Pacific Journal of Mathematics

THE LATTICE OF PRETOPOLOGIES ON AN ARBITRARY SET S

ALLAN MATLOCK WEBER CARSTENS

Vol. 29, No. 1

May 1969

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Allan M. Carstens

The structure of the lattice of pretopologies on the set S, unlike that of the lattice of topologies on S (a proper sublattice of the former), has not been closely examined. We establish that pretopologies may be identified with products of certain filters in a natural way. From this identification, we are able to determine much of the structure of this lattice.

We show that $(p(S), \leq)$, the lattice of pretopologies (pretopologies in the sense of Kent [2; p. 126]) on the set S, is order isomorphic to a sublattice of filters on S^s (using Bourbaki's [1; p. 61-63] approach to filters). From this, we deduce that $(p(S), \leq)$ is complete, atomic, coatomic, modular, distributive, and compactly generated; S being finite is both necessary and sufficient for the lattice to be co-compactly generated and complemented (in which case it has a unique complement). It is infinitely distributive only in the trivial case of S being finite. (The lattice terminology is that of Szász [3] with the exception of coatomic which we use rather than dually atomic and co-compactly generated which is used for the notion dual to that of compactly generated.)

1. The isomorphism φ . A pretopology p on a set S is completely determined by a specification of the neighborhood filter $\eta_p(x)$ of each x in S. These neighborhoods necessarily satisfy $\eta_p(x) \leq \overline{x}$, where \overline{x} is the principal filter generated by $\{x\}$. For each $x \in S$, let $\underline{F}(x) = \{\mathfrak{F}: \mathfrak{F} \geq \overline{x}, \mathfrak{F} \text{ a filter on } S\}$, and let $\underline{F} = \prod_{x \in S} \underline{F}(x)$ (Bourbaki [1; p. 69-70]); both ordered by $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F} \subset \mathfrak{G}$. Then \underline{F} is a subset of the set of filters on S^s . Indeed, it is easily seen that \underline{F} , with this ordering, is a sublattice of the lattice of filters on S^s . For given $\mathfrak{F}, \mathfrak{G} \in \underline{F}$, we have $\mathfrak{F} \wedge \mathfrak{G} = \{F \cup G: F \in \mathfrak{F}, G \in \mathfrak{G}\}$ and $\mathfrak{F} \vee \mathfrak{G} = \{F \cap G: F \in \mathfrak{F}, G \in \mathfrak{G}\}$ ($F \cap G \neq \phi$ since $\prod_{x \in S} \{x\} \in F, G$).

Given a pretopology p, we define φ by $\varphi(p) = \prod_{x \in S} \eta_p(x)$. Then φ is easily seen to be a one-to-one mapping from the pretopologies on S onto \underline{F} . Furthermore, if p, q are pretopologies on S, the following will be equivalent:

(1) $p \leq q$; (2) $\eta_p(x) \leq \eta_q(x)$ for all x in S; and (3) $\prod_{x \in S} \eta_p(x) \leq \prod_{x \in S} \eta_q(x)$. Thus we have THEOREM 1. φ is an order isomorphism from the lattice of pretopologies on S onto the sublattice (\underline{F}, \leq) of filters on S^s .

2. The structure of $(\underline{F}(x), \leq)$. We shall deduce the structure of (\underline{F}, \leq) from an examination of the structures of the lattices $(\underline{F}(x), \leq)$, for each x.

It follows readily from the definition of \wedge and \vee in $(\underline{F}(x), \leq)$ that this lattice is complete, modular, and distributive. The remaining propositions of this section further describe its structure.

PROPOSITION 1. $(\underline{F}(x), \leq)$ is atomic. Its atoms are precisely those elements of the form $\overline{S\setminus\{a\}}$ for $a \neq x$. (\overline{A} denotes the filter of all super-sets of A in S).

Proof. Given $\mathfrak{F} \neq 0 \equiv \overline{S}$ in $\underline{F}(x)$, select $A \in \mathfrak{F}$, $A \neq S$. Then there exists an $a \neq x$ in $\underline{S \setminus A}$, and $\overline{S \setminus \{a\}} \leq \mathfrak{F}$.

To show that $S \setminus \{a\}$ is an atom of $(\underline{F}(x), \leq)$ for $a \neq x$, let $\mathfrak{F} < S \setminus \{a\}$. Then $S \setminus \{a\} \subset F$ for all $F \in \mathfrak{F}$, and $F \subset S \setminus \{a\}$ for no $F \in \mathfrak{F}$. Thus $\mathfrak{F} = \overline{S} = 0$.

