THE LATTICE OF PRETOPOLOGIES ON AN ARBITRARY SET $S$

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The structure of the lattice of pretopologies on the set S, unlike that of the lattice of topologies on S (a proper sublattice of the former), has not been closely examined. We establish that pretopologies may be identified with products of certain filters in a natural way. From this identification, we are able to determine much of the structure of this lattice.

We show that \((p(S), \leq)\), the lattice of pretopologies (pretopenologies in the sense of Kent [2; p. 126]) on the set S, is order isomorphic to a sublattice of filters on \(S^S\) (using Bourbaki's [1; p. 61-63] approach to filters). From this, we deduce that \((p(S), \leq)\) is complete, atomic, coatomic, modular, distributive, and compactly generated; S being finite is both necessary and sufficient for the lattice to be co-compactly generated and complemented (in which case it has a unique complement). It is infinitely distributive only in the trivial case of S being finite. (The lattice terminology is that of Szász [3] with the exception of coatomic which we use rather than dually atomic and co-compactly generated which is used for the notion dual to that of compactly generated.)

1. The isomorphism \(\varphi\). A pretopology \(p\) on a set \(S\) is completely determined by a specification of the neighborhood filter \(\eta_p(x)\) of each \(x\) in \(S\). These neighborhoods necessarily satisfy \(\eta_p(x) \leq \bar{x}\), where \(\bar{x}\) is the principal filter generated by \(\{x\}\). For each \(x \in S\), let \(F(x) = \{\mathfrak{F}; \mathfrak{F} \geq \bar{x}, \mathfrak{F}\ \text{a filter on} \ S\}\), and let \(F = \Pi_{x \in S} F(x)\) (Bourbaki [1; p. 69-70]); both ordered by \(\mathfrak{F} \leq \mathfrak{G}\) if and only if \(\mathfrak{F} \subseteq \mathfrak{G}\). Then \(F\) is a subset of the set of filters on \(S^S\). Indeed, it is easily seen that \(F\), with this ordering, is a sublattice of the lattice of filters on \(S^S\). For given \(\mathfrak{F}, \mathfrak{G} \in F\), we have \(\mathfrak{F} \land \mathfrak{G} = \{F \cup G; F \in \mathfrak{F}, G \in \mathfrak{G}\}\) and \(\mathfrak{F} \lor \mathfrak{G} = \{F \cap G; F \in \mathfrak{F}, G \in \mathfrak{G}\}\) \((F \cap G \neq \emptyset\) since \(\Pi_{x \in S} \{x\} \in F, G\).

Given a pretopology \(p\), we define \(\varphi\) by \(\varphi(p) = \Pi_{x \in S} \eta_p(x)\). Then \(\varphi\) is easily seen to be a one-to-one mapping from the pretopologies on \(S\) onto \(F\). Furthermore, if \(p, q\) are pretopologies on \(S\), the following will be equivalent:

\begin{enumerate}
  \item \(p \leq q;\)
  \item \(\eta_p(x) \leq \eta_q(x)\) for all \(x\) in \(S\); and
  \item \(\Pi_{x \in S} \eta_p(x) \leq \Pi_{x \in S} \eta_q(x)\).
\end{enumerate}

Thus we have
THEOREM 1. \( \varphi \) is an order isomorphism from the lattice of pretopologies on \( S \) onto the sublattice \( (F, \leq) \) of filters on \( S \).

2. The structure of \( (F(x), \leq) \). We shall deduce the structure of \( (F, \leq) \) from an examination of the structures of the lattices \( (F(x), \leq) \), for each \( x \).

It follows readily from the definition of \( \land \) and \( \lor \) in \( (F(x), \leq) \) that this lattice is complete, modular, and distributive. The remaining propositions of this section further describe its structure.

PROPOSITION 1. \( (F(x), \leq) \) is atomic. Its atoms are precisely those elements of the form \( S \setminus \{a\} \) for \( a \neq x \). \( (A \) denotes the filter of all super-sets of \( A \) in \( S \)).

Proof. Given \( \mathcal{G} \neq 0 = \mathcal{S} \in F(x) \), select \( A \in \mathcal{G}, \ A \neq S \). Then there exists an \( a \neq x \in S, A \subseteq S \), and \( S \setminus \{a\} \leq \mathcal{G} \).

