

# Pacific Journal of Mathematics

**RELATIONS ON MINIMAL HYPERSURFACES**

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## RELATIONS ON MINIMAL HYPERSURFACES

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DEDICATED TO THE MEMORY OF CHARLES LOEWNER

In the theory of nonparametric minimal surfaces there is a transformation which replaces a minimal surface by a certain type of convex surface. Construction of this transformation depends on the exactness of certain differential one-forms, a consequence of the minimal surface equation. In this article analogous systems of  $(n-1)$ -forms are introduced on a minimal  $n$ -hypersurface. This leads to new tensors and to relations between them.

Let  $u = u(x, y)$  satisfy the minimal hypersurface equation

$$(1 + p^2 + q^2)(r + t) = rp^2 + 2spq + tq^2.$$

It is known (see Radó [6], pp. 57-60) that if we set

$$w^2 = 1 + p^2 + q^2, \quad \alpha = dx + p du, \quad \beta = dy + q du,$$

then

$$d\left(\frac{\alpha}{w}\right) = 0, \quad d\left(\frac{\beta}{w}\right) = 0.$$

Also if we define  $P$  and  $Q$  by

$$dP = \frac{\alpha}{w}, \quad dQ = \frac{\beta}{w},$$

then

$$d(P dx + Q dy) = 0,$$

hence there is a function  $U$  satisfying

$$dU = P dx + Q dy.$$

The function  $U$  has Hessian

$$\frac{\partial^2 U}{\partial x^2} \frac{\partial^2 U}{\partial y^2} - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2 = 1$$

and by Jörgens [4, Th. 2],  $U$  must be a quadratic polynomial if  $u$  is defined on the whole plane. This yields another proof of Bernstein's theorem. Nitsche [5] gave an alternative proof of Jörgen's result, Flanders [2] pushed the proof, not the theorem, to  $n$ -dimensions, and Calabi [1] pushed Jörgen's theorem to five dimensions with smooth-

ness requirements.

This paper is a partial attempt to extend the formal transition from  $u$  to  $U$  to more than two dimensions.

2. **Notation.** Let  $u = u(x_1, \dots, x_n)$  be  $C''$  on a domain in  $E^n$ . Set

$$p_i = \frac{\partial u}{\partial x_i}, \quad r_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad w^2 = 1 + \sum p_i^2.$$

The mean curvature of the graph of  $u$  is

$$H = \frac{-1}{nw^3} [w^2 \sum r_{ii} - \sum p_i r_{ij} p_j].$$

(See Flanders [3, p. 126].) This graph is a minimal hypersurface if  $H = 0$ , i.e.,

$$w^2 \sum r_{ii} = \sum p_i r_{ij} p_j.$$

We introduce the matrices

$$\begin{aligned} d\mathbf{x} &= (dx_1, \dots, dx_n), & \mathbf{p} &= (p_1, \dots, p_n), \\ \mathbf{R} &= \| r_{ij} \|, & \mathbf{B} &= \mathbf{I} + {}^t\mathbf{p}\mathbf{p}. \end{aligned}$$

The minimal hypersurface equation is

$$(2.1) \quad w^2 \operatorname{tr}(\mathbf{R}) = \mathbf{p}\mathbf{R} {}^t\mathbf{p}.$$

We set

$$\begin{aligned} \alpha_i &= dx_i + p_i du = dx_i + \sum p_i p_j dx_j, \\ \boldsymbol{\alpha} &= (\alpha_1, \dots, \alpha_n). \end{aligned}$$