PROPOSITION 2. $(\underline{F}(x), \leq)$ is coatomic. Its coatoms are precisely those $\mathfrak{F} = \overline{x} \wedge \mathfrak{U}$ where $\mathfrak{U} \neq \overline{x}$ is an ultrafilter.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be distinct from $1 \equiv \overline{x}$. Then since \mathfrak{F} is not an ultrafilter, there must be at least two ultrafilters above \mathfrak{F} . One of these, say \mathfrak{U} , must be distinct from \overline{x} . Then $\mathfrak{F} \leq \overline{x} \wedge \mathfrak{U}$.

To show that $\overline{x} \wedge \mathfrak{U}$ is a coatom of $(\underline{F}(x), \leq)$, assume there is an $\mathfrak{F} \in \underline{F}(x)$ with $\overline{x} \wedge \mathfrak{U} < \mathfrak{F} < \overline{x}$. Since $\mathfrak{F} < \overline{x}$, $\{F \setminus \{x\}: F \in \mathfrak{F}\}$ is a base for some filter \mathfrak{G} . Clearly $\overline{x} \wedge \mathfrak{G} = \mathfrak{F}$. Now, for each $U \in \mathfrak{U}$, there exists $F \in \mathfrak{F}$ such that $F \subset U \subset \{x\}$, since $\overline{x} \wedge \mathfrak{U} < \mathfrak{F}$. Thus $F \setminus \{x\} \subset U$. Hence $\mathfrak{G} \geq \mathfrak{U}$. But \mathfrak{U} is an ultrafilter. Consequently $\mathfrak{G} = \mathfrak{U}$. Thus, we must conclude that $\overline{x} \wedge \mathfrak{U} = \overline{x} \wedge \mathfrak{G} = \mathfrak{F}$, a contradiction.

PROPOSITION 3. $\mathfrak{F} \in \underline{F}(x)$ is compact if and only if $\mathfrak{F} = \overline{A}$ for some $A \subset S$ with $x \in A$. Consequently $(\underline{F}(x), \leq)$ is compactly generated.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be compact. Observe that $\mathfrak{F} = \bigvee \{\overline{F}: F \in \mathfrak{F}\}$. Thus $\mathfrak{F} \leq \bigvee_{i=1}^{n} \overline{F}_{i}$ for some choice of n and $F_{i} \in \mathfrak{F}(i = 1, \dots, n)$. But since filters include finite intersections of their members, $\mathfrak{F} \geq \bigcap_{i=1}^{n} \overline{F}_{i} \equiv \bigvee_{i=1}^{n} \overline{F}_{i}$. Thus $\mathfrak{F} \equiv \bigcap_{i=1}^{n} \overline{F}_{i}$.

Conversely, let $\mathfrak{F} = \overline{A}$ and let $\mathfrak{F} \leq \bigvee_{\tau \in r} \mathfrak{F}_{\tau}$. Then since $A \in \mathfrak{F}_{\tau}$, there exists $\Gamma_0 \subset \Gamma$ (Γ_0 finite), and $F_{\tau} \in \mathfrak{F}_{\tau}$ such that $\bigcap_{\tau \in r} F_{\tau} \subset A$. Thus $\mathfrak{F} \leq \bigvee_{\tau \in \Gamma_0} F_{\Gamma}$. For any $\mathfrak{G} \in \underline{F}(x)$, we have

$$\begin{array}{l} \bigvee \{ \mathfrak{F} \colon \mathfrak{F} \leq \mathfrak{G}, \, \mathfrak{F} \, \mathrm{compact} \} \leq \mathfrak{G} = \, \bigvee \{ \overline{G} \colon G \in \mathfrak{G} \} \\ & \leq \, \bigvee \{ \mathfrak{F} \colon \mathfrak{F} \leq \mathfrak{G}, \, \mathfrak{F} \, \mathrm{compact} \} \end{array}$$

thus $\mathfrak{G} = \mathbf{V}{\mathfrak{F}: \mathfrak{F} \leq \mathfrak{G}, \mathfrak{F} \text{ compact}}$ and $(\underline{F}(x), \leq)$ is compactly generated.

PROPOSITION 4. $\mathfrak{F} \in \underline{F}(x)$ is co-compact if and only if $\mathfrak{F} = \overline{A}$ where A is some finite subset of S containing x. Consequently $(\underline{F}(x), \leq)$ is co-compactly generated if and only if S is finite.

