

# Pacific Journal of Mathematics

**ON MINIMAL COMPLEXES**

JOSEPH ZAKS

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An  $n$ -complex  $K$  is called p.w.l. minimal in  $E^d$  if each proper subcomplex of  $K$  is p.w.l. embeddable in  $E^d$ . The main purpose of this paper is to prove that for each  $n \geq 2$ , and each  $d$ ,  $n + 1 \leq d \leq 2n$ , there are countably many nonhomeomorphic  $n$ -complexes, each one of which is p.w.l. minimal in  $E^d$  and is not p.w.l. embeddable there. From general position arguments it follows that if an  $n$ -complex  $K$  is p.w.l. minimal in  $E^{2n}$ , then for each  $x \in |K|$ ,  $|K| - \{x\}$  is embeddable topologically in  $E^{2n}$ ; if an  $n$ -complex  $K$  is p.w.l. minimal in  $E^{n+d}$  and is not embeddable there, then the dimension of each maximal simplex of  $K$  is at least  $d$ .

Here  $E^d$  denotes the Euclidean  $d$ -space, an  $n$ -complex is a finite  $n$ -dimensional simplicial complex.  $|K|$  denotes the underlying point set of the complex  $K$  in some  $E^d$ , and in case where there is no confusion,  $|K|$  will be replaced by  $K$ .  $C_m^n$  denotes the complete  $n$ -complex with  $m$  vertices.

A subset  $X$  of  $E^d$  is called *cellular* if there exists a sequence  $\{Q_i\}_{i=1}^\infty$  of closed  $d$ -cells, such that  $Q_{i+1} \subset \text{Int } Q_i$ , for each  $i$ , and  $X = \bigcap_{i=1}^\infty Q_i$ ; where  $\text{Int}$  means interior.

A *cellular decomposition*  $G$  of  $E^n$  is an upper semicontinuous (u.s.c.) decomposition of  $E^n$ , such that each element of  $G$  is cellular; an u.s.c. decomposition is finite if it has only finitely many nondegenerate elements, see [1].

2. There are precisely two 1-complexes which are p.w.l. minimal in  $E^2$  and are not topologically embeddable there: these are the two Kuratowski's nonplanar graphs, [6].

B. Grünbaum proved in [3] that all the  $n$ -complexes of certain form are not embeddable in  $E^{2n}$ , and that one of them, for each  $n$ , is geometrically minimal in  $E^{2n}$ , where the geometrically minimal in  $E^d$  means that each proper subcomplex can be rectilinearly (= affine on each simplex) embedded in  $E^d$ . All of these  $n$ -complexes were proved by J. Zaks, in [10], to be p.w.l. minimal in  $E^{2n}$ , and in certain cases, for each  $n$ , to be geometrically minimal there. B. Grünbaum proved in [4] that, indeed, each one of these  $n$ -complexes is geometrically minimal in  $E^{2n}$ .

However, the number of these  $n$ -complexes is finite, for each  $n$ . Related to these results, we have the following.

**THEOREM 1.** *For each  $n \geq 2$ , there are countably many non-homeomorphic  $n$ -complexes, each one of which is p.w.l. minimal in  $E^{2n}$  and is not p.w.l. embeddable there.*

Moreover, we extend this to a

**COROLLARY 1.** *For each  $n \geq 2$  and each  $d$ ,  $n + 1 \leq d \leq 2n$ , there are countably many nonhomeomorphic  $n$ -complexes, each one of which is p.w.l. minimal in  $E^d$  and is not p.w.l. embeddable there.*

3. *Proof of Theorem 1.* For the proof of this theorem, we need certain lemmas, which seem to be obvious; our proofs make use of some heavy techniques from combinatorial topology, see [8], [9] and [11].

**LEMMA 1.** *A polyhedral disk  $D$  in  $E^n$  is cellular.*

*Proof.* Let  $K$  be a triangulation of  $D$ . There exists a triangulation  $T$  of  $E^n$  and a subdivision  $K^1$  of  $K$  such that  $K^1$  is a subcomplex of  $T$ . The disk  $D$  with the triangulation  $K^1$  can be shelled, by [8], hence  $K^1$  collapses to a triangle, and therefore  $K^1$  is collapsible, see [9], [11]. Using a theorem of J.H.C. Whitehead, [9], it follows that  $\text{st}(\beta^2 K^1, \beta^2 T)$ —the star of  $K^1$  in  $T$ , taken in the second barycentric subdivision of  $T$ —is an  $n$ -cell. Let  $Q_i = \text{st}(\beta^{2i} K^1, \beta^{2i} T)$ , then all the  $Q_i - s$  are  $n$ -cells,  $Q_{i+1} \subset \text{Int } Q_i$  and  $D = \bigcap_{i=1}^{\infty} Q_i$ : Therefore  $D$  is cellular.

