta(Se \cap Sf)$. By Theorem 3.5 this is equivalent to

 $(Se \lor Sf) \lor (Se \cap Sf) = Se \lor Sf = (Se \cap Sf) \lor Sg$

for some $Sg \in \ker \Theta$.

Thus by Lemma 7, p. 36, of [5] we have:

COROLLARY 3.7. Compatible congruences are permutable.

By Theorem 3.5 every congruence compatible with S is determined by its kernel. Since, by Theorem 3.4, Θ_{ρ} is compatible with S for any A - P congruence ρ on S we know that Θ_{ρ} is determined by its kernel. By the following lemma, Θ_{ρ} is also uniquely determined by ker ρ .

LEMMA 3.8. Let ρ be an A - P congruence on S. Then $x \in \ker \rho$ if and only if $LR(x) \in \ker \Theta_{\rho}$.

Proof. Let $x \in \ker \rho$. Then $x\rho 0 \Rightarrow xy\rho 0$ for any $y \in S$. Thus $R(x/\rho) = S/\rho$ so that $LR(x/\rho) = L(S/\rho) = (0/\rho)$, i.e., $LR(x) \in \ker \Theta_{\rho}$. If we let $LR(x) \in \ker \Theta_{\rho}$ then $LR(x/\rho) = (0/\rho)$. Thus $R(x/\rho) = RLR(x/\rho) = R(0/\rho) = S/\rho$ which gives $x/\rho = 0/\rho$ and we have $x \in \ker \rho$.

That Θ should be determined completely by ker ρ is unexpected since an A - P congruence need not be determined by its kernel. For clearly the congruence ω given by $x\omega y$ if and only if x = y is an A - P congruence with kernel {0} as is the congruence ρ_0 given by $x\rho_0 y$ if and only if $\phi_x = \phi_y$. Clearly ρ_0 is not generally equal to ω . It turns out that ρ_0 is the largest A - P congruence with kernel {0}. Our next project shall be to start with a congruence Θ compatible with S and determine the existance of an A - P congruence λ on S, such that $\Theta = \Theta_{\lambda}$. By lemma 3.8 we shall have to construct λ so that ker $\lambda = \{x \in S : LR(x) \in \ker \Theta\}$.

For any congruence Θ on $\mathscr{L}(S)$ let Se/Θ denote the equivalence class of $\mathscr{L}(S)/\Theta$ containing Se.

LEMMA 3.9. Let Θ be a congruence compatible with S. For each $x \in S$ define Φ_x ; $\mathscr{L}(S)/\Theta \to \mathscr{L}(S)/\Theta$ by $(Se/\Theta)\Phi_x = (Se\phi_x)/\Theta$. Then Φ_x is residuated with residual Φ_x^+ given by $(Se/\Theta)\Phi_x^+ = (Se\phi_x^+)/\Theta$.

Proof. Clearly Φ_x and Φ_x^+ are well defined since Θ is compatible with S. We shall use Lemma 1.3 (iv), i.e., we shall show that the inverse image of a principal ideal is principal. Let $Sf/\Theta \in [(0)/\Theta, Se/\Theta]\Phi_x^{-1}$. Then $(Sf/\Theta)\Phi_x = (Sf\phi_x)/\Theta = (Sf\phi_x)/\Theta \cap Se/\Theta = (Sf\phi_x \cap Se)/\Theta$. This gives $Sf\phi_x\Theta(Sf\phi_x \cap Se)$ so by compatibility with S,

 $Sf \subseteq Sf \phi_x \phi_x^+ \Theta(Sf \phi_x \cap Se) \phi_x^+ \subseteq Se \phi_x^+$.

Thus in $\mathscr{L}(S)/\Theta$, $Sf/\Theta \subseteq (Sf\phi_x\phi_x^+)/\Theta = (Sf\phi_x \cap Se)\phi_x^+/\Theta \subseteq (Se/\Theta)\Phi_x^+$, i.e., $[(0)/\Theta, Se/\Theta]\Phi_x^{-1} \subseteq [(0)/\Theta, (Se/\Theta)\Phi_x^+]$. Now let $Sf/\Theta \subseteq (Se/\Theta)\Phi_x^+$. Then

 $Sf/artheta \,=\, Sf/artheta \,\cap \, [(Se/artheta) artheta_x^+] \,=\, Sf/artheta \,\cap \, (Se\phi_x^+)/artheta \,=\, (Sf \cap \, Se\phi_x^+)/artheta$

i.e., $Sf\Theta(Sf \cap Se\phi_x^+)$. By compatibility with S

$$Sf\phi_x heta[(Sf\cap Se\phi_x^+)\phi_x] \subseteq Se\phi_x^+\phi_x \subseteq Se$$
 .

