ON MINIMAL COMPLEXES

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An $n$-complex $K$ is called p.w.l. minimal in $E^d$ if each proper subcomplex of $K$ is p.w.l. is embeddable in $E^d$. The main purpose of this paper is to prove that for each $n \geq 2$, and each $d, n + 1 \leq d \leq 2n$, there are countably many nonhomeomorphic $n$-complexes, each one of which is p.w.l. minimal in $E^d$ and is not p.w.l. embeddable there. From general position arguments it follows that if an $n$-complex $K$ is p.w.l. minimal in $E^{2n}$, then for each $x \in |K|$, $|K| - \{x\}$ is embeddable topologically in $E^{2n}$; if an $n$-complex $K$ is p.w.l. minimal in $E^{n+d}$ and is not embeddable there, then the dimension of each maximal simplex of $K$ is at least $d$.

Here $E^d$ denotes the Euclidean $d$-space, an $n$-complex is a finite $n$-dimensional simplicial complex. $|K|$ denotes the underlying point set of the complex $K$ in some $E^d$, and in case where there is no confusion, $|K|$ will be replaced by $K$. $C^m_n$ denotes the complete $n$-complex with $m$ vertices.

A subset $X$ of $E^d$ is called cellular if there exists a sequence $\{Q_i\}_{i=1}^\infty$ of closed $d$-cells, such that $Q_{i+1} \subset \text{Int} Q_i$, for each $i$, and $X = \bigcap_{i=1}^\infty Q_i$; where Int means interior.

A cellular decomposition $G$ of $E^n$ is an upper semicontinuous (u.s.c.) decomposition of $E^n$, such that each element of $G$ is cellular; an u.s.c. decomposition is finite if it has only finitely many nondegenerate elements, see [1].

2. There are precisely two 1-complexes which are p.w.l. minimal in $E^2$ and are not topologically embeddable there: these are the two Kuratowski's nonplanar graphs, [6].

B. Grünbaum proved in [3] that all the $n$-complexes of certain form are not embeddable in $E^{2n}$, and that one of them, for each $n$, is geometrically minimal in $E^{2n}$, where the geometrically minimal in $E^d$ means that each proper subcomplex can be rectilinearly (= affine on each simplex) embedded in $E^d$. All of these $n$-complexes were proved by J. Zaks, in [10], to be p.w.l. minimal in $E^{2n}$, and in certain cases, for each $n$, to be geometrically minimal there. B. Grünbaum proved in [4] that, indeed, each one of these $n$-complexes is geometrically minimal in $E^{2n}$.

However, the number of these $n$-complexes is finite, for each $n$. Related to these results, we have the following.
THEOREM 1. For each \( n \geq 2 \), there are countably many non-homeomorphic \( n \)-complexes, each one of which is p.w.l. minimal in \( E^{2n} \) and is not p.w.l. embeddable there.

Moreover, we extend this to a

COROLLARY 1. For each \( n \geq 2 \) and each \( d, n + 1 \leq d \leq 2n \), there are countably many nonhomeomorphic \( n \)-complexes, each one of which is p.w.l. minimal in \( E^d \) and is not p.w.l. embeddable there.

3. Proof of Theorem 1. For the proof of this theorem, we need certain lemmas, which seem to be obvious; our proofs make use of some heavy techniques from combinatorial topology, see [8], [9] and [11].

LEMMA 1. A polyhedral disk \( D \) in \( E^n \) is cellular.

Proof. Let \( K \) be a triangulation of \( D \). There exists a triangulation \( T \) of \( E^n \) and a subdivision \( K' \) of \( K \) such that \( K' \) is a subcomplex of \( T \). The disk \( D \) with the triangulation \( K' \) can be shelled, by [8], hence \( K' \) collapses to a triangle, and therefore \( K' \) is collapsible, see [9], [11]. Using a theorem of J.H.C. Whitehead, [9], it follows that \( \text{st}(\beta^2 K', \beta^2 T) \)—the star of \( K' \) in \( T \), taken in the second barycentric subdivision of \( T \)—is an \( n \)-cell. Let \( Q_i = \text{st}(\beta^2 K', \beta^2 T) \), then all the \( Q_i - s \) are \( n \)-cells, \( Q_{i+1} \subset \text{Int} Q_i \) and \( D = \bigcap_{i=1}^{n} Q_i \). Therefore \( D \) is cellular.

LEMMA 2. If \( G \) is a finite cellular decomposition of \( E^n \), then the decomposition space \( E^n/G \) of \( G \) is homeomorphic to \( E^n \).

