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In the theory of nonparametric minimal surfaces there is a transformation which replaces a minimal surface by a certain type of convex surface. Construction of this transformation depends on the exactness of certain differential one-forms, a consequence of the minimal surface equation. In this article analogous systems of $(n-1)$-forms are introduced on a minimal n-hypersurface. This leads to new tensors and to relations between them.

Let $u = u(x, y)$ satisfy the minimal hypersurface equation

$$(1 + p^2 + q^2)(r + t) = rp^2 + 2spq + tq^2.$$

It is known (see Radó [6], pp. 57–60) that if we set

$$w^2 = 1 + p^2 + q^2,\ \alpha = dx + p\,du,\ \beta = dy + q\,du,$$

then

$$d\left(\frac{\alpha}{w}\right) = 0\,, \qquad d\left(\frac{\beta}{w}\right) = 0\,.$$

Also if we define P and Q by

$$dP = \frac{\alpha}{w}\,, \qquad dQ = \frac{\beta}{w}\,,$$

then

$$d(P\,dx + Q\,dy) = 0\,,$$

hence there is a function U satisfying

$$dU = P\,dx + Q\,dy\,.$$

The function U has Hessian

$$\frac{\partial^2 U}{\partial x^2}\frac{\partial^2 U}{\partial y^2} - \left(\frac{\partial^2 U}{\partial x \partial y}\right)^2 = 1$$

and by Jörgens [4, Th. 2], U must be a quadratic polynomial if u is defined on the whole plane. This yields another proof of Bernstein's theorem. Nitsche [5] gave an alternative proof of Jörgen's result, Flanders [2] pushed the proof, not the theorem, to n-dimensions, and Calabi [1] pushed Jörgen's theorem to five dimensions with smoothness requirements.

This paper is a partial attempt to extend the formal transition from u to U to more than two dimensions.

2. Notation. Let $u = u(x_1, \cdots, x_n)$ be C'' on a domain in E^n. Set

$$p_i = \frac{\partial u}{\partial x_i}, \qquad r_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad w^2 = 1 + \sum p_i^2 .$$

The mean curvature of the graph of u is

$$H = \frac{-1}{nw^3}[w^2 \sum r_{ii} - \sum p_i r_{ij} p_j] .$$

(See Flanders [3, p. 126].) This graph is a minimal hypersurface if $H = 0$, i.e.,

$$w^2 \sum r_{ii} = \sum p_i r_{ij} p_j .$$

We introduce the matrices

$$\begin{aligned} d\boldsymbol{x} &= (dx_1, \cdots, dx_n) , & \boldsymbol{p} &= (p_1, \cdots, p_n) , \\ R &= \| r_{ij} \| , & B &= I + {}^t\boldsymbol{pp} . \end{aligned}$$

The minimal hypersurface equation is

$$w^2 \operatorname{tr}(R) = \boldsymbol{p}R\,{}^t\boldsymbol{p} . \tag{2.1}$$

We set

$$\begin{aligned} \alpha_i &= dx_i + p_i du = dx_i + \sum p_i p_j dx_j , \\ \boldsymbol{\alpha} &= (\alpha_1, \cdots, \alpha_n) . \end{aligned}$$

Hence

$$\boldsymbol{\alpha} = dx\,\mathbf{B} . \tag{2.2}$$

3. Relations. Since $\boldsymbol{p}^t\boldsymbol{p} = w^2 - 1$ we have

$$({}^t\boldsymbol{pp})^2 = (w^2 - 1)({}^t\boldsymbol{pp}) .$$

It follows that

$$B^2 - (w^2 + 1)B + w^2 I = 0 . \tag{3.1}$$

The characteristic roots of the rank zero or one matrix ${}^t\boldsymbol{pp}$ are 0 with multiplicity $n - 1$ and $(w^2 - 1)$. It follows that the roots of B are 1 with multiplicity $n - 1$ and w^2. This gives us

$$| B | = w^2 . \tag{3.2}$$

From (3.1) we have

$$B^{-1} = \frac{1}{w^2}[(w^2+1)I - B] = I - \frac{1}{w^2}\,{}^t\boldsymbol{p}\boldsymbol{p}\,. \tag{3.3}$$

and for the matrix of cofactors,

$$\operatorname{cof} B = (w^2+1)I - B = w^2 I - {}^t\boldsymbol{p}\boldsymbol{p}\,. \tag{3.4}$$

We note that B and this matrix $\operatorname{cof} B$ are positive definite.