Hence $(Sf/\Theta)\Phi_x = (Sf\phi_x)/\Theta = (Sf \cap Se\phi_x^+)\phi_x/\Theta \subseteq Se/\Theta$. Therefore,

$$[(0)/\Theta, Se/\Theta] arPsi_x^{-1} = [(0)/\Theta, (Sf/\Theta) arPsi_x^+]$$

and by Lemma 1.3 (iv), Φ_x is residuated with residual Φ_x^+ .

For any equivalence relation E on $\mathscr{L}(S)$ we can define a left congruence λ_E on S by taking $x\lambda_E y$ if and only if $(Se\phi_x)E(Se\phi_y)$ for all $Se \in \mathscr{L}(S)$. Similarly, $x\rho_E y$ if and only if $(Se\phi_x^+)E(Se\phi_x^+)$ for all $Se \in \mathscr{L}(S)$, defines a right congruence on S.

LEMMA 3.10. If Θ is a congruence compatible with S then $\lambda_{\theta} = \rho_{\theta}$. Thus λ_{θ} is a congruence on S.

Proof. By definition $x\lambda_{\theta}y$ if and only if $\Phi_x = \Phi_y$. But $\Phi_x = \Phi_y$ if and only if $\Phi_x^+ = \Phi_y^+$ which is equivalent to $x\rho_{\theta}y$.

THEOREM 3.11. Let Θ be a congruence compatible with S. Then λ_{θ} is an A - P congruence on S.

Proof. We know that λ_{θ} is an A - P congruence if and only if $L(y/\lambda_{\theta}) \subseteq L(y)/\lambda_{\theta}$ and $R(y/\lambda_{\theta}) \subseteq R(y)/\lambda_{\theta}$ for all $y \in S$. We shall start with $x/\lambda_{\theta} \in L(y/\lambda_{\theta})$ and show that $x/\lambda_{\theta} = xe/\lambda_{\theta}$ where Se = L(y). This, of course, is equivalent to $\Phi_x = \Phi_{xe}$.

Let $x/\lambda_{\theta} \in L(y/\lambda_{\theta})$ so that $xy \in \ker \lambda_{\theta}$. Thus $Sf\phi_{xy}\Theta Sf\phi_0 = (0)$ for all $Sf \in \mathscr{L}(S)$. In particular, $S\phi_{xy}\Theta(0)$ so for any $Sf \in \mathscr{L}(S)$,

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$$Sf\phi_x \subseteq S\phi_x \subseteq S\phi_x\phi_y\phi_y^+ = (S\phi_{xy}\phi_y^+) \mathcal{E}(0)\phi_y^+ = L(y) = Se$$
 .

Thus $Sf\phi_x = (Sf\phi_x \cap S\phi_{xy}\phi_x^+) \Theta(Se \cap Sf\phi_x)$. Now if $Sg \subseteq Se$ then g = geso $Sg\phi_e = LR(ge) = LR(g) = Sg$. Thus applying ϕ_e to both sides of the above gives $Sf\phi_{xe} = Sf\phi_x\phi_e\Theta(Se \cap Sf\phi_x)\phi_e = Se \cap Sf\phi_x$. By transitivity $Sf\phi_{xe}\Theta Sf\phi_x$ and so $xe\lambda_\theta x$. Since $xe \in L(y)$ this gives $x/\lambda_\theta \in L(y)/\lambda_\theta$.

The argument to show $R(y/\lambda_{\theta}) = R(y)/\lambda_{\theta}$ is exactly dual to the above but will be included. We have $x/\lambda_{\theta} \in R(y/\lambda_{\theta})$ if and only if $yx \in \ker \lambda_{\theta}$. By Lemma 3.10 and the definition of ρ_{θ} this is equivalent to $Sf\phi_{yx}^+ \Theta Sf\phi_0^+ = L(0) = S$ for all $Sf \in \mathscr{L}(S)$. In particular $(0)\phi_{yx}^+ \Theta S$. By Lemma 1.3 (i) $\phi_{yx}^+ = \phi_x^+ \phi_y^+$ so for any $Sf \in \mathscr{L}(S)$ we have

$$Sf\phi_x^+ \supseteq (0)\phi_x^+ \supseteq (0)\phi_x^+\phi_y^+\phi_y^+ = (0)\phi_{yx}^+\phi_y^-\Theta S\phi_y$$
.

