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**A BILATERAL GENERATING FUNCTION FOR THE  
ULTRASPHERICAL POLYNOMIALS**

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## A BILATERAL GENERATING FUNCTION FOR THE ULTRASPHERICAL POLYNOMIALS

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The following differentiation formula for the Ultraspherical polynomials  $P_n^\lambda(x)$  was given by Tricomi:

$$(1.1) \quad P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n(x^2-1)^{\lambda+1/2n}}{n!} D^n(x^2-1)^{-\lambda}.$$

The object of this paper is to point out that the formula of Tricomi leads us to the following bilateral generating function for the Ultraspherical polynomials:

THEOREM.

If  $F(x, t) = \sum_{m=0}^{\infty} a_m t^m P_m^\lambda(x)$ ,

then

$$(1.2) \quad \rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right) = \sum_{r=0}^{\infty} t^r b_r(y) P_r^\lambda(x),$$

where

$$b_r(y) = \sum_{m=0}^{\infty} \binom{r}{m} a_m y^m, \quad \text{and} \quad \rho = (1 - 2xt + t^2)^{1/2}.$$

Starting from the formula (1.2), one can derive a large number of bilateral generating functions for the Ultraspherical polynomials by attributing different values to  $a_m$ .

2. Proof of the main formula (1.2). We first note from (1.1) that

$$(2.1) \quad \left(\frac{1}{\sqrt{x^2-1}}\right)^n P_n^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right) = \frac{(-1)^n}{n!} (x^2-1)^\lambda D^n(x^2-1)^{-\lambda}.$$

Now let

$$F\left(\frac{x}{\sqrt{x^2-1}}, \frac{t}{\sqrt{x^2-1}}\right) = \sum_{m=0}^{\infty} a_m \left(\frac{t}{\sqrt{x^2-1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right)$$

be a given generating function for  $P_n^\lambda(x)$ . Replacing  $t$  by  $ty$  and multiplying both sides by  $(x^2-1)^{-\lambda}$  and then operating  $e^{-tD}$ , we get

$$(2.2) \quad \begin{aligned} & e^{-tD}(x^2-1)^{-\lambda} F\left(\frac{x}{\sqrt{x^2-1}}, \frac{ty}{\sqrt{x^2-1}}\right) \\ &= e^{-tD}(x^2-1)^{-\lambda} \sum_{m=0}^{\infty} a_m \left(\frac{ty}{\sqrt{x^2-1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2-1}}\right). \end{aligned}$$

Since we know that

$$(2.3) \quad e^{-tD}f(x) = f(x - t),$$

the left member of (2.2) is equal to

$$\{(x - t)^2 - 1\}^{-\lambda} F\left(\frac{x - t}{\sqrt{(x - t)^2 - 1}}, \frac{ty}{\sqrt{(x - t)^2 - 1}}\right).$$

But the right member of (2.2) is equal to

$$\begin{aligned} & \sum_{m=0}^{\infty} a_m (ty)^m e^{-tD} (x^2 - 1)^{-\lambda} \left(\frac{1}{\sqrt{x^2 - 1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right) \\ &= \sum_{m=0}^{\infty} a_m (ty)^m e^{-tD} \left\{ \frac{(-1)^m}{m!} D^m (x^2 - 1)^{-\lambda} \right\} \\ &= \sum_{m=0}^{\infty} a_m (ty)^m \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} D^r \left\{ \frac{(-1)^m}{m!} D^m (x^2 - 1)^{-\lambda} \right\} \\ &= \sum_{m=0}^{\infty} a_m y^m \sum_{r=0}^{\infty} \frac{(-t)^{r+m}}{r! m!} D^{r+m} (x^2 - 1)^{-\lambda} \\ &= (x^2 - 1)^{-\lambda} \sum_{m=0}^{\infty} a_m y^m \sum_{r=0}^{\infty} \binom{r+m}{m} t^{r+m} \left(\frac{1}{\sqrt{x^2 - 1}}\right)^{r+m} P_{r+m}^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right) \\ &= (x^2 - 1)^{-\lambda} \sum_{r=0}^{\infty} \left(\frac{t}{\sqrt{x^2 - 1}}\right)^r P_r^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right) \sum_{m=0}^r \binom{r}{m} a_m y^m. \end{aligned}$$

It follows therefore that: If

$$F\left(\frac{x}{\sqrt{x^2 - 1}}, \frac{t}{\sqrt{x^2 - 1}}\right) = \sum_{m=0}^{\infty} a_m \left(\frac{t}{\sqrt{x^2 - 1}}\right)^m P_m^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right),$$

then

$$(2.4) \quad \begin{aligned} & \left\{ \frac{(x - t)^2 - 1}{x^2 - 1} \right\}^{-\lambda} F\left(\frac{x - t}{\sqrt{(x - t)^2 - 1}}, \frac{ty}{\sqrt{(x - t)^2 - 1}}\right) \\ &= \sum_{r=0}^{\infty} \left(\frac{t}{\sqrt{x^2 - 1}}\right)^r b_r(y) P_r^\lambda\left(\frac{x}{\sqrt{x^2 - 1}}\right), \end{aligned}$$

where  $b_r(y) = \sum_{m=0}^r \binom{r}{m} a_m y^m$ . Now changing  $x(x^2 - 1)^{-1/2}$  into  $x$  and then  $t$  into  $t(x^2 - 1)^{-1/2}$ , we obtain the theorem mentioned in the introduction.