We next establish the relations

$$\boldsymbol{p} \wedge {}^t\boldsymbol{\alpha} = w^2 du\,, \tag{3.5}$$

$$d\boldsymbol{\alpha} = d\boldsymbol{p} \wedge du\,, \tag{3.6}$$

$$\boldsymbol{\alpha} \wedge {}^t d\boldsymbol{x} = 0\,. \tag{3.7}$$

For

$$\boldsymbol{p} \wedge {}^t\boldsymbol{\alpha} = \boldsymbol{p} \wedge ({}^t d\boldsymbol{x} + {}^t\boldsymbol{p}\,du) = du + (w^2-1)du = w^2 du,$$
$$d\boldsymbol{\alpha} = d(d\boldsymbol{x} + \boldsymbol{p}du) = d\boldsymbol{p} \wedge du,$$

and

$$\boldsymbol{\alpha} \wedge {}^t d\boldsymbol{x} = (d\boldsymbol{x} + du\boldsymbol{p}) \wedge {}^t d\boldsymbol{x} = du \wedge du = 0\,.$$

For convenience we shall set

$$M = M(u) = w^2 \sum r_{ii} - \sum p_i r_{ij} p_j\,. \tag{3.8}$$

When there is no danger of misinterpretation we shall omit the wedge ($\wedge$) in exterior products. Finally we use the abbreviation

$$d\tau = dx_1 \cdots dx_n$$

for the volume element of E^n.

We next introduce the usual star (adjoint operator) $*$. (See Flanders [3, pp. 15–17; pp. 82 ff.].) With this we have

$$*du = \sum (-1)^{i-1} p_i dx_1 \cdots \widehat{dx_i} \cdots dx_n\,,$$
$$\begin{aligned} d\Big(\frac{1}{w}*du\Big) &= -\frac{1}{w^3}(w\,dw \wedge *du) + \frac{1}{w}d*du \\ &= \frac{1}{w}(\sum r_{ii}d\tau) - \frac{1}{w^3}(\sum p_i dp_i \wedge *du) \\ &= \frac{1}{w^3}[w^2 \sum r_{ii}d\tau - \sum p_i r_{ij} dx_j p_k * dx_k] \\ &= \frac{1}{w^3}[w^2 \sum r_{ii} - \sum p_i r_{ij} p_j]d\tau \end{aligned}$$

and so

$$d\left(\frac{1}{w}*du\right) = \frac{1}{w^3}M(u)d\tau\ . \tag{3.9}$$

The components of the vector $*dx$ are the $(n-1)$-forms

$$(-1)^{i-1}dx_1 \cdots \widehat{dx_i} \cdots dx_n\ .$$

We seek the corresponding expressions in the α_i. We introduce the notation

$$\boldsymbol{\alpha}^* = (\cdots,\ (-1)^{i-1}\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n,\ \cdots)\ . \tag{3.10}$$

Since $\boldsymbol{\alpha} = d\boldsymbol{x}B$ we have

$$(\cdots,\ \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n,\ \cdots) = (\cdots,\ dx_1 \cdots \widehat{dx_j} \cdots dx_n,\ \cdots)(\wedge^{n-1}B)\ .$$