Hence

$$(2.2) \quad \boldsymbol{\alpha} = dx \mathbf{B}.$$

3. **Relations.** Since  $\mathbf{p} {}^t\mathbf{p} = w^2 - 1$  we have

$$({}^t\mathbf{p}\mathbf{p})^2 = (w^2 - 1)({}^t\mathbf{p}\mathbf{p}).$$

It follows that

$$(3.1) \quad \mathbf{B}^2 - (w^2 + 1)\mathbf{B} + w^2\mathbf{I} = \mathbf{0}.$$

The characteristic roots of the rank zero or one matrix  ${}^t\mathbf{p}\mathbf{p}$  are 0 with multiplicity  $n - 1$  and  $(w^2 - 1)$ . It follows that the roots of  $\mathbf{B}$  are 1 with multiplicity  $n - 1$  and  $w^2$ . This gives us

$$(3.2) \quad |\mathbf{B}| = w^2.$$

From (3.1) we have

$$(3.3) \quad B^{-1} = \frac{1}{w^2}[(w^2 + 1)I - B] = I - \frac{1}{w^2} {}^t \mathbf{p} \mathbf{p} .$$

and for the matrix of cofactors,

$$(3.4) \quad \text{cof } B = (w^2 + 1)I - B = w^2 I - {}^t \mathbf{p} \mathbf{p} .$$

We note that  $B$  and this matrix  $\text{cof } B$  are positive definite.

We next establish the relations

$$(3.5) \quad \mathbf{p} \wedge {}^t \boldsymbol{\alpha} = w^2 du ,$$

$$(3.6) \quad d\boldsymbol{\alpha} = d\mathbf{p} \wedge du ,$$

$$(3.7) \quad \boldsymbol{\alpha} \wedge {}^t d\mathbf{x} = \mathbf{0} .$$

For

$$\begin{aligned} \mathbf{p} \wedge {}^t \boldsymbol{\alpha} &= \mathbf{p} \wedge ({}^t d\mathbf{x} + {}^t \mathbf{p} du) = du + (w^2 - 1)du = w^2 du, \\ d\boldsymbol{\alpha} &= d(d\mathbf{x} + \mathbf{p} du) = d\mathbf{p} \wedge du, \end{aligned}$$

and

$$\boldsymbol{\alpha} \wedge {}^t d\mathbf{x} = (d\mathbf{x} + d\mathbf{p}) \wedge {}^t d\mathbf{x} = du \wedge du = \mathbf{0} .$$

For convenience we shall set

$$(3.8) \quad M = M(u) = w^2 \sum r_{ii} - \sum p_i r_{ij} p_j .$$

When there is no danger of misinterpretation we shall omit the wedge ( $\wedge$ ) in exterior products. Finally we use the abbreviation

$$d\tau = dx_1 \cdots dx_n$$

for the volume element of  $E^n$ .

We next introduce the usual star (adjoint operator)  $*$ . (See Flanders [3, pp. 15-17; pp. 82 ff.]) With this we have

$$\begin{aligned} *du &= \sum (-1)^{i-1} p_i dx_1 \cdots \widehat{dx_i} \cdots dx_n , \\ d\left(\frac{1}{w} *du\right) &= -\frac{1}{w^3} (wdw \wedge *du) + \frac{1}{w} d*du \\ &= \frac{1}{w} (\sum r_{ii} d\tau) - \frac{1}{w^3} (\sum p_i dp_i \wedge *du) \\ &= \frac{1}{w^3} [w^2 \sum r_{ii} d\tau - \sum p_i r_{ij} dx_j p_k *dx_k] \\ &= \frac{1}{w^3} [w^2 \sum r_{ii} - \sum p_i r_{ij} p_j] d\tau \end{aligned}$$

and so

$$(3.9) \quad d\left(\frac{1}{w} *du\right) = \frac{1}{w^3} M(u) d\tau .$$

The components of the vector  $*dx$  are the  $(n-1)$ -forms

$$(-1)^{i-1} dx_1 \cdots \widehat{dx}_i \cdots dx_n .$$

We seek the corresponding expressions in the  $\alpha_i$ . We introduce the notation

$$(3.10) \quad \alpha^* = (\cdots, (-1)^{i-1} \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_n, \cdots) .$$

