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CRITICAL POINTS ON RIM-COMPACT SPACES

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In this note we prove that all the points of a rim-compact space X at which X is not locally compact are critical points for any local dynamical system defined on X. When a local system is global this result is obtained by extending the global system π on X to a global system ρ on the Freudenthal compactification Y of X, then showing that Y-X is a critical set for ρ, and, finally, observing that $\overline{Y\text{-}X}$ contains all the points of X at which X is not locally compact. This weaker result will appear in the author's doctoral dissertation and requires the use of general extension theorems proven there. For this paper, we isolate those parts of the thesis which are pertinent to our theorem.

DEFINITION 1. [1]. A (continuous) local dynamical system on a topological space Z is an object π satisfying the following conditions: (R is the set of real numbers with the usual topology)

(1) π is a continuous partial map from $Z \times R$ into Z.

(2) For every z in Z there are (bounds) α_z and ω_z such that $-\infty \leqq \alpha_z < 0$, $0 < \omega_z \leqq +\infty$ and $\pi(z, t)$ is defined if and only if t is in (α_z, ω_z).

(3) The domain of π is open in $Z \times R$.

(4) For each z in Z, $\pi(z, 0) = z$.

(5) When $\pi(z, t)$ is defined, $\pi(\pi(z, t), t') = \pi(z, t+t')$ whenever either side is defined.

A local dynamical system π on Z is global if $\alpha_z = -\infty$ and $\omega_z = +\infty$ for all $z \in Z$. Condition (4) is called the initial value condition, and condition (5) the additive condition. Because of the additive condition, it has become conventional to write $\pi(z, t)$ as $z\pi t$. Thus the equality in condition (5) is written $(z\pi t)\pi t' = z\pi(t + t')$. To prove that a point z is a critical point of π, it suffices to show there is an $\varepsilon > 0$ such that $z = z\pi t$ for all $t \in [0, \varepsilon)$.

DEFINITION 2. A topological space X is said to be rim-compact if and only if it is T_2 and each point of X has a fundamental system of neighborhoods with compact boundaries.

In this paper X denotes a rim-compact space and Y denotes the Freudenthal compactification of X, ([2], p. 111). Y is T_2 and every point in Y has a fundamental system of neighborhoods with compact boundaries entirely in X. These and compactness are the only properties of Y that will be used. Convergence of a net $\{y_i\}$ indexed by the directed set I is written as $\{y_i\} \xrightarrow{i} y$; thus a net $\{y^i_j \mid j \in J^i\}$, indexed by J^i, converging to the point y_i is written as $\{y^i_j\} \xrightarrow{j} y_i$; and convergence of the composite net is written as $\{y^i_j\} \xrightarrow{ij} y$.

THEOREM. *For a local dynamical system π on a rim-compact space X, each point in X which does not have a compact neighborhood is a critical point of π.*

Proof. Consider X as a subspace of its Freudenthal compactification, Y. All of our work is done in the space Y, with the following notational convention: the letter x denotes an element of X, the letter b denotes an element of Y-X, and the letter y denotes an element of Y (all of these may be equipped with various indices).

Let x be any point which does not have a compact neighborhood in X. There is a net $\{b_i\} \xrightarrow{i} x$, with an index set I, since for any compact neighborhood N_x of x we have that $N_x \cap X$ is not compact, which implies $N_x \cap Y\text{-}X \neq \varnothing$. Also, since X is dense in Y, for each b_i there is a net $\{x^i_j\} \xrightarrow{j} b_i$ with index set J^i. Choose $t > 0$ so small that $x\pi t$ is defined, and assume $x \neq x\pi t$. Because $\{x^i_j\} \xrightarrow{ij} x$, eventually $\{x^i_j\pi t\}_{i,j}$ is defined. More precisely, there is an antiresidual subset I' of I and an antiresidual subset of J^i for each i in I' such that $\{x^i_j\pi t\}$ is defined. Without loss of generality, let I and J^i be the antiresidual subsets described above. Again, there must be an antiresidual subset of I for which the net $\{x^i_j\pi t\}_{Ji}$ does not converge to b_i. If not, there would be a cofinal subset of I such that $\{x^i_j\pi t\} \xrightarrow{j} b_i$ and thus a subnet of $\{x^i_j\pi t\} \xrightarrow{ij} x$, which would imply $x = x\pi t$. Once more, let I be the new antiresidual subset.

Each b_i has a neighborhood N_i such that some subnet of $\{x^i_j\pi t\}_{Ji}$ is not in N_i and that its boundary ∂N_i is compact and contained in X. Denote this subnet by $\{x^i_j\pi t\}$. From the connectedness of $x^i_j\pi[0, t]$ there is a τ^i_j in $(0, t)$ such that $x^i_j\pi\tau^i_j \in \partial N_i$. As ∂N_i is compact and $\{\tau^i_j\}_{Ji}$ is bounded, there are subnets $\{x^i_j\pi\tau^i_j\} \xrightarrow{j} x^i$ in ∂N_i and $\{\tau^i_j\} \xrightarrow{j} \tau^i$. Let $\{x^i_j\pi\tau^i_j\}_{Ji}$ and $\{\tau^i_j\}_{Ji}$ denote these subnets. Since Y is compact and $\{\tau^i\}$ is bounded there are subnets of $\{x^i\} \xrightarrow{i} y \in Y$ and of $\{\tau^i\} \xrightarrow{i} \tau \in [0, t]$, again denoted by $\{x^i\}$ and $\{\tau^i\}$. Thus, we have $\{x^i_j\pi\tau^i_j\} \xrightarrow{ij} y$ and $\{\tau^i_j\} \xrightarrow{ij} \tau$. However, since $x\pi\tau$ is defined and $\{x^i_j\} \xrightarrow{ij} x$, we obtain $y = x\pi\tau$. From condition (3) of Definition 1, this implies there is an i such that $x^i\pi(-\tau^i)$ is defined. For this i, $\{x^i_j\pi\tau^i_j\} \xrightarrow{j} x^i$, $\{-\tau^i_j\} \xrightarrow{j} -\tau^i$ implies $\{(x^i_j\pi\tau^i_j)\pi - \tau^i_j\} \xrightarrow{j} x^i\pi - \tau^i$ which is in X. But, $\{(x^i_j\pi\tau^i_j)\pi - \tau^i_j\} = \{x^i_j\} \xrightarrow{j} b_i \in Y\text{-}X$. This contradiction yields our assertion.

COROLLARY. *Let a space Z have a rim-compact but not locally compact open subset. Then every local dynamical system on Z has a critical point.*

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