**LEMMA 2.** *If  $G$  is a finite cellular decomposition of  $E^n$ , then the decomposition space  $E^n/G$  of  $G$  is homeomorphic to  $E^n$ .*

This lemma is a particular and simple case of L. V. Keldysh's Theorem 1 of [5], because of the finiteness of  $G$ . We would like to mention the difference between the usual definition of cellularity, and that of [5]. Theorem 1 of [5] was proved later as part of Theorem 1.4 of [7].

It follows from Lemma 2 that if  $\alpha$  is a polyhedral simple (closed-) arc in the interior of an  $n$ -simplex  $\delta^n$  in  $E^d$ , then the space obtained from  $\delta^n$  by shrinking  $\alpha$  to a point (" $\delta^n$  modulo  $\alpha$ ") is homeomorphic to  $\delta^n$ . This will be stated as

**LEMMA 3.** *Let  $\delta^n$  be an  $n$ -simplex in  $E^d$ , and let  $G$  be a finite u.s.c. decomposition of  $\delta^n$  having only polyhedral simple arcs in  $\text{Int } \delta^n$  for its nondegenerate elements, then  $\delta^n/G$  is homeomorphic to  $\delta^n$ .*

LEMMA 4. For each  $n$ ,  $C_{2n+3}^n$  is not embeddable topologically in  $E^{2n}$ ; however, there exists a maps  $f: C_{2n+3}^n \rightarrow E^{2n}$ , which is affine on each simplex, and has only one inverse set, which contains only two points. (An inverse set of a map  $f: X \rightarrow Y$  is  $f^{-1}(f(x))$ , provided  $f^{-1}(f(x)) \neq \{x\}$ .)

The nonembeddability of  $C_{2n+3}^n$  in  $E^{2n}$  is a well known result, due to A. Flores [2], and the map  $f$  is described in [3], [4] (see also [10]).

LEMMA 5. For each  $n$  and each point  $x \in |C_{2n+3}^n|$ ,  $|C_{2n+3}^n| - \{x\}$  is embeddable in  $E^{2n}$ .

This lemma will later be extended, see Theorem 2.

*Proof.* In the case where  $x$  is an interior point of some  $n$ -simplex, we can use the map  $f$  as given in Lemma 4. Otherwise, let  $V_x$  be a small neighborhood of  $x$  in  $C_{2n+3}^n$ . By pushing each point of  $V_x - \{x\}$  away from  $x$ , it follows that  $|C_{2n+3}^n| - \{x\}$  is homeomorphic to a subset of  $|C_{2n+3}^n| - \{y\}$ , where  $y$  is an interior point of some  $n$ -simplex, hence, by the first part of this proof,  $|C_{2n+3}^n| - \{y\}$  is embeddable in  $E^{2n}$ , and therefore  $|C_{2n+3}^n| - \{x\}$  is embeddable there, too. This completes the proof of Lemma 5.

*Proof of Theorem 1.* For each  $n \geq 2$ , let us first define inductively a sequence  $\{K^n(m)\}_{m=1}^\infty$  of  $n$ -complexes as follows: let  $\delta^n$  be a fixed  $n$ -simplex of  $C_{2n+3}^n$ .  $K^n(1)$  is obtained from  $C_{2n+3}^n$  as follows:

Step 1. Subdivide  $C_{2n+3}^n$  in such a way that  $\delta^n$  will contain as a subcomplex a simple arc  $A_1^n A_2^n A_3^n A_4^n$ , consisting of three edges, all of them in  $\text{Int } \delta^n$ , and both of  $A_1^n$  and  $A_4^n$  are in the star of no vertex in the new complex.

Step 2. Identify  $A_1^n = A_4^n$ ,

Step 3. Add a new triangle  $B$ , having the new circuit  $A_1^n A_2^n A_3^n$  as its boundary.

$K^n(m)$  is obtain from  $K^n(m - 1)$  by a similar way, where we pick the new arc of Step 1 to be disjoint from all the previously added triangles  $B$  of Step 3, and keep the triangles  $B$  of Step 3 untouched.

Since  $n \geq 2$ , and we add only 2-simplexes,  $K^n(m)$  is an  $n$ -complex.

*Main claim.* For each  $m$ ,  $K^n(m)$  is not p.w.l. embeddable in  $E^{2n}$ .

*Proof.* Suppose this is false, then for some  $n$  and some  $m$  we have a p.w.l. embedding  $f$

$$f: K^n(m) \rightarrow E^{2n} .$$

Let  $B_1, \dots, B_m$  be the added triangles of  $K^n(m)$ , as described in Step 3, and let  $G$  be the decomposition of  $E^{2n}$ , having  $f(B_i)$ ,  $1 \leq i \leq m$ , as the only non-degenerate elements.