Proof. Let $\mathfrak{F} \in \underline{F}(x)$ be co-compact and let $T = S \setminus \{x\}$. Observe that $\mathfrak{F} \geq \overline{S} \equiv \bigwedge_{a \in T} \{x, a\}$. Consequently for some n and $a_i \in T$ $(i = 1, \dots, n)$, $\mathfrak{F} \geq \bigwedge_{i=1}^n \{x, a_i\} = \bigcup_{i=1}^n \{x, a_i\}$. Thus $\bigcup_{i=1}^n \{x, a_i\} \in \mathfrak{F}$. But any filter containing a finite set B can be expressed as \overline{A} for some $A \subseteq B$. Thus $\mathfrak{F} = \overline{A}$ for some $A \subset \bigcup_{i=1}^n \{x, a_i\}$.

Conversely, let $\mathfrak{F} = \overline{A}$ where $A = \{x, a_1, a_2 \cdots a_n\}$, and suppose that $\mathfrak{F} \geq \bigwedge_{\gamma \in \Gamma} \mathfrak{F}_{\gamma}$. Then we may select $F_{\gamma} \in \mathfrak{F}_{\gamma}$ such that $\bigcup_{\gamma \in \Gamma} F_{\gamma} \supseteq A$. Select γ_i so that $a_i \in F_{\gamma_i}$. Then $\mathfrak{F} \geq \bigwedge_{i=1}^n \mathfrak{F}_{\gamma_i}$.

If S is finite, each $\mathfrak{F} \in \underline{F}(x)$ is of the form A with A finite, so $(\underline{F}(x), \leq)$ will consist only of co-compact elements and hence be cocompactly generated. Observe however, that $\wedge_{\tau \in \Gamma} \overline{A_{\tau}} = \bigcup_{\tau \in \Gamma} \overline{A_{\tau}}$ for arbitrary filters. Thus, in particular, the only elements of $(\underline{F}(x), \leq)$ which will be co-compactly generated are the principal filters. Consequently $(\underline{F}(x), \leq)$ is not co-compactly generated when S is infinite.

PROPOSITION 5. $\mathfrak{F} \in \underline{F}(x)$ has a complement \mathfrak{G} if and only if $\mathfrak{F} = \overline{A}$. In this case \mathfrak{G} is unique and $\mathfrak{G} = (S \setminus A) \cup \{x\}$. Consequently $(\underline{F}(x), \leq)$ is complemented if and only if S is finite.

Proof. Let $\mathfrak{F} = \overline{A}$. If $\mathfrak{G} = (S \setminus A) \cup \{x\}$, then $\mathfrak{F} \wedge \mathfrak{G} = \overline{S} \equiv 0$ and $\mathfrak{F} \vee \mathfrak{G} = \overline{x} \equiv 1$. Thus \mathfrak{G} is a complement of \mathfrak{F} . Let \mathfrak{G}' be any complement of \mathfrak{F} . Then since $\mathfrak{F} \wedge \mathfrak{G}' = \overline{S}$, $(S \setminus A) \subset \{x\}$ must be in \mathfrak{G}' . But $\mathfrak{F} \vee \mathfrak{G}' = \overline{x}$, so no proper subset of $(\underline{S} \setminus \underline{A}) \cup \{x\}$ may be in \mathfrak{G}' . Consequently $\mathfrak{G}' = (\overline{S} \setminus A \cup \{x\}) = \mathfrak{G}$.

Suppose on the other hand, that \mathfrak{F} is not principal. Let $A = \bigcap \mathfrak{F}$. Then $A \neq \phi$ since $x \in A$. Suppose that \mathfrak{G} is a complement of \mathfrak{F} . Then for each $F \in \mathfrak{F}$, $G \in \mathfrak{G}$, we have $F \cup G = S$, since $\mathfrak{F} \land \mathfrak{G} = \overline{S}$. Thus $B = (S \setminus A) \cup \{x\}$ must be a subset of every G in \mathfrak{G} . Observe that any F in \mathfrak{F} will contain A as a proper subset since \mathfrak{F} is nonprincipal. Thus any $F \in \mathfrak{F}$ will include points of B distinct from x. Hence for each $G \in \mathfrak{G}$, $F \vee G$ will contain at least two points. But this violates the requirement that $\mathfrak{F} \vee \mathfrak{G} = \overline{x}$. Therefore \mathfrak{F} can not have a complement.