Thus $Sf\phi_x^+ = (Sf\phi_x^+ \bigvee (0)\phi_{yx}^+\phi_y)\Theta(S\phi_y \bigvee Sf\phi_x^+)$. Let eS = R(y) and note that $S\phi_y = LR(y) = L(e)$. Now $L(e) \subseteq Sg$ implies $eS = RL(e) \supseteq R(g) =$ g^rS so $g^r = eg^r$. Thus $Sg\phi_e^+ = L(eg^r) = L(g^r) = LR(g) = Sg$. Since $L(e) = S\phi_y \subseteq S\phi_y \bigvee Sf\phi_x^+$, applying ϕ_e^+ to both sides of the above gives; $Sf\phi_{ex}^+ = Sf\phi_x^+\phi_e^+\Theta(S\phi_y \bigvee Sf\phi_x^+)\phi_e^+ = S\phi_y \bigvee Sf\phi_x^+$. By transitivity $Sf\phi_x^+\Theta Sf\phi_{ex}^+$ so by Lemma 3.10 $x\lambda_{\phi}ex$. Since $ex \in R(y)$ this gives $x/\lambda_{\phi} \in R(y)/\lambda_{\phi}$. Thus λ_{ϕ} is an A - P congruence.

By Theorem 3.11 every congruence Θ compatible with S gives rise to an A - P congruence λ_{θ} on S.

LEMMA 3.12. Let Θ be a congruence compatible with S. Then $x \in \ker \lambda_{\theta}$ if and only if $LR(x) \in \ker \Theta$.

Proof. Let $x \in \ker \lambda_{\theta}$, i.e., $Se\phi_x \Theta Se\phi_0 = (0)$ for all $Se \in \mathscr{L}(S)$. Taking Se = S gives $LR(x) \in \ker \Theta$. Let $LR(x) \in \ker \Theta$. The for any $Se \in \mathscr{L}(S)$ $Se\phi_x \subseteq S\phi_x = LR(x)\Theta(0) = Se\phi_0$ so $x \in \ker \lambda_{\theta}$.

THEOREM 3.13. Let Θ be a congruence compatible with S and $\rho = \lambda_{\theta}$. Then $\Theta_{\rho} = \Theta$.

Proof. By Theorem 3.11 ρ is an A - P congruence so by Theorem 3.4 θ_{ρ} is compatible with S. By Lemma 3.8 and 3.12 ker $\theta_{\rho} = \ker \theta$. Thus by Theorem 3.5 $\theta_{\rho} = \theta$.

We now show that λ_{θ} is the largest A - P congruence which induces Θ .

COROLLARY 3.14. Let Θ be a congruence compatible with S. If ρ is an A - P congruence on S such that ker $\rho = \ker \lambda_{\theta}$, then $\rho \leq \lambda_{\theta}$.

Proof. Let $x \rho y$. Then for any $Se \in \mathcal{L}(S)$, $ex \rho e y$ so

$$(Se\phi_x) heta_
ho = LR((ex)/
ho) = LR((ey)/
ho) = (Se\phi_y) heta_
ho$$
 .

Thus $Se\phi_x \Theta_{\rho} Se\phi_y$ and since $\Theta_{\rho} = \Theta$ this gives $x\lambda_{\theta} y$.

4. Compatible ideals. In this section ideals which are kernels of congruences compatible with S are characterised. Clearly if Θ is a congruence compatible with S and $J = \ker \Theta$ then $J\phi_x \subseteq J$ for all $x \in S$. Since ϕ_x preserves join (Lemma 1.3 (iii)) the following is clear.

LEMMA 4.1. Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteq J$ for all $x \in S$. Define a relation R on $\mathscr{L}(S)$ by Se R Sf if and only if there is an $Sg \in J$ such that Se $\bigvee Sg = Sf \bigvee Sg$. Then R is an equivalence relation and Se R Sf \Rightarrow (Se ϕ_x) R (Sf ϕ_x) for all $x \in S$.

In order to find an additional condition on J which will assure that the relation R defined in Lemma 4.1 is compatible with S, it will be valuable to look at certain residuated maps on the lattice $I(\mathscr{L}(S))$ of all ideals of $\mathscr{L}(S)$.