To show that \( S \setminus \{a\} \) is an atom of \( (F(x), \leq) \) for \( a \neq x \), let \( \mathcal{G} < S \setminus \{a\} \).

Then \( S \setminus \{a\} \subseteq F \) for all \( F \in \mathcal{G} \), and \( F \subseteq S \setminus \{a\} \) for no \( F \in \mathcal{G} \). Thus \( \mathcal{G} = S = 0 \).

PROPOSITION 2. \( (F(x), \leq) \) is coatomic. Its coatoms are precisely those \( \mathcal{G} = x \land \mathcal{U} \) where \( \mathcal{U} \neq x \) is an ultrafilter.

Proof. Let \( \mathcal{G} \in F(x) \) be distinct from \( 1 = \mathcal{X} \). Then since \( \mathcal{G} \) is not an ultrafilter, there must be at least two ultrafilters above \( \mathcal{G} \). One of these, say \( \mathcal{U} \), must be distinct from \( \mathcal{X} \). Then \( \mathcal{G} \subseteq \mathcal{X} \land \mathcal{U} \).

To show that \( \mathcal{X} \land \mathcal{U} \) is a coatom of \( (F(x), \leq) \), assume there is an \( \mathcal{H} \in F(x) \) with \( x \land \mathcal{U} < \mathcal{H} < \mathcal{X} \). Since \( \mathcal{G} < \mathcal{X} \), \( F \setminus \{x\}: F \in \mathcal{H} \) is a base for some filter \( \mathcal{G} \). Clearly \( x \land \mathcal{G} = \mathcal{X} \). Now, for each \( U \in \mathcal{U} \), there exists \( F \in \mathcal{H} \) such that \( F \subseteq U \subseteq \{x\} \), since \( x \land \mathcal{U} < \mathcal{G} \). Thus \( F \setminus \{x\} \subseteq U \). Hence \( \mathcal{G} \geq \mathcal{U} \). But \( \mathcal{U} \) is an ultrafilter. Consequently \( \mathcal{G} = \mathcal{U} \). Thus, we must conclude that \( \mathcal{X} \land \mathcal{U} = \mathcal{X} \land \mathcal{G} = \mathcal{X}, \) a contradiction.

PROPOSITION 3. \( \mathcal{G} \in F(x) \) is compact if and only if \( \mathcal{G} = \mathcal{A} \) for some \( A \subseteq S \) with \( x \in A \). Consequently \( (F(x), \leq) \) is compactly generated.

Proof. Let \( \mathcal{G} \in F(x) \) be compact. Observe that \( \mathcal{G} = \bigvee \{F: F \in \mathcal{J}\} \).

Thus \( \mathcal{G} \leq \bigvee_{i=1}^{n} F_i \) for some choice of \( n \) and \( F_i \in \mathcal{G} \) \( (i = 1, \ldots, n) \). But since filters include finite intersections of their members, \( \mathcal{G} \leq \bigcap_{i=1}^{n} F_i = \bigvee_{i=1}^{n} F_i \).

Conversely, let \( \mathcal{G} = \mathcal{A} \) and let \( \mathcal{G} \leq \bigvee_{i \in \mathcal{J}} \mathcal{G}_i \). Then since \( A \in \mathcal{G} \), there exists \( I_0 \subseteq \mathcal{I} \) \( (I_0 \) finite, and \( F_i \in \mathcal{G}_i \) such that \( \bigcap_{i \in \mathcal{I}} F_i \subseteq A \).
Thus \( \mathcal{F} \leq \bigvee_{\tau \in \tau} F_\tau \).

For any \( \mathcal{G} \in F(x) \), we have

\[
\bigvee \{\mathcal{G} : \mathcal{F} \leq \mathcal{G}, \mathcal{F} \text{ compact} \} \leq \mathcal{G} = \bigvee \{G : G \in \mathcal{G} \}
\]

\[
\leq \bigvee \{\mathcal{G} : \mathcal{F} \leq \mathcal{G}, \mathcal{F} \text{ compact} \}
\]

thus \( \mathcal{G} = \bigvee \{\mathcal{G} : \mathcal{F} \leq \mathcal{G}, \mathcal{F} \text{ compact} \} \) and \( (F(x), \leq) \) is compactly generated.