### 3. Some applications of the theorem.

(A) First we consider the generating function of Truesdell:

$$(3.1) \quad e^{xt} \circ F_1\left(-; \lambda + \frac{1}{2}; \frac{t^2(x^2 - 1)}{4}\right) = \sum_{m=0}^{\infty} \frac{t^m}{(2\lambda)_m} P_m^\lambda(x).$$

Thus if we take  $a_m = 1/(2\lambda)_m$  in our theorem, we obtain

$$\rho^{-2} \exp\left\{\frac{yt(x - t)}{\rho^2}\right\} \circ F_1\left(-; \lambda + \frac{1}{2}; \frac{y^2 t^2 (x^2 - 1)}{4\rho^4}\right) = \sum_{r=0}^{\infty} t^r b_r(y) P_r^\lambda(x).$$

But we notice that

$$b_r(y) = {}_1F_1(-r; 2\lambda; -y) = \frac{r!}{(2\lambda)_r} L_r^{(2\lambda-1)}(-y) .$$

Hence we derive the following generating function of Weisner [3].

$$\begin{aligned} (3.2) \quad & \rho^{-2\lambda} \exp \left\{ \frac{-yt(x-t)}{\rho^2} \right\} {}_0F_1 \left( -; \lambda + \frac{1}{2}; \frac{y^2 t^2 (x^2 - 1)}{4\rho^4} \right) \\ & = \sum_{r=0}^{\infty} \frac{r! L_r^{(2\lambda-1)}(y)}{(2\lambda)_r} t^r P_r^\lambda(x) . \end{aligned}$$

Thus we remark that the bilateral generating function of Weisner is a particular case of our theorem. Moreover we have obtained the theorem by a method different from that used by Weisner or from that used by Rainville [2].

(B) If we consider the formula of Brafman:

$$\begin{aligned} (3.3) \quad & (1 - xt)^{-\gamma} {}_2F_1 \left( \frac{1}{2} \gamma, \frac{1}{2} \gamma + \frac{1}{2}; \lambda + \frac{1}{2}; \frac{t^2(x^2 - 1)}{(1 - xt)^2} \right) \\ & = \sum_{m=0}^{\infty} \frac{(\gamma)_m t^m}{(2\lambda)_m} P_m^\lambda(x) , \end{aligned}$$

then we put  $a_m = (\gamma)_m / (2\lambda)_m$  in our theorem and we obtain

$$\begin{aligned} (3.4) \quad & \rho^{2(\gamma-\lambda)} \{ \rho^2 + yt(x-t) \}^{-\gamma} {}_2F_1 \left( \frac{1}{2} \gamma, \frac{1}{2} \gamma + \frac{1}{2}; \lambda + \frac{1}{2}; \frac{y^2 t^2 (x^2 - 1)}{(\rho^2 + yt(x-t))^2} \right) \\ & = \sum_{r=0}^{\infty} {}_2F_1(-r, \gamma; 2\lambda; y) t^r P_r^\lambda(x) . \end{aligned}$$

(C) Next we consider the following generating function of Bateman:

$$\begin{aligned} (3.5) \quad & {}_0F_1 \left( -; \lambda + \frac{1}{2}; \frac{t(x-1)}{2} \right) {}_0F_1 \left( -; \lambda + \frac{1}{2}; \frac{t(x+1)}{2} \right) \\ & = \sum_{m=0}^{\infty} \frac{t^m}{(2\lambda)_m \left( \lambda + \frac{1}{2} \right)_m} P_m^\lambda(x) . \end{aligned}$$

Here we set  $a_m = 1 / \{ (2\lambda)_m (\lambda + 1/2)_m \}$  in our theorem and we derive

$$\begin{aligned} (3.6) \quad & \rho^{-2\lambda} {}_0F_1 \left( -; \lambda + \frac{1}{2}; \frac{yt(t-x+\rho)}{2\rho^2} \right) {}_0F_1 \left( -; \lambda + \frac{1}{2}; \frac{yt(t-x-\rho)}{2\rho^2} \right) \\ & = \sum_{r=1}^{\infty} {}_1F_2 \left( -r; 2\lambda, \lambda + \frac{1}{2}; y \right) t^r P_r^\lambda(x) . \end{aligned}$$

(D) Lastly if we consider the following generating function of Brafman:

$$\begin{aligned}
 & {}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1-t-\rho}{2}\right)x \\
 (3.7) \quad & {}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{1+t-\rho}{2}\right) \\
 & = \sum_{m=0}^{\infty} \frac{(\gamma)_m (2\lambda - \gamma)_m}{(2\lambda)_m \left(\lambda + \frac{1}{2}\right)_m} t^m P_m^\lambda(x);
 \end{aligned}$$

we put

$$a_m = \frac{(\gamma)_m (2\lambda - \gamma)_m}{(2\lambda)_m \left(\lambda + \frac{1}{2}\right)_m}$$

in our theorem and thus we obtain

$$\begin{aligned}
 & \rho^{-2\lambda} {}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{\rho + yt - \omega}{2\rho}\right)x \\
 (3.8) \quad & {}_2F_1\left(\gamma, 2\lambda - \gamma; \lambda + \frac{1}{2}; \frac{\rho - yt - \omega}{2\rho}\right) \\
 & = \sum_{r=0}^{\infty} {}_3F_2\left(-r, \gamma, 2\lambda - \gamma; 2\lambda, \lambda + \frac{1}{2}; y\right) t^r P_r^\lambda(x);
 \end{aligned}$$

where

$$\omega = [1 - 2xt(1-y) + t^2(1-y)^2]^{1/2}.$$

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