LEMMA 4.2. For each $x \in S$ let $\hat{\phi}_x : I(\mathscr{L}(S) \to I(\mathscr{L}(S)))$ be given by $I\hat{\phi}_x = \{Se \in \mathscr{L}(S) : Se \subseteq Sf\phi_x \text{ for some } Sf \in I\}$. Then $\hat{\phi}_x$ is residuated with residual $\hat{\phi}_x^+$ given by $I\hat{\phi}_x^+ = \{Se \in \mathscr{L}(S) : Se \subseteq Sf\phi_x^+ \text{ for some } Sf \in I\}$.

Proof. Clearly $I\hat{\phi}_x$ and $I\hat{\phi}_x^+$ are ideals. Also $\hat{\phi}_x$ and $\hat{\phi}_x^+$ are clearly isotone. Now since $Sf \subseteq Sf\phi_x\phi_x^+$, $Sf \in I$ implies $Sf \in I\hat{\phi}_x\hat{\phi}_x^+$. Thus $I \subseteq I\hat{\phi}_x\hat{\phi}_x^+$. Similarly $Sf \in I\hat{\phi}_x\hat{\phi}$ implies $Sf \subseteq Sg\phi_x^+\phi_x$ for some $Sg \in I$. Thus $Sf \subseteq Sg\phi_x^+\phi_x \subseteq Sg \in I$ so we have $I\hat{\phi}_x^+\hat{\phi}_x \subseteq I$.

We will make use of the residuated maps $\hat{\phi}_x$ to characterise ideals which are kernels of congruences compatible with S.

LEMMA 4.3. Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteq J$. Then for any $I \in I(\mathscr{L}(S)), I\hat{\phi}_x^+ \lor J \subseteq (I \lor J)\hat{\phi}_x^+$.

Proof. Recall that by Lemma 1.3 (iv), for any residuated map ϕ on a lattice L and any $a, b \in L, a\phi \leq b$ if and only if $a \leq b\phi^+$. Now $(I\hat{\phi}_x^+ \bigvee J)\hat{\phi}_x = I\hat{\phi}_x^+\hat{\phi}_x \bigvee J\hat{\phi}_x \subseteq I \bigvee J$ since $I\hat{\phi}_x^+\hat{\phi}_x \subseteq I$ and $J\hat{\phi}_x \subseteq J$. Thus $I\hat{\phi}_x^+ \bigvee J \subseteq (I \lor J)\hat{\phi}_x^+$.

COROLLARY 4.4. Let J be an ideal of $\mathscr{L}(S)$ such that $J\phi_x \subseteq J$. Then $J \subseteq J\phi_x^+$ and for any $I \in I(\mathscr{L}(S))$ we have B. J. THORNE

$$I \widehat{\phi}_x^+ ig V J \subseteq I \widehat{\phi}_x^+ ig V J \widehat{\phi}_x^+ \subseteq (I ig V J) \widehat{\phi}_x^+$$
 .

The next theorem indicates what all of this has to do with congruences compatible with S.

LEMMA 4.5. Let Θ be a congruence compatible with S and $J = \ker \Theta$. Then for any $I \in I(\mathscr{L}(S))$, and any $x \in S$,

$$I\phi_x^+ \bigvee J = I\phi_x^+ \bigvee J\phi_x^+ = (I \bigvee J)\phi_x^+$$
 .

Proof. By Corollary 4.4 we need only show that $(I \lor J)\phi_x^+ \subseteq I\hat{\phi}_x^+ \lor J$. Thus let $Se \in I \lor J$. Then there is an $Sf \in I$ and an $Sg \in J$ such that $Se \subseteq Sf \lor Sg$. Since $Sg\theta(0)$ we have $Sf \lor Sg\theta Sf$. Thus, by compatibility with $S, Se\phi_x^+ \subseteq (Sf \lor Sg)\phi_x^+\theta Sf\phi_x^+$. By Theorem 3.5 there is an $Sh \in J$ such that $(Sf \lor Sg)\phi_x^+ \lor Sh = Sf\phi_x^+ \lor Sh$. This gives $Se\phi_x^+ \subseteq (Sf \lor Sg)\phi_x^+ \lor Sh = Sf\phi_x^+ \lor Sh$. This $If \lor J, f_x^+ \subseteq I\hat{\phi}_x^+ \lor J$.

Without further justification we make the following definition.

DEFINITION 4.6. An ideal J of $\mathscr{L}(S)$ is called *compatible with* S if for all $x \in S$, $J\phi_x \subseteq J$ and, for all $I \in I(\mathscr{L}(S))$, $I\hat{\phi}_x^+ \bigvee J = (I \bigvee J)\hat{\phi}_x^+$.