Now $\wedge^{n-1}B$ is the matrix of $(n-1)$-rowed minors of the (symmetric) matrix B. Alternating the signs changes this to cof B, hence

$$\boldsymbol{\alpha}^* = (*d\boldsymbol{x})(\operatorname{cof} B)\ . \tag{3.11}$$

THEOREM 1. *We have*

$$\boldsymbol{\alpha}^* \wedge {}^t d\boldsymbol{p} = M(u)d\tau\ . \tag{3.12}$$

Proof. By (3.11)

$$\begin{aligned}\boldsymbol{\alpha}^* \wedge {}^t d\boldsymbol{p} &= (*d\boldsymbol{x})(\operatorname{cof} B)(R^t d\boldsymbol{x}) \\ &= \operatorname{tr}\,[(\operatorname{cof} B)R]d\tau\ .\end{aligned}$$

By (3.4) and (3.8),

$$\begin{aligned}\operatorname{tr}\,[(\operatorname{cof} B)R] &= \operatorname{tr}\,[w^2R - {}^t\boldsymbol{p}\boldsymbol{p}R] \\ &= w^2 \operatorname{tr} R - \boldsymbol{p}R^t\boldsymbol{p} \\ &= M(u)\ .\end{aligned}$$

LEMMA. *We have*

$$(wdw)\boldsymbol{\alpha}^* = \boldsymbol{p}R(\operatorname{cof} B)d\tau\ , \tag{3.13}$$

$$d\boldsymbol{\alpha}^* = [\boldsymbol{p}R - (\operatorname{tr} R)\boldsymbol{p}]d\tau\ . \tag{3.14}$$

Proof. We have

$$wdw = \boldsymbol{p}^t d\boldsymbol{p} = \boldsymbol{p}R^t d\boldsymbol{x}$$

hence

$$\begin{aligned}(wdw)\boldsymbol{\alpha}^* &= \boldsymbol{p}R({}^t d\boldsymbol{x})(*d\boldsymbol{x})(\operatorname{cof} B)\\ &= \boldsymbol{p}R(d\tau I)(\operatorname{cof} B)\\ &= \boldsymbol{p}R(\operatorname{cof} B)d\tau\ .\end{aligned}$$

We avoid some signs by transposing and have

$$\begin{aligned}{}^t(\boldsymbol{\alpha}^*) &= (\operatorname{cof} B)\,{}^t(*d\boldsymbol{x}) = (w^2 I - {}^t\boldsymbol{p}\boldsymbol{p})\,{}^t(*d\boldsymbol{x})\ ,\\ {}^t(d\boldsymbol{\alpha}^*) &= [2wdwI - d({}^t\boldsymbol{p}\boldsymbol{p})]\,{}^t(*d\boldsymbol{x})\\ &= [2d\boldsymbol{x}R\,{}^t\boldsymbol{p} - {}^t\boldsymbol{p}d\boldsymbol{x}R - R\,{}^t d\boldsymbol{x}\boldsymbol{p}]\,{}^t(*d\boldsymbol{x})\\ &= [2R{}^t\boldsymbol{p} - {}^t\boldsymbol{p}(trR) - R\,{}^t\boldsymbol{p}]d\tau\\ &= [R\,{}^t\boldsymbol{p} - (trR)\,{}^t\boldsymbol{p}]d\tau\ .\end{aligned}$$

Equation (3.14) follows.

We now state the main result of this section.

THEOREM 2. *We have*

$$d\left(\frac{1}{w}\boldsymbol{\alpha}^*\right) = \frac{1}{w^3}M(u)\boldsymbol{p}d\tau\ . \tag{3.15}$$

Proof. By (3.13),

$$\begin{aligned}(wdw)\boldsymbol{\alpha}^* &= \boldsymbol{p}R(w^2 I - {}^t\boldsymbol{p}\boldsymbol{p})d\tau\\ &= w^2\boldsymbol{p}Rd\tau - (\boldsymbol{p}R\,{}^t\boldsymbol{p})\boldsymbol{p}d\tau\ .\end{aligned}$$

Using (3.14) we have

$$\begin{aligned}(wdw)\boldsymbol{\alpha}^* - w^2 d\boldsymbol{\alpha}^* &= w^2(\operatorname{tr} R)\boldsymbol{p}d\tau - (\boldsymbol{p}R\,{}^t\boldsymbol{p})\boldsymbol{p}d\tau\\ &= M(u)\boldsymbol{p}d\tau\ ,\end{aligned}$$

and the result follows.