This lemma is a particular and simple case of L. V. Keldysh's Theorem 1 of [5], because of the finiteness of \( G \). We would like to mention the difference between the usual definition of cellularity, and that of [5]. Theorem 1 of [5] was proved later as part of Theorem 1.4 of [7].

It follows from Lemma 2 that if \( \alpha \) is a polyhedral simple (closed-) arc in the interior of an \( n \)-simplex \( \delta^n \) in \( E^d \), then the space obtained from \( \delta^n \) by shrinking \( \alpha \) to a point ("\( \delta^n \) modulo \( \alpha' \)) is homeomorphic to \( \delta^n \). This will be stated as

LEMMA 3. Let \( \delta^n \) be an \( n \)-simplex in \( E^d \), and let \( G \) be a finite u.s.c. decomposition of \( \delta^n \) having only polyhedral simple arcs in \( \text{Int} \delta^n \) for its nondegenerate elements, then \( \delta^n/G \) is homeomorphic to \( \delta^n \).
LEMMA 4. For each $n$, $C^{n}_{2n+3}$ is not embeddable topologically in $E^{2n}$; however, there exists a maps $f: C^{n}_{2n+3} \to E^{2n}$, which is affine on each simplex, and has only one inverse set, which contains only two points. (An inverse set of a map $f: X \to Y$ is $f^{-1}(f(x))$, provided $f^{-1}(f(x)) \neq \{x\}$.)

The nonembeddability of $C^{n}_{2n+3}$ in $E^{2n}$ is a well known result, due to A. Flores [2], and the map $f$ is described in [3], [4] (see also [10]).

LEMMA 5. For each $n$ and each point $x \in C^{n}_{2n+3}$, $|C^{n}_{2n+3} \setminus \{x\}$ is embeddable in $E^{2n}$.

This lemma will later be extended, see Theorem 2.

Proof. In the case where $x$ is an interior point of some $n$-simplex, we can use the map $f$ as given in Lemma 4. Otherwise, let $V_x$ be a small neighborhood of $x$ in $C^{n}_{2n+3}$. By pushing each point of $V_x \setminus \{x\}$ away from $x$, it follows that $|C^{n}_{2n+3} \setminus \{x\}$ is homeomorphic to a subset of $|C^{n}_{2n+3} \setminus \{y\}$, where $y$ is an interior point of some $n$-simplex, hence, by the first part of this proof, $C^{n}_{2n+3} \setminus \{y\}$ is embeddable in $E^{2n}$, and therefore $|C^{n}_{2n+3} \setminus \{x\}$ is embeddable there, too. This completes the proof of Lemma 5.

Proof of Theorem 1. For each $n \geq 2$, let us first define inductively a sequence $(K^{n}(m))_{m=1}^{m}$ of $n$-complexes as follows: let $\delta^{n}$ be a fixed $n$-simplex of $C^{n}_{2n+3}$. $K^{n}(1)$ is obtained from $C^{n}_{2n+3}$ as follows:

Step 1. Subdivide $C^{n}_{2n+3}$ in such a way that $\delta^{n}$ will contain as a subcomplex a simple arc $A^{s}_{1} A^{s}_{2} A^{s}_{3} A^{s}_{4}$, consisting of three edges, all of them in Int $\delta^{n}$, and both of $A^{s}_{1}$ and $A^{s}_{4}$ are in the star of no vertex in the new complex.

Step 2. Identify $A^{s}_{1} = A^{s}_{4}$.

Step 3. Add a new triangle $B$, having the new circuit $A^{s}_{2} A^{s}_{3} A^{s}_{4}$ as its boundary.

$K^{n}(m)$ is obtain from $K^{n}(m-1)$ by a similar way, where we pick the new arc of Step 1 to be disjoint from all the previously added triangles $B$ of Step 3, and keep the triangles $B$ of Step 3 untouched.

Since $n \geq 2$, and we add only 2-simplexes, $K^{n}(m)$ is an $n$-complex.

Main claim. For each $m$, $K^{n}(m)$ is not p.w.l. embeddable in $E^{2n}$.

Proof. Suppose this is false, then for some $n$ and some $m$ we have a p.w.l. embedding $f$
Let \(B_1, \ldots, B_m\) be the added triangles of \(K^n(m)\), as described in Step 3, and let \(G\) be the decomposition of \(E^{2n}\), having \(f(B_i), 1 \leq i \leq m\), as the only non-degenerate elements.