By Lemma 1, each  $f(B_i)$  is cellular in  $E^{2n}$ , since  $f$  is a p.w.l. embedding; therefore  $G$  is a cellular decomposition of  $E^{2n}$ , and it is finite, hence by Lemma 2 there exists a homeomorphism  $h: E^{2n}/G \rightarrow E^{2n}$ . Let  $p: E^{2n} \rightarrow E^{2n}/G$  be the natural projection, related to the decomposition  $G$ .

Let  $g: C_{2n+3}^n \rightarrow K^n(m)$  be the map which identifies the  $m$  pairs of points, as described in Step 2, and is the identity elsewhere.

In the following diagram

$$C_{2n+3}^n \xrightarrow{g} K^n(m) \xrightarrow{f} E^{2n} \xrightarrow{p} E^{2n}/G \xrightarrow{h} E^{2n} ,$$

the map  $pfh$  shrinks the  $m$  polygonal simple arcs, as described in Step 1, each one to a point, hence  $pfh(\delta^n)$  is an  $n$ -cell, by Lemma 3, therefore  $pfh(C_{2n+3}^n)$  is homeomorphic to  $C_{2n+3}^n$ , and as a result  $hpfh(C_{2n+3}^n)$  is a subset of  $E^{2n}$  which is homeomorphic to  $C_{2n+3}^n$ . This contradicts Lemma 4, and hence completes the proof of the main claim.

Next, for each  $n \geq 2$ , let  $\{\tilde{K}^n(m)\}_{m=1}^\infty$  be the sequence, obtained from  $\{K^n(m)\}_{m=1}^\infty$  as follows: if  $K^n(m)$  is p.w.l. minimal in  $E^{2n}$ , we let  $\tilde{K}^n(m) = K^n(m)$ ; otherwise we define  $\tilde{K}^n(m)$  to be a subcomplex of  $K^n(m)$  which is not p.w.l. embeddable in  $E^{2n}$ , and is p.w.l. minimal there. Using Lemmas 4 and 5, and the fact that our construction of  $K^n(m)$  from  $C_{2n+3}^n$  can be performed in a small neighborhood of any point of  $C_{2n+3}^n$ , it follows that the only simplexes of  $K^n(m)$  which are not in  $\tilde{K}^n(m)$  are triangles, among the ones added in Step 3. In particular, no point, which is the identification of two points of  $\delta^n$ , by Step 2, can be deleted. These  $m$  points of  $\tilde{K}^n(m)$  have neighborhoods which are topologically different from neighborhoods of other points of  $\tilde{K}^n(m)$ ; therefore if  $m \neq m'$ ,  $\tilde{K}^n(m)$  and  $\tilde{K}^n(m')$  are not homeomorphic, and the proof of Theorem 1 is completed.

From Corollary 2 it will follow that for  $n \geq 3$ , no one of the  $m$  added triangles of  $K^n(m)$ , by Step 3, appears in  $\tilde{K}^n(m)$ , and  $\tilde{K}^n(m)$  is just the result of identifying  $m$  pairs of points in  $\text{Int } \delta$ , each pair to a point. Probably, this is the case for  $n = 2$ , too.

In order to obtain other  $n$ -complexes, each one of which is p.w.l. minimal in  $E^{2n}$  and is not p.w.l. embeddable there, for  $n \geq 2$ , we can use more than just one  $n$ -simplex of  $C_{2n+3}^n$ , or we can take, to begin with, any other  $n$ -complex from the list in [3], since they all share the needed properties that  $C_{2n+3}^n$  does, by [3], [4], [10] and Theorem 2, here. Moreover, we can identify more than two point in our Step

2, but then we have to make some obvious alterations.

*Proof of Corollary 1.* Let us first observe that if a complex  $K$  is not p.w.l. embeddable in  $E^d$ , then  $KVC_1^0$  is not p.w.l. embeddable in  $E^{d+1}$  and  $KVC_n^{n-1}$  is not p.w.l. embeddable in  $E^{d+n}$ , where  $KVL$  is the join complex of  $K$  and  $L$ , see [11]. Moreover, if  $K$  is p.w.l. minimal in  $E^d$ , then  $KVC_1^0$  is p.w.l. minimal in  $E^{d+1}$ , by [10], and therefore  $KVC_n^{n-1}$  is p.w.l. minimal in  $E^{d+n}$ .

Let  $n$  and  $d$  be given, where  $n \geq 2$  and  $n + 2 \leq d \leq 2n$ . A sequence  $\{L^{n;d}(m)\}_{m=1}^\infty$  of nonhomeomorphic  $n$ -complexes, each one of which being p.w.l. minimal in  $E^d$  and not p.w.l. embeddable there, can be obtained as follows:  $L^{n;d}(m) = \tilde{K}^{d-n}(m)VC_{2n-d}^{2n-d-1}$ , where  $\tilde{K}^{d-n}(m)$  is given by Theorem 1, which is applicable since  $d - n \geq 2$ .