Since  $\alpha = dxB$  we have

$$(\cdots, \alpha_1 \cdots \widehat{\alpha}_i \cdots \alpha_n, \cdots) = (\cdots, dx_1 \cdots \widehat{dx}_j \cdots dx_n, \cdots) (\wedge^{n-1} B) .$$

Now  $\wedge^{n-1} B$  is the matrix of  $(n-1)$ -rowed minors of the (symmetric) matrix  $B$ . Alternating the signs changes this to  $\text{cof } B$ , hence

$$(3.11) \quad \alpha^* = (*dx)(\text{cof } B) .$$

**THEOREM 1.** *We have*

$$(3.12) \quad \alpha^* \wedge {}^t d\mathbf{p} = M(u) d\tau .$$

*Proof.* By (3.11)

$$\begin{aligned} \alpha^* \wedge {}^t d\mathbf{p} &= (*dx)(\text{cof } B)(R^t dx) \\ &= \text{tr} [(\text{cof } B)R] d\tau . \end{aligned}$$

By (3.4) and (3.8),

$$\begin{aligned} \text{tr} [(\text{cof } B)R] &= \text{tr} [w^2 R - {}^t \mathbf{p} \mathbf{p} R] \\ &= w^2 \text{tr } R - \mathbf{p} R^t \mathbf{p} \\ &= M(u) . \end{aligned}$$

**LEMMA.** *We have*

$$(3.13) \quad (wdw)\alpha^* = \mathbf{p} R (\text{cof } B) d\tau ,$$

$$(3.14) \quad d\alpha^* = [\mathbf{p} R - (\text{tr } R)\mathbf{p}] d\tau .$$

*Proof.* We have

$$wdw = \mathbf{p}^t d\mathbf{p} = \mathbf{p} R^t dx$$

hence

$$\begin{aligned}
(wdw)\alpha^* &= pR({}^t dx)(*dx)(\text{cof } B) \\
&= pR(d\tau I)(\text{cof } B) \\
&= pR(\text{cof } B)d\tau .
\end{aligned}$$

We avoid some signs by transposing and have

$$\begin{aligned}
{}^t(\alpha^*) &= (\text{cof } B) {}^t(*dx) = (w^2 I - {}^t p p) {}^t(*dx) , \\
{}^t(d\alpha^*) &= [2wdwI - d({}^t p p)] {}^t(*dx) \\
&= [2dxR {}^t p - {}^t p dxR - R {}^t dx p] {}^t(*dx) \\
&= [2R {}^t p - {}^t p(\text{tr } R) - R {}^t p] d\tau \\
&= [R {}^t p - (\text{tr } R) {}^t p] d\tau .
\end{aligned}$$

Equation (3.14) follows.

We now state the main result of this section.

**THEOREM 2.** *We have*

$$(3.15) \quad d\left(\frac{1}{w}\alpha^*\right) = \frac{1}{w^3}M(u)p d\tau .$$

*Proof.* By (3.13),

$$\begin{aligned}
(wdw)\alpha^* &= pR(w^2 I - {}^t p p)d\tau \\
&= w^2 p R d\tau - (p R {}^t p) p d\tau .
\end{aligned}$$

Using (3.14) we have

$$\begin{aligned}
(wdw)\alpha^* - w^2 d\alpha^* &= w^2(\text{tr } R)p d\tau - (p R {}^t p) p d\tau \\
&= M(u)p d\tau ,
\end{aligned}$$

and the result follows.

**COROLLARY.** *If the graph of  $u$  is a minimal hypersurface, then*

$$d\left(\frac{1}{w}\alpha^*\right) = 0 .$$

We close this section with the proof of one other relation :

$$(3.16) \quad du\alpha^* = p d\tau .$$

By (3.5),

$$(w^2 du)\alpha^* = p {}^t \alpha \alpha^* = p(\alpha_1 \cdots \alpha_n) .$$

But  $\alpha_1 \cdots \alpha_n = |B| d\tau = w^2 d\tau$  and (3.16) follows.