THEOREM 4.7. An ideal J of $\mathcal{L}(S)$ is compatible with S if and only if it is the kernel of a congruence compatible with S.

Proof. By Lemma 4.5 the kernel of a congruence compatible with S is an ideal compatible with S. Conversely let J be an ideal compatible with S and define Θ by $Se\Theta Sf$ if and only if there is an $Sg \in J$ such that $Se \bigvee Sg = Sf \bigvee Sg$. By Lemma 4.1 Θ is an equivalence relation such that $Se\Theta Sf$ implies $Se\phi_x \Theta Sf\phi_x$ for all $x \in S$. Let $Se \bigvee Sg = Sf \bigvee Sg$. By $Sg \in J$, i.e., let $Se\Theta Sf$. Note that

$$(Sf \bigvee Sg)\phi_x^+ \in ([(0), Sf] \bigvee J)\widehat{\phi}_x^+ = [(0), Sf\phi_x^+] \bigvee J$$

and $(Se \lor Sg)\phi_x^+ \in ([(0), Se] \lor J)\hat{\phi}_x^+ = [(0), Se\phi_x^+] \lor J$. Thus there are $Sh, Sh' \in J$ such that

$$Se\phi_x^+ \subseteq (Se \lor Sg)\phi_x^+ \subseteq Se\phi_x^+ \lor Sh$$

and

$$Sf\phi_x^+ \subseteq (Sf \lor Sg)\phi_x^+ \subseteq Sf\phi_x^+ \lor Sh'$$

Thus $Se\phi_x^+ \bigvee Sh = (Se \bigvee Sg)\phi_x^+ \bigvee Sh$ and $Sh' \bigvee (Sf \bigvee Sg)\phi_x^+ = Sf\phi_x^+ \bigvee Sh'$. It follows that $Sf\phi_x^+ \bigvee (Sh \bigvee Sh') = Se\phi_x^+ \bigvee (Sh \lor Sh')$ and since $Sh \bigvee Sh' \in J$ we have $Se\phi_x^+ \Theta Sf\phi_x^+$. Thus by Lemma 3.3 Θ is a con-

gruence compatible with S.

Note that in the proof of Theorem 4.7 the only use made of the hypothesis $(I \bigvee J)\hat{\phi}_x^+ = I\hat{\phi}_x^+ \bigvee J$ was for *I* a principal ideal. This observation together with Lemma 4.5 gives.

COROLLARY 4.8. Let J be an ideal such that $J\phi_x \subseteq J$ for all $x \in S$. Then J is compatible with S if and only if for any principal ideal $I \in I(\mathcal{L}(S))$ $(I \bigvee J)\hat{\phi}_x^+ = I\hat{\phi}_x^+ \bigvee J$ for all $x \in S$.

By Corollary 4.8 the situation with ideals compatible with S is analogus to that with standard ideals. An ideal J of a lattice L is standard if $(I \lor J) \land K = (I \land K) \lor (J \land K)$ for all $I, K \in I(L)$. By Theorem 2, p. 30, of [5] an ideal is standard if and only if the above holds for all principal ideals $I, K \in I(L)$. This similarity is not surprising since by Corollary 3.6, Theorem 4.7, and Theorem 2 of [5] any ideal compatible with S is a standard ideal. In fact the definition of ideal compatible with S is closely related to the definition of standard ideal. To see this we need the following :

LEMMA 4.9. For any $I \in I(\mathscr{L}(S))$ and any $Se \in \mathscr{L}(S)$, $I \cap [(0), Se] = I \hat{\phi}_e^+ \hat{\phi}_e$.

 $\begin{array}{ll} Proof. & \text{Clearly } I \cap [(0), Se] = \{Sf \in \mathscr{L}(S) : Sf \sqsubseteq Sg \cap Se, \text{ for some} \\ Sg \in I\}. & \text{But } Sg \cap Se = Sg\phi_e^+\phi_e \text{ so } I \cap [(0), Se] = I\hat{\phi}_e^+\hat{\phi}_e. \end{array}$

For any ideal for which $J\phi_x \subseteq J$ Corollary 4.4 gives

$$I\hat{\phi}_x^+ \bigvee J \subseteq I\hat{\phi}_x^+ \bigvee J\hat{\phi}_x^+ \subseteq (I \bigvee J)\phi_x^+$$

for all $I \in I(L)$. Now $I\hat{\phi}_x^+ \bigvee J\hat{\phi}_x^+ = (I \bigvee J)\hat{\phi}_x^+$ implies

$$(I \widehat{\phi}_x^+ igvee J \widehat{\phi}_x^+) \phi_x = I \widehat{\phi}_x^+ \widehat{\phi}_x igvee J \widehat{\phi}_x^+ \widehat{\phi}_x = (I igvee J) \widehat{\phi}_x^+ \widehat{\phi}_x$$
 .