$L^{n;d}(m)$  is an  $n$ -complex, because  $(d - n) + (2n - d - 1) + 1 = n$ ; it is not p.w.l. embeddable in  $E^d$ , because  $\tilde{K}^{d-n}(m)$  is not p.w.l. embeddable in  $E^{2(d-n)}$ , and  $2(d - n) + (2n - d) = d$ .

For the case where  $n \geq 2$  and  $d = n + 1$ , it is obviously enough to deal with  $n = 2$  and  $d = 3$ :  $L^{2;3}(m)$  is the following complex: We take a triangulated orientable closed 2-manifold of genus  $m$ , which contains the shap "X" as a subcomplex, having  $OA, OB, OC$  and  $OD$  as edges, where  $A, B, C, D$  are in a clockwise order. We add two new vertices  $P$  and  $Q$ , the four triangles  $POA, POC, QOB, QOD$ , together with their faces, and we add the edge  $PQ$ . It is very easy to verify that  $L^{2;3}(m)$  is not p.w.l. embeddable in  $E^3$ , and that it is p.w.l. minimal there. (Moreover, if  $x$  is an interior point of one of the added triangles, then  $L^{2;3}(m) - \{x\}$  is still not embeddable in  $E^3$ . Compare this comment with Theorem 2.)

4. The following is an extension to Lemma 5:

**THEOREM 2.** *If an  $n$ -complex  $K$  is p.w.l. minimal in  $E^{2n}$ , then for each point  $x \in |K|$ ,  $|K| - \{x\}$  is embeddable in  $E^{2n}$ .*

*Proof.* As it was shown in the proof of Lemma 5, we can assume, without loss of generality, that  $x \in \text{Int } \delta$ , where  $\delta$  is a maximal simplex of  $K$ . Since  $K$  is p.w.l. minimal in  $E^{2n}$ , let  $f: |K - \delta| \rightarrow E^{2n}$  be a p.w.l. and general position embedding. Let  $A \in E^{2n}$  be in general position with respect to  $f(|K - \delta|)$ .

Let  $F: |K| \rightarrow E^{2n}$  be defined as follows

$$F(z) = \begin{cases} f(z) & \text{if } z \in |K - \delta| \\ \lambda A + (1 - \lambda)f(z') & \text{if } z \in \text{Int } \delta \text{ and } z = \lambda b_\delta + (1 - \lambda)z', \\ & \text{where } b_\delta \text{ is the barycenter of } \delta, \\ & \text{and } z' \in |Bd\delta|, \quad 0 < \lambda \leq 1. \end{cases}$$

$F$  is a well defined immersion (= locally embedding) of  $|K|$ , and its singularities are those of  $F(\text{Int } \delta)$ , together with the possible intersections of  $F(|\delta|)$  with  $F(|K - \delta|)$ . Let  $s$  be the dimension of  $\delta$ , then from general position arguments it follows that the dimensions of these singularities are either  $\leq 2s - 2n$  or  $\leq s + t - 2n$ , for some  $1 \leq t \leq n$ , and since  $s \leq n$ , they are  $\leq 0$ . Hence the singularities of  $F$  consists of finitely many points, each point  $z$  of which has at least one point of  $F^{-1}(z)$  in  $\text{Int } \delta$ .

Therefore, there exists a  $t$ ,  $0 < t < 1$ , such that  $F$  is an embedding when restricted to

$$|K| - \{\lambda b_s + (1 - \lambda)x \mid t < \lambda \leq 1 \text{ and } x \in |\text{Bd } \delta|\},$$

which is homeomorphic to  $|K| - \{x\}$ , and the proof is completed.

**COROLLARY 2.** *If an  $n$ -complex  $K$  is p.w.l. minimal in  $E^{n+d}$  and is not embeddable there, then the dimension of each maximal simplex of  $K$  is at least  $d$ .*

In particular, if  $d = n$ , then these dimensions are equal to  $n$ .

*Proof.* Let  $\delta$  be a maximal  $s$ -simplex of  $K$ , which among all the maximal simplexes of  $K$  is of minimal dimension.

Let  $F: |K| \rightarrow E^{n+d}$  be the extension of a p.w.l. and general position embedding of  $K - \delta$  in  $E^{n+d}$ , similar to the one described in the proof Theorem 2. The dimensions of the singularities of  $F$  are  $\leq s + t - (n + d)$ , with  $s \leq t \leq n$ ; however it is never  $\leq -1$  because  $K$  is not embeddable in  $E^{n+d}$ . Therefore

$$\min_{s \leq t \leq n} [s + t - (n + d)] = 2s - n - d \geq 0$$

and since  $s \leq n$  it follows that  $s \geq d$ , and the proof is completed.

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