4. **Minimal hypersurfaces.** In this section we assume  $u$  is defined on a contractible domain and that  $M(u) = 0$  so that the graph of  $u$  is a minimal hypersurface.

By the corollary above, each of the  $(n - 1)$ -forms

$$\frac{1}{w}\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n$$

is closed. Hence there exist  $(n - 2)$ -forms  $\omega_i$  ( $i = 1, \dots, n$ ) such that

$$(4.1) \quad d\omega_j = \frac{(-1)^{j-1}}{w}\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n \quad (j = 1, \dots, n).$$

**THEOREM 3.** *For each  $i, j$  we have*

$$(4.2) \quad d(\omega_j dx_j - \omega_i dx_i) = 0.$$

*Proof.* We multiply the relation (3.7) by

$$\alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n$$

to derive

$$\begin{aligned} (\alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n)(\alpha_i dx_i + \alpha_j dx_j) &= 0, \\ (-1)^{i+1}(\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n)dx_i + (-1)^j(\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n)dx_j &= 0, \\ (-1)^{i+1}(-1)^{j-1}d\omega_j dx_i + (-1)^j(-1)^{i-1}d\omega_i dx_j &= 0. \end{aligned}$$

and the result follows.

**COROLLARY.** *There exist  $(n - 1)$ -forms  $\eta_{ij}$  such that*

$$\eta_{ij} + \eta_{ji} = 0$$

and

$$(4.3) \quad d\eta_{ij} = \omega_i dx_j - \omega_j dx_i \quad (i, j = 1, \dots, n).$$

There are too many choices of the  $\omega_i$  and  $\eta_{ij}$ . We should expect progress on Bernstein's Theorem in higher dimension if a way were found of limiting these forms to families with finitely many parameters.

To take one step in this direction we use the operators  $\delta$ ,  $\Delta$ . (See Flanders [2], pp. 136 ff.) One known fact is that the Poisson equation

$$\Delta f = y$$

has a solution on  $E^n$  for any continuous  $y$ . This implies that if  $\beta$  is a  $p$ -form on  $E^n$ , then

$$\Delta\alpha = \beta$$

has a solution  $\alpha$ .

Now consider the  $(n - 2)$ -form  $\omega_i$ . We may write

$$\omega_i = \Delta\lambda_i = d\delta\lambda_i + \delta d\lambda_i$$

hence

$$d\omega_i = d\delta d\lambda_i.$$

Thus we may replace  $\omega_i$  by  $\delta d\lambda_i$ . Now  $\lambda_i$  is determined up to an  $(n - 2)$ -form  $\mu_i$  such that  $d\delta d\mu_i = 0$ . There are, unfortunately, still too many of these when  $n \geq 3$ .

REMARK. If  $f$  is any function on the hypersurface, its Laplacian relative to the hypersurface is

$$(4.4) \quad \bar{\Delta}f = \frac{1}{w} \sum \frac{\partial}{\partial x_i} \left( \frac{1}{w} \sum_j (w^2 \delta_{ij} - p_i p_j) \frac{\partial f}{\partial x_j} \right).$$

(Here  $\bar{\Delta}$  is the Beltrami operator.) We apply this to  $f = x$  and use (3.11) to obtain

$$(4.5) \quad w(\bar{\Delta}x) = d\left(\frac{1}{w}\alpha^*\right).$$

We also apply (4.4) to  $f = u$ :

$$\begin{aligned} w(\bar{\Delta}u) &= \sum \frac{\partial}{\partial x_i} \left[ \frac{1}{w} \sum_j (w^2 \delta_{ij} - p_i p_j) p_j \right] \\ &= \sum \frac{\partial}{\partial x_i} \left[ \frac{1}{w} (w^2 p_i - (w^2 - 1) p_i) \right] \\ &= \sum \frac{\partial}{\partial x_i} \left( \frac{p_i}{w} \right). \end{aligned}$$

These formulas verify the well-known fact that on a minimal hypersurface each of the euclidean coordinate functions  $x_1, \dots, x_n$ ,  $u$  is harmonic.