COROLLARY. *If the graph of u is a minimal hypersurface, then*

$$d\left(\frac{1}{w}\boldsymbol{\alpha}^*\right) = 0\ .$$

We close this section with the proof of one other relation:

$$du\boldsymbol{\alpha}^* = \boldsymbol{p}d\tau\ . \tag{3.16}$$

By (3.5),

$$(w^2 du)\boldsymbol{\alpha}^* = \boldsymbol{p}\,{}^t\boldsymbol{\alpha}\boldsymbol{\alpha}^* = \boldsymbol{p}(\alpha_1 \cdots \alpha_n)\ .$$

But $\alpha_1 \cdots \alpha_n = |B|\,d\tau = w^2 d\tau$ and (3.16) follows.

4. Minimal hypersurfaces. In this section we assume u is defined on a contractible domain and that $M(u) = 0$ so that the graph of u is a minimal hypersurface.

By the corollary above, each of the $(n-1)$-forms

$$\frac{1}{w}\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n$$

is closed. Hence there exist $(n-2)$-forms ω_i $(i = 1, \cdots, n)$ such that

$$d\omega_j = \frac{(-1)^{j-1}}{w}\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n \qquad (j = 1, \cdots, n)\ . \tag{4.1}$$

THEOREM 3. *For each i, j we have*

$$d(\omega_i dx_j - \omega_j dx_i) = 0\ . \tag{4.2}$$

Proof. We multiply the relation (3.7) by

$$\alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n$$

to derive

$$(\alpha_1 \cdots \hat{\alpha}_i \cdots \hat{\alpha}_j \cdots \alpha_n)(\alpha_i dx_i + \alpha_j dx_j) = 0\ ,$$
$$(-1)^{i+1}(\alpha_1 \cdots \hat{\alpha}_j \cdots \alpha_n)dx_i + (-1)^j(\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n)dx_j = 0\ ,$$
$$(-1)^{i+1}(-1)^{j-1}d\omega_j dx_i + (-1)^j(-1)^{i-1}d\omega_i dx_j = 0\ .$$

and the result follows.

COROLLARY. *There exist $(n-1)$-forms η_{ij} such that*

$$\eta_{ij} + \eta_{ji} = 0$$

and

$$d\eta_{ij} = \omega_i dx_j - \omega_j dx_i \qquad (i,j = 1, \cdots, n)\ . \tag{4.3}$$

There are too many choices of the ω_i and η_{ij}. We should expect progress on Bernstein's Theorem in higher dimension if a way were found of limiting these forms to families with finitely many parameters.

To take one step in this direction we use the operators δ, Δ. (See Flanders [2], pp. 136 ff.) One known fact is that the Poisson equation

$$\Delta f = y$$

has a solution on E^n for any continuous y. This implies that if β is a p-form on E^n, then

$$\Delta\alpha = \beta$$

has a solution α.

Now consider the $(n-2)$-form ω_i. We may write

$$\omega_i = \Delta\lambda_i = d\delta\lambda_i + \delta d\lambda_i$$

hence

$$d\omega_i = d\delta d\lambda_i \, .$$

Thus we may replace ω_i by $\delta d\lambda_i$. Now λ_i is determined up to an $(n-2)$-form μ_i such that $d\delta d\mu = 0$. There are, unfortunately, still too many of these when $n \geqq 3$.

REMARK. If f is any function on the hypersurface, its Laplacian relative to the hypersurface is

$$\