Taking x = e with $Se \in \mathcal{L}(S)$ and applying Lemma 4.9 this becomes

 $(I \cap [(0), Se]) \bigvee (J \cap [(0), Se]) = (I \bigvee J) \cap [(0), Se]$.

Thus if we had required only $I\hat{\phi}_{e}^{+} \bigvee J\hat{\phi}_{e}^{+} = (I \bigvee J)\hat{\phi}_{e}^{+}$ for all e such that $Se \in \mathscr{L}(S)$ we would have J a standard ideal. However, to define an ideal compatible with S we require the stronger condition that $I\hat{\phi}_{x}^{+} \bigvee J = (I \bigvee J)\hat{\phi}_{x}^{+}$ and not only for all idempotents x such that $Sx \in \mathscr{L}(S)$ but for all $x \in S$.

5. Compatible elements. An element a of a lattice L is called standard if $x \wedge (a \vee y) = (x \wedge a) \vee (x \wedge y)$ for all $x, y \in L$. By Lemma 4, p. 32 of [5] an element is standard if and only if the principal

ideal it generates is a standard ideal.

DEFINITION 5.1. An element Se of $\mathcal{L}(S)$ is compatible with S if [(0), Se] is an ideal compatible with S. Let Θ_{Se} denote the congruence compatible with S having [(0), Se] as kernel.

Note that by Corollary 3.6 every element compatible with S is a standard element of $\mathscr{L}(S)$.

It will be convenient to look at co-kernels of congruences compatible with S.

LEMMA 5.2. Let Θ be a congruence compatible with S. Then $LR(x) \in \ker \Theta$ if and only if $L(x) \in \operatorname{co-ker} \Theta$.

Proof. Let $Sf = LR(x) \in \ker \Theta$. Then $Sf\phi_x^+\Theta(0)\phi_x^+ = L(x)$. But $Sf\phi_x^+ = L(xf^r)$ and since $f^rS = R(f) = R(Sf) = RLR(x) = R(x)$ we have $Sf\phi_x^+ = L(0) = S$. Thus $L(x) \in \operatorname{co-ker} \Theta$. Conversity let $L(x)\Theta S$ and note that $L(x)\phi_x = LRL(x)\phi_x = LR(x^l)\phi_x = LR(x^l)\phi_x = LR(x^l)\phi_x$.

$$(0) = L(x)\phi_x \Theta S\phi_x = LR(x), \quad \text{i.e.,} \quad LR(x) \in \ker \Theta .$$

LEMMA 5.3. Let Se be compatible with S. Then Seⁱ is a complement of Se and $[Se^i, S] = \operatorname{co-ker} \Theta_{Se}$.

Proof. Clearly $Se \cap Se^{i} = (0)$. By Lemma 5.2, $Se^{i}\Theta_{se}S$ so, by Theorem 3.5, $Se^{i} \bigvee Se = S$. Thus we clearly have $[Se^{i}, S] \subseteq \text{co-ker } \Theta_{se}$. Let $Sf \in \text{co-ker } \Theta_{se}$. Then $Sf \bigvee Se = S$ and since Se is standard we have $Se^{i} = Se^{i} \cap (Sf \bigvee Se) = (Se^{i} \cap Sf) \bigvee (Se^{i} \cap Se) = Se^{i} \cap Sf$. Thus $Se^{i} \subseteq Sf$, i.e., co-ker $\Theta_{se} = [Se^{i}, S]$.

We now wish to characterise elements compatible with S.

LEMMA 5.4. Let Se be compatible with S. Then e is central in S and eS = RL(e).