By Lemma 1, each \(f(B_i)\) is cellular in \(E^{2n}\), since \(f\) is a p.w.l. embedding; therefore \(G\) is a cellular decomposition of \(E^{2n}\), and it is finite, hence by Lemma 2 there exists a homeomorphism \(h: E^{2n}/G \rightarrow E^{2n}\). Let \(p: E^{2n} \rightarrow E^{2n}/G\) be the natural projection, related to the decomposition \(G\).

Let \(g: C^{n+3}_{2n} \rightarrow K^n(m)\) be the map which identifies the \(m\) pairs of points, as described in Step 2, and is the identity elsewhere.

In the following diagram

\[
C^{n+3}_{2n} \xrightarrow{g} K^n(m) \xrightarrow{f} E^{2n} \xrightarrow{p} E^{2n}/G \xrightarrow{h} E^{2n},
\]

the map \(pf\) shrinks the \(m\) polygonal simple arcs, as described in Step 1, each one to a point, hence \(pf(\delta^n)\) is an \(n\)-cell, by Lemma 3, therefore \(pf(C^{n}_{2n+3})\) is homeomorphic to \(C^n_{2n+3}\), and as a result \(hpf(C^{n}_{2n+3})\) is a subset of \(E^{2n}\) which is homeomorphic to \(C^{n}_{2n+3}\). This contradictsLemma 4, and hence completes the proof of the main claim.

Next, for each \(n \geq 2\), let \(\{K^n(m)\}_{m=1}^{\infty}\) be the sequence, obtained from \(\{K^n(m)\}_{m=1}^{\infty}\) as follows: if \(K^n(m)\) is p.w.l. minimal in \(E^{2n}\), we let \(\tilde{K}^n(m) = K^n(m)\); otherwise we define \(\tilde{K}^n(m)\) to be a subcomplex of \(K^n(m)\) which is not p.w.l. embeddable in \(E^{2n}\), and is p.w.l. minimal there. Using Lemmas 4 and 5, and the fact that our construction of \(K^n(m)\) from \(C^{n+3}_{2n+3}\) can be performed in a small neighborhood of any point of \(C^{n+3}_{2n+3}\), it follows that the only simplexes of \(K^n(m)\) which are not in \(\tilde{K}^n(m)\) are triangles, among the ones added in Step 3. In particular, no point, which is the identification of two points of \(\delta^n\), by Step 2, can be deleted. These \(m\) points of \(\tilde{K}^n(m)\) have neighborhoods which are topologically different from neighborhoods of other points of \(K^n(m)\); therefore if \(m \neq m', \tilde{K}^n(m)\) and \(\tilde{K}^n(m')\) are not homeomorphic, and the proof of Theorem 1 is completed.

From Corollary 2 it will follow that for \(n \geq 3\), no one of the \(m\) added triangles of \(K^n(m)\), by Step 3, appears in \(\tilde{K}^n(m)\), and \(\tilde{K}^n(m)\) is just the result of identifying \(m\) pairs of points in \(\text{Int} \, \delta\), each pair to a point. Probably, this is the case for \(n = 2\), too.

In order to obtain other \(n\)-complexes, each one of which is p.w.l. minimal in \(E^{2n}\) and is not p.w.l. embeddable there, for \(n \geq 2\), we can use more than just one \(n\)-simplex of \(C^{n}_{2n+3}\), or we can take, to begin with, any other \(n\)-complex from the list in [3], since they all share the needed properties that \(C^{n}_{2n+3}\) does, by [3], [4], [10] and Theorem 2, here. Moreover, we can identify more than two point in our Step
Proof of Corollary 1. Let us first observe that if a complex $K$ is not p.w.l. embeddable in $E^d$, then $K_{VC_i}$ is not p.w.l. embeddable in $E^{d+1}$ and $KVC_{n-1}$ is not p.w.l. embeddable in $E^{d+n}$, where $KVL$ is the join complex of $K$ and $L$, see [11]. Moreover, if $K$ is p.w.l. minimal in $E^d$, then $KVC_i$ is p.w.l. minimal in $E^{d+1}$, by [10], and therefore $KVC_{n-1}$ is p.w.l. minimal in $E^{d+n}$.

Let $n$ and $d$ be given, where $n \geq 2$ and $n + 2 \leq d \leq 2n$. A sequence $\{L^{n,d}(m)\}_{m=1}^{\infty}$ of nonhomeomorphic $n$-complexes, each one of which being p.w.l. minimal in $E^d$ and not p.w.l. embeddable there, can be obtained as follows: $L^{n,d}(m) = \hat{K}^{d-n}(m)VC_{2n-d-1}$, where $\hat{K}^{d-n}(m)$ is given by Theorem 1, which is applicable since $d - n \geq 2$.