5. Equations in component form. We shall restate the results of § 4 in component form. As in that section we assume  $M(u) = 0$ . We set

$$(5.1) \quad G = \frac{1}{w} (\text{cof } B) = \|g_{ij}\|$$

so that (4.1) and (3.11) become



$$(5.2) \quad d\omega_i = g_{ij} * dx_j ,$$

where we use the summation convention as we shall in this section.  
We write

$$(5.3) \quad \omega_i = \frac{1}{2} a_{ijk} * (dx_j dx_k) , \quad a_{ijk} + a_{ikj} = 0 .$$

Now (5.2) may be rewritten as

$$(5.4) \quad \frac{\partial a_{ijk}}{\partial x_k} = g_{ij} .$$

This is obtained by a direct calculation which hinges on the following readily checked relations :

$$(5.5) \quad \begin{aligned} dx_k \wedge *(dx_j dx_k) &= *dx_k , \\ dx_j \wedge *(dx_j dx_k) &= - *dx_j . \end{aligned}$$

Next we set

$$(5.6) \quad (-1)^{n-2} \eta_{ij} = \frac{1}{2} b_{ijk} *(dx_k dx_i) ,$$

where

$$(5.7) \quad \begin{aligned} b_{ijk} + b_{jik} &= 0 \\ b_{ijk} + b_{jlk} &= 0 . \end{aligned}$$

In this notation the relations (4.3) become

$$(5.8) \quad \frac{\partial b_{ijk}}{\partial x_l} = a_{jik} - a_{ijk} .$$

Combined with the skew-symmetry of  $a_{ijk}$  in the second and third indices, this yields in the usual way

$$(5.9) \quad a_{ijk} = \frac{\partial c_{ijkl}}{\partial x_l}$$

where

$$(5.10) \quad c_{ijkl} = \frac{1}{2} (-b_{ijk} + b_{kij} - b_{kji}) .$$

These relations imply

$$(5.11) \quad b_{ijk} = -c_{ijk} - c_{jik} .$$

The skew-symmetries in (5.7) thus are equivalent to

$$(5.12) \quad \begin{aligned} c_{ijkl} + c_{jkil} + c_{ijlk} + c_{jlik} &= 0, \\ c_{ijkl} + c_{jkil} + c_{jilk} + c_{ikjl} &= 0. \end{aligned}$$

Equations (5.9) and (5.4) combine to yield

$$(5.13) \quad \frac{\partial^2 c_{ijkl}}{\partial x_k \partial x_l} = g_{ij}.$$

The minimal hypersurface equation  $M(u) = 0$  may be interpreted as integrability conditions for (5.13) with the side conditions (5.14).

We may cut down the number of variables by introducing

$$(5.14) \quad \begin{aligned} h_{ijkl} &= \frac{1}{4}(c_{ijkl} + c_{ijlk} + c_{jilk} + c_{jlik}) \\ &= \frac{1}{4}(b_{ikjl} + b_{jljk} + b_{jkil} + b_{iljk}). \end{aligned}$$

Then we have

$$(5.15) \quad \begin{aligned} h_{ijkl} &= h_{jikl} \\ h_{ijkl} &= h_{ijlk} \end{aligned}$$

while (5.13) implies

$$(5.16) \quad \frac{\partial^2 h_{ijkl}}{\partial x_k \partial x_l} = g_{ij}.$$

In addition to the symmetries in (5.15) the quantities  $h$  satisfy

$$(5.17) \quad h_{ijkl} = h_{klij} = 0$$

and

$$(5.18) \quad h_{ijkl} + h_{jkil} + h_{kijl} = 0.$$

These are easy consequences of (5.14) and (5.7). The relations (5.15), (5.17), (5.18) span all relations in the  $h$ 's. To see this we must count dimensions. The space of tensors  $(b)$  subject to (5.7) has dimension  $n^2(n-1)^2/4$ . The nullity of the mapping  $(b) \rightarrow (h)$  given by (5.14) is determined by finding independent solutions of