**Proposition 4.** \( \mathcal{F} \in F(x) \) is co-compact if and only if \( \mathcal{F} = \overline{A} \) where \( A \) is some finite subset of \( S \) containing \( x \). Consequently \( (F(x), \leq) \) is co-compactly generated if and only if \( S \) is finite.

**Proof.** Let \( \mathcal{F} \in F(x) \) be co-compact and let \( T = S \{x\} \). Observe that \( \mathcal{F} \geq \overline{S} = \Lambda_{x \in \tau} \{x, a_i\} \). Consequently for some \( n \) and \( a_i \in T (i = 1, \ldots, n) \), \( \mathcal{F} \geq \Lambda_{i=1}^n \{x, a_i\} = \bigcup_{i=1}^n \{x, a_i\} \). Thus \( \bigcup_{i=1}^n \{x, a_i\} \in \mathcal{F} \). But any filter containing a finite set \( B \) can be expressed as \( \overline{A} \) for some \( A \subseteq B \). Thus \( \mathcal{F} = \overline{A} \) for some \( A \subseteq \bigcup_{i=1}^n \{x, a_i\} \).

Conversely, let \( \mathcal{F} = \overline{A} \) where \( A = \{x, a_1, a_2 \ldots a_n\} \), and suppose that \( \mathcal{F} \geq \Lambda_{\tau \in \tau} \mathcal{F}_\tau \). Then we may select \( F_\tau \in \mathcal{F}_\tau \) such that \( \bigcup_{\tau \in \tau} F_\tau \supseteq A \). Select \( \gamma_i \) so that \( a_i \in F_{\gamma_i} \). Then \( \mathcal{F} \geq \Lambda_{i=1}^n \mathcal{F}_{\gamma_i} \).

If \( S \) is finite, each \( \mathcal{F} \in F(x) \) is of the form \( \overline{A} \) with \( A \) finite, so \( (F(x), \leq) \) will consist only of co-compact elements and hence be co-compactly generated. Observe however, that \( \bigwedge_{\tau \in \tau} \overline{A}_\tau = \bigcup_{\tau \in \tau} \overline{A}_\tau \) for arbitrary filters. Thus, in particular, the only elements of \( (F(x), \leq) \) which will be co-compactly generated are the principal filters. Consequently \( (F(x), \leq) \) is not co-compactly generated when \( S \) is infinite.

**Proposition 5.** \( \mathcal{F} \in F(x) \) has a complement \( \mathcal{G} \) if and only if \( \mathcal{F} = \overline{A} \). In this case \( \mathcal{G} \) is unique and \( \mathcal{G} = (S \{A\} \cup \{x\}) \). Consequently \( (F(x), \leq) \) is complemented if and only if \( S \) is finite.

**Proof.** Let \( \mathcal{F} = \overline{A} \). If \( \mathcal{G} = (S \{A\} \cup \{x\}) \), then \( \mathcal{F} \wedge \mathcal{G} = \overline{S} = \emptyset \) and \( \mathcal{F} \vee \mathcal{G} = \overline{x} = 1 \). Thus \( \mathcal{G} \) is a complement of \( \mathcal{F} \). Let \( \mathcal{G}' \) be any complement of \( \mathcal{F} \). Then \( \mathcal{G}' \wedge \mathcal{G} = \overline{S} \), \( (S \{A\} \cup \{x\}) \) must be in \( \mathcal{G}' \). But \( \mathcal{F} \vee \mathcal{G}' = \overline{x} \), so no proper subset of \( (S \{A\} \cup \{x\}) \) may be in \( \mathcal{G}' \). Consequently \( \mathcal{G}' = (S \{A\} \cup \{x\}) = \mathcal{G} \).

Suppose on the other hand, that \( \mathcal{F} \) is not principal. Let \( A = \bigcap \mathcal{F} \). Then \( A \neq \phi \) since \( x \in A \). Suppose that \( \mathcal{G} \) is a complement of \( \mathcal{F} \). Then for each \( F \in \mathcal{F} \), \( G \in \mathcal{G} \), we have \( F \cup G = S \), since \( \mathcal{F} \wedge \mathcal{G} = \overline{S} \). Thus \( B = (S \{A\} \cup \{x\}) \) must be a subset of every \( G \) in \( \mathcal{G} \). Observe that any \( F \in \mathcal{F} \) will contain \( A \) as a proper subset since \( \mathcal{F} \) is nonprincipal. Thus any \( F \in \mathcal{F} \) will include points of \( B \) distinct from \( x \). Hence for each \( G \in \mathcal{G} \), \( F \vee G \) will contain at least two points. But this violates
the requirement that $\mathcal{F} \vee \mathcal{G} = \bar{x}$. Therefore $\mathcal{F}$ can not have a complement.