Proof. By Lemma 5.2, $Se^{l} \in \text{co-ker } \Theta_{Se}$. Since $Se^{l} = LR(e^{l}) = L(e^{lr})$ applying Lemma 5.2 again gives $LR(e^{lr}) \in \text{ker } \Theta_{Se}$, i.e., $LR(e^{lr}) \subseteq Se$. Thus $e^{lr} = e^{lr}e$. But $e^{l}e = 0$ implies $e \in R(e^{l})$ so $e = e^{lr}e$. Thus $e = e^{lr}$ so $eS = R(e^{l}) = RL(e)$. By Lemma 5.3, $Se^{l}\phi_{x}^{+} \supseteq Se^{l}$ and $Se^{l}\phi_{x}^{+} = L(xe^{lr}) = L(xe)$. Thus $RL(xe) \subseteq R(e^{l}) = RL(e) = eS$ so xe = exe. But $Se\phi_{x} \subseteq Se$ so ex = exe = xe, i.e., e is central in S.

We can use any central idempotent of S to induce an A - P congruence on S as follows:

LEMMA 5.5. Let e be central in S and define a relation ρ on S

by xpy if and only if xe = ye. Then ρ is an A - P congruence on S and ker $\rho = Se^i$.

Proof. Clearly ρ is a congruence on S. Let $y/\rho \in L(x/\rho)$. Then $0/\rho = (y/\rho)(x/\rho) = (yx)/\rho$ so yxe = 0. But yxe = (ye)x so $ye \in L(x)$. Thus ye = (ye)e gives $y/\rho = (ye)/\rho \in L(x)/\rho$. Similarly $R(x/\rho) \subseteq R(x)/\rho$. Clearly $x/\rho = 0/\rho$ if and only if $x \in L(e) = Se^i$.

LEMMA 5.6. If e is central in S then Se^{i} is compatible with S.

Proof. Since e is central $x \rho y$ if and only if xe = ye is an A - P congruence with kernel Se^i . By Lemma 3.8, $LR(x) \in \ker \Theta_{\rho}$ if and only if $x \in Se^i$. But $x \in Se^i$ if and only if $x = xe^i$ if and only if $LR(x) \subseteq Se^i$. Thus ker $\Theta_{\rho} = [(0), Se^i]$ so that Se^i is compatible with S.

We can now characterise elements compatible with S as follows:

THEOREM 5.7. Let $Se \in \mathcal{L}(S)$. Then Se is compatible with S if and only if e is central in S.

Proof. Let e be central in S. By Lemma 5.6, Se^{i} is compatible with S. Now L(e) = R(e) so $Se^{i} = e^{r}S$. Thus $e^{i} = e^{r}e^{i} = e^{r}$. By Lemma 5.6, $Se^{i} = Se^{r}$ compatible with S gives Se^{ri} compatible with S. But $Se^{ri} = LR(e^{ri}) = LR(e) = Se$. Thus Se is compatible with S. The converse is Lemma 5.4.

Note that Se is compatible with S if and only if Se^i is compatible with S. Thus, by Lemma 5.3, if either Se or Se^i is compatible with S then Se and Se^i are standard elements of $\mathscr{L}(S)$ which are complements. Thus by Theorem 7.3, p. 300, of [6] we have.

THEOREM 5.8. If either Se or Se^{i} is compatible with S then:

- (i) Both Se and Seⁱ are compatible with S.
- (ii) Both Se and Seⁱ are central in $\mathcal{L}(S)$.
- (iii) Θ_{s_e} and $\Theta_{s_e^l}$ are complements in $\Theta(\mathscr{L}(S))$.

COROLLARY 5.9. Let $Se \in \mathcal{L}(S)$. Then if e is central in S, Se is central in $\mathcal{L}(S)$.

5. The lattice of compatible congruences. From the formula for meet and join in $\theta(L)$ (see Theorem 1.7) it is clear that both the meet and the join of any family of congruences compatible with S are congruences compatible with S. Thus, applying Theorem 1.8, we have.

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THEOREM 6.1. The lattice $\mathcal{O}_{s}(\mathscr{L}(S))$ of all congruence compatible with S is a subcomplete sublattice of $\mathcal{O}(\mathscr{L}(S))$. Thus $\mathcal{O}_{s}(\mathscr{L}(S))$ is an uppercontinuous distributive lattice.

It follows from [1], Theorem 15, p. 147, that $\theta_s(\mathscr{L}(S))$ is pseudocomplemented. If $\theta \in \theta_s(\mathscr{L}(S))$ we shall use θ^* to denote the pseudocomplent of θ in $\theta(\mathscr{L}(S))$ and θ' to denote the pseudo-complement of θ in $\theta_s(\mathscr{L}(S))$.

In [9], Theorem 4.17 (iii), it is shown that for a complete relatively complemented lattice $L, \Theta(L)$ is a *Stone lattice* in the sense that every pseudo-complement has a complement. The remainder of this section is devoted to showing that for suitable choice of S, $\Theta_{S}(\mathscr{L}(S))$ is a Stone lattice.