$$(5.19) \quad (ijkl) + (jlik) + (jkil) + (iljk) = 0$$

where we abbreviate  $(ijkl) = b_{ijkl}$ . We need consider only  $(ijkl)$  where  $i < j$  and  $k < l$ , using (5.7) to determine the others. By (5.19),

$$4(1212) = 0, \quad (1212) = 0.$$

The  $(ijkl)$  with three distinct indices are represented by (say) indices

1, 1, 2, 3 and this gives us (1213) and (1312). But by (5.19),

$$2(1213) + 2(1312) = 0 ,$$

hence we are free to choose only one of these. We thus have  $3\binom{n}{3}$  degrees of freedom in choosing  $(ijkl)$  with three distinct indices. If there are four distinct indices, say 1, 2, 3, 4, the quantities we consider are these six :

$$(1234) , (1324) , (1423) , (2314) , (2413) , (3412) .$$

The relations (5.19) are seen to yield two independent relations amongst these :

$$\begin{aligned} (1234) + (3412) - (2314) - (1423) &= 0 , \\ (1234) + (3412) + (1234) + (2413) &= 0 . \end{aligned}$$

This means that with all indices distinct we have  $4\binom{n}{4}$  degrees of freedom. Thus the desired nullity is

$$3\binom{n}{3} + 4\binom{n}{4}$$

and the rank equals dimension of the  $(h)$  space is

$$\frac{n^2(n-1)^2}{4} - 3\binom{n}{3} - 4\binom{n}{4} = \frac{n^2(n^2-1)}{12} .$$

On the other hand, the space of  $(h)$  tensors subject to (5.15), (5.17), and (5.18) has precisely the same dimensions. To see this we use (5.15) and (5.17) to limit the parameter to those  $(ijkl)$  for which  $i \leq j$ ,  $k \leq l$ , and  $(ij) \leq (kl)$  in lexicographic order. (Now  $(ijkl)$  denotes  $h_{ijkl}$ .) By (5.17),  $(1111) = 0$  and  $(1112) = 0$ . With two distinct indices we need only consider (1212) and (1122). By (5.17) these are related by

$$(1122) + 2(1212) = 0 .$$

Thus with only two distinct indices we have  $\binom{n}{2}$  degrees of freedom. With three distinct indices, say 1, 1, 2, 3, the only possibilities, (1123) and (1213), are again related by

$$(1123) + 2(1213) = 0 .$$

We thus have  $3\binom{n}{3}$  degrees of freedom in this case. Finally with four distinct indices, say 1, 2, 3, 4, the three possibilities, (1234), (1324), and (1423), are related by

$$(1234) + (1324) + (1423) = 0$$

so we have  $2\binom{n}{4}$  degrees of freedom in this case. In total the space of  $(h)$  we are considering has dimension

$$\binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4} = \frac{n^2(n^2 - 1)}{12}.$$

This completes our proof that the relations (5.15), (5.17), and (5.18) span all relations between the  $h$ 's. In the course of the proof we have obtained a set of independent parameters for the  $(h)$  space:

$$\begin{array}{ll} h_{ijij} & (i < j), \\ h_{ijik} & (i < j < k), \\ h_{ijkl}, h_{ikjl} & (i < j < k < l). \end{array}$$

This result  $n^2(n^2 - 1)/12$  is certainly better than the number of  $b$ 's (or  $c$ 's), namely  $n^2(n - 1)^2/4$ . When  $n = 2$ , both numbers are one so that equations (5.16) only involve a single unknown function  $h = h_{1212}$ . This is what makes a proof of Bernstein's Theorem along the lines discussed in the introduction work.

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Received February 8, 1968. This research was supported by National Science Foundation Grant GP 6388. Reproduction for any purpose by the United States Government is permitted.

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Vol. 29, No. 1

May, 1969

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