We conclude this section with a discussion of infinite distributivity. Let $\mathcal{F}, \mathcal{G}, \in F(x)$ ($\gamma \in \Gamma$) be arbitrary. Then, since filter joins are given by finite intersections, we have $\mathcal{F} \wedge (\bigvee_{\tau \in \Gamma} \mathcal{G}_\tau) = \bigvee_{\tau \in \Gamma} (\mathcal{F} \wedge \mathcal{G}_\tau)$. We also have $\mathcal{F} \vee (\bigwedge_{\tau \in \Gamma} \mathcal{G}_\tau) \leq \bigwedge_{\tau \in \Gamma} (\mathcal{F} \vee \mathcal{G}_\tau)$. However, if $S$ is not finite, we need not have equality. A particular example can be found by letting $\Gamma = S$, $\mathcal{G}_x = \{\gamma, x\}$, and $\mathcal{F} = \{A : x \in A, A$ cofinite}. For in this case $\{s\} \in \bigwedge_{\tau \in \Gamma} (\mathcal{F} \vee \mathcal{G}_\tau)$ is not cofinite. Thus $(F(x), \leq)$ is distributive only in the trivial case where $S$, and consequently $F(x)$, is finite.

3. Structure of $(F, \leq)$ and $(p(S), \leq)$. The results of § 2 carry directly over to the lattice $(F, \leq)$. For letting $\mathcal{F} = \prod_{x \in S} \mathcal{F}_x$, $\mathcal{G} = \prod_{x \in S} \mathcal{G}_x$ with $\mathcal{F}_x, \mathcal{G}_x \in F(x)$ for each $x$, we see that $\mathcal{F} \leq \mathcal{G}$ if and only if $\mathcal{F}_x \leq \mathcal{G}_x$ in $(F(x), \leq)$ for each $x$, while $\mathcal{F} \wedge \mathcal{G} = \prod_{x \in S} (\mathcal{F}_x \wedge \mathcal{G}_x)$ and $\mathcal{F} \vee \mathcal{G} = \prod_{x \in S} (\mathcal{F}_x \vee \mathcal{G}_x)$. We summarize these results in the following proposition. Each of its components follows from the corresponding result in § 2.

**Proposition 6.** 1. $(F, \leq)$ is complete, modular, and distributive. It is infinitely distributive only in the trivial case of $S$, and consequently $F$, being finite.

2. $(F, \leq)$ is atomic (coatomic). $\mathcal{F} = \prod_{x \in S} \mathcal{F}_x \in F$ is an atom (coatom) if and only if $\mathcal{F}_x = \mathcal{S}$ for $x \neq s$ ($\mathcal{F}_x = \bar{x}$ for $x \neq s$) and $\mathcal{F}_s$ is an atom of $F(s)$ (a coatom of $(F(s))$).

3. $\mathcal{F}_x \in F$ is compact (co-compact) if and only if $\mathcal{F}_x$ is compact (co-compact) for each $x \in S$ and $\mathcal{F}_x = \mathcal{S}$ ($\mathcal{F}_x = \bar{x}$) except for most a finite number of the $x \in S$.

4. $(F, \leq)$ is compactly generated.

5. $(F, \leq)$ is co-compactly generated if and only if $S$ is finite.

6. $\mathcal{F}$ has a complement $\mathcal{G} = \prod_{x \in S} \mathcal{G}_x$ if and only if $\mathcal{F}_x$ and $\mathcal{G}_x$ are complements for each $x \in S$.

7. $(F, \leq)$ is complemented if and only if $S$ is finite. In this case complements will be unique.

Using the isomorphism $\varphi$, these results immediately carry over to $(p(S), \leq)$. Thus we have

**Theorem 2.** $(p(S), \leq)$ is always complete, modular, distributive, atomic, coatomic, and compactly generated. It is complemented (and has unique complements), co-compactly generated, and infinitely distributive if and only if $S$ is finite.
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