We first look at the left and right annihilators of the kernel of an A - P congruence.

LEMMA 6.2. Let ρ be an A - P congruence on S and $J = \ker \rho$. Then L(J) = R(J).

Proof. Let $x \in J$ and $y \in L(J)$. If $z \in J$ then xyz = 0. Thus $J \subseteq R(xy)$ so that $L(J) \supseteq LR(xy)$. Let LR(xy) = Sf and note that $f \in L(J)$. Since J is an ideal, $xy \in J$, i.e., $xy/\rho = 0/\rho$. Thus

$$f/
ho \in LR(xy)/
ho = LR(xy/
ho) = LR(0/
ho) = (0/
ho)$$

so $f \in J$. But then we have $f \in J \cap L(J)$ so $f = f^2 = 0$. This gives LR(xy) = (0) which implies xy = 0. Thus $L(J) \subseteq R(J)$. By symmetry $R(J) \subseteq L(J)$ so R(J) = L(J).

Recall that a semigroup S is a complete Baer semigroup if the left and right annihilators of an arbitrary subset of S are principal left and right ideals generated by idempotents. Also (Theorem 1.6) as S ranges over all complete Baer semigroups $\mathscr{L}(S)$ ranges over all complete lattices.

LEMMA 6.3. Let S be a complete Baer semigroup, Θ a congruence compatible with S, and Se = \cap co-ker Θ . Then Se is compatible with S.

Proof. Let $J = \ker \lambda_{\theta}$. By Lemmas 5.2 and 3.12, $x \in J$ if and only if $L(x) \in \operatorname{co-ker} \theta$. Thus $L(J) \subseteq Se$ since $L(J) \subseteq L(x)$ for all $x \in J$. But $Se \subseteq L(x)$ for all $x \in J$ gives $Se \subseteq L(J)$. Thus Se = L(J). Now by Lemma 6.2, L(J) = R(J) and since S is a complete Baer semigroup there is an idempotent $f \in S$ such that fS = R(J). Then

fS = Se so e = fe = f. Since Se = eS is an ideal we have ex = exe = xe for all $x \in S$. Thus e is central in S so by Theorem 5.7, Se is compatible with S.

We can now characterise the kernel of the pseudo-complement of a congruence compatible with a complete Baer semigroup.

THEOREM 6.4. Let S be a complete Baer semigroup and Θ a congruence compatible with S. Then ker Θ^* is a principal ideal generated by an element of $\mathscr{L}(S)$ which is compatible with S.

Proof. Let $Se = \cap$ co-ker θ and $J = \ker \lambda_{\theta}$. By Lemma 6.3, Se is compatible with S. But Se = L(J) = R(J) and $x \in J$ if and only if $LR(x) \in \ker \theta$ gives $Se \cap Sf = (0)$ for all $Sf \in \ker \theta$. Thus ker $\theta_{Se} \cap \ker \theta = (0)$ so by Theorem 3.5, $\theta_{Se} \wedge \theta = \omega$. By definition of pseudo-complement we have $\theta_{Se} \leq \theta^*$ so $[(0), Se] = \ker \theta_{Se} \subseteq \ker \theta^*$. Now let $Sg \in \text{co-ker }\theta$ and $Sf \in \ker \theta^*$. Then $(Sf \cap Sg)\theta(Sf \cap S) = Sf$ and $(Sf \cap Sg)\theta^*(0)$. Since $(0)\theta^*Sf$ we have $(Sf \cap Sg) \equiv Sf(\theta \wedge \theta^*)$. This gives $Sf \cap Sg = Sf$ so $Sf \subseteq Sg$. Thus $Sf \subseteq Se$ and $\ker \theta^* \subseteq [(0), Se]$. We, therefore, have $\ker \theta^* = [(0), Se]$ and since Se is compatible with S this completes the proof.

We clearly have $\Theta' \subseteq \Theta^*$. Since ker Θ^* is a principal ideal generated by an element Se compatible with S, it is clear that $\Theta' = \Theta_{Se}$. By Theorem 5.8, Seⁱ is compatible with S and Θ_{Se^i} is a complement of Θ_{Se} in $\Theta_S(\mathscr{L}(S))$.

THEOREM 6.5. Let S be a complete Baer semigroup. Then $\Theta_{s}(\mathscr{L}(S))$ is a Stone lattice.

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