$L^{n,d}(m)$ is an $n$-complex, because $(d - n) + (2n - d - 1) + 1 = n$; it is not p.w.l. embeddable in $E^d$, because $\hat{K}^{d-n}(m)$ is not p.w.l. embeddable in $E^{2(d-n)}$, and $2(d - n) + (2n - d) = d$.

For the case where $n \geq 2$ and $d = n + 1$, it is obviously enough to deal with $n = 2$ and $d = 3$: $L^{2,3}(m)$ is the following complex: We take a triangulated orientable closed 2-manifold of genus $m$, which contains the shape "X" as a subcomplex, having $OA, OB, OC$ and $OD$ as edges, where $A, B, C, D$ are in a clockwise order. We add two new vertices $P$ and $Q$, the four triangles $POA$, $POC$, $QOB$, $QOD$, together with their faces, and we add the edge $PQ$. It is very easy to verify that $L^{2,3}(m)$ is not p.w.l. embeddable in $E^3$, and that it is p.w.l. minimal there. (Moreover, if $x$ is an interior point of one of the added triangles, then $L^{2,3}(m) - \{x\}$ is still not embeddable in $E^3$. Compare this comment with Theorem 2.)

4. The following is an extension to Lemma 5:

**Theorem 2.** If an $n$-complex $K$ is p.w.l. minimal in $E^{2n}$, then for each point $x \in |K|$, $|K| - \{x\}$ is embeddable in $E^{2n}$.

**Proof.** As it was shown in the proof of Lemma 5, we can assume, without loss of generality, that $x \in \text{Int} \delta$, where $\delta$ is a maximal simplex of $K$. Since $K$ is p.w.l. minimal in $E^{2n}$, let $f: |K - \delta| \to E^{2n}$ be a p.w.l. and general position embedding. Let $A \in E^{2n}$ be in general position with respect to $f(|K - \delta|)$.

Let $F: |K| \to E^{2n}$ be defined as follows

\[
F(z) = \begin{cases} 
  f(z) & \text{if } z \in |K - \delta| \\
  \lambda A + (1 - \lambda)f(z') & \text{if } z \in \text{Int} \delta \text{ and } z = \lambda b_\delta + (1 - \lambda)z', \\
    \end{cases}
\]

where $b_\delta$ is the barycenter of $\delta$, and $z' \in |Bd\delta|$, $0 < \lambda \leq 1$. 

$F$ is a well defined immersion (= locally embedding) of $|K|$, and its singularities are those of $F(|\partial \delta|)$, together with the possible intersections of $F(|\partial \delta|)$ with $F(|K-\delta|)$. Let $s$ be the dimension of $\delta$, then from general position arguments it follows that the dimensions of these singularities are either $\leq 2s - 2n$ or $\leq s + t - 2n$, for some $1 \leq t \leq n$, and since $s \leq n$, they are $\leq 0$. Hence the singularities of $F$ consists of finitely many points, each point $z$ of which has at least one point of $F^{-1}(z)$ in $\text{Int}\ \delta$.

Therefore, there exists a $t$, $0 < t < 1$, such that $F$ is an embedding when restricted to 

$$|K| - \{\lambda b_t + (1 - \lambda)x \mid t < \lambda \leq 1 \text{ and } x \in |Bd\delta|\},$$

which is homeomorphic to $|K| - \{x\}$, and the proof is completed.

**Corollary 2.** If an $n$-complex $K$ is p.w.l. minimal in $E^{n+d}$ and is not embeddable there, then the dimension of each maximal simplex of $K$ is at least $d$.

In particular, if $d = n$, then these dimensions are equal to $n$.

**Proof.** Let $\delta$ be a maximal $s$-simplex of $K$, which among all the maximal simplexes of $K$ is of minimal dimension.

Let $F: |K| \rightarrow E^{n+d}$ be the extension of a p.w.l. and general position embedding of $K-\delta$ in $E^{n+d}$, similar to the one described in the proof Theorem 2. The dimensions of the singularities of $F$ are $\leq s + t - (n + d)$, with $s \leq t \leq n$; however it is never $\leq - 1$ because $K$ is not embeddable in $E^{n+d}$. Therefore

$$\min_{s \leq t \leq n} [s + t - (n + d)] = 2s - n - d \geq 0$$

and since $s \leq n$ it follows that $s \geq d$, and the proof is completed.

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