We conclude this section with a discussion of infinite distributivity. Let $\mathfrak{F}, \mathfrak{F}_r \in \underline{F}(x) \ (\gamma \in \Gamma)$ be arbitrary. Then, since filter joins are given by finite intersections, we have $\mathfrak{F} \land (\bigvee_{r \in \Gamma} \mathfrak{F}_r) = \bigvee_{r \in \Gamma} (\mathfrak{F} \land \mathfrak{F}_r)$. We also have $\mathfrak{F} \lor (\bigwedge_{r \in \Gamma} \mathfrak{F}_r) \leq \bigwedge_{r \in \Gamma} (\mathfrak{F} \lor \mathfrak{F}_r)$. However, if \underline{S} is not finite, we need not have equality. A particular example can be found by letting $\Gamma = S, \mathfrak{F}_r = \{\gamma, x\}$, and $\mathfrak{F} = \{A: x \in A, A \text{ cofinite}\}$. For in this case $\{s\} \in \bigwedge_{r \in \Gamma} (\mathfrak{F} \lor \mathfrak{F}_r)$ is not cofinite. Thus $(\underline{F}(x), \leq)$ is distributive only in the trivial case where S, and consequently $\underline{F}(x)$, is finite.

3. Structure of (\underline{F}, \leq) and $(\underline{p}(S), \leq)$. The results of § 2 carry directly over to the lattice (\underline{F}, \leq) . For letting $\mathfrak{F} = \prod_{x \in S} \mathfrak{F}_x, \mathfrak{G} = \prod_{x \in S} \mathfrak{G}_x$, $\mathfrak{G} = \prod_{x \in S} \mathfrak{G}_x, \mathfrak{G} = \mathfrak{G}_x \mathfrak{G}_x$ with $\mathfrak{F}_x, \mathfrak{G}_x \in \underline{F}(x)$ for each x, we see that $\mathfrak{F} \leq \mathfrak{G}$ if and only if $\mathfrak{F}_x \leq \mathfrak{G}_x$ in $(\underline{F}(x), \leq)$ for each x, while $\mathfrak{F} \wedge \mathfrak{G} = \prod_{x \in S} (\mathfrak{F}_x \wedge \mathfrak{G}_x)$ and $\mathfrak{F} \vee \mathfrak{G} = \prod_{x \in S} (\mathfrak{F}_x \vee \mathfrak{G}_x)$. We summarize these results in the following proposition. Each of its components follows from the corresponding result in § 2.

PROPOSITION 6. 1. (\underline{F}, \leq) is complete, modular, and distributive. It is infinitely distributive only in the trivial case of S, and consequently \underline{F} , being finite.

2. (\underline{F}, \leq) is atomic (coatomic). $\mathfrak{F} = \prod_{x \in S} \mathfrak{F}_x \in \underline{F}$ is an atom (coatom) if and only if $\mathfrak{F}_x = \overline{S}$ for $x \neq s$ ($\mathfrak{F}_x = \overline{x}$ for $x \neq s$) and \mathfrak{F}_s is an atom of $\underline{F}(s)$ (a coatom of $(\underline{F}(s))$.

3. $\mathfrak{F} \in \underline{F}$ is compact (co-compact) if and only if \mathfrak{F}_x is compact (co-compact) for each $x \in \overline{S}$ and $\mathfrak{F}_x = \overline{S}(\mathfrak{F}_x = \overline{x})$ except for most a finite number of the $x \in S$.

4. (\underline{F}, \leq) is compactly generated.

5. (\overline{F}, \leq) is co-compactly generated if and only if S is finite.

6. \mathfrak{F} has a complement $\mathfrak{G} = \prod_{x \in S} \mathfrak{G}_x$ if and only if \mathfrak{F}_x and \mathfrak{G}_x are complements for each $x \in S$.

7. (\underline{F}, \leq) is complemented if and only if S is finite. In this case complements will be unique.

Using the isomorphism φ , these results immediately carry over to $(p(S), \leq)$. Thus we have

THEOREM 2. $(\underline{p}(S), \leq)$ is always complete, modular, distributive, atomic, coatomic, and compactly generated. It is complemented (and has unique complements), co-compactly generated, and infinitely distributive if and only if S is finite.

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Received March 31, 1968, and in revised form June 17, 1968.

WASHINGTON STATE UNIVERSITY

Pacific Journal of Mathematics Vol. 29, No. 1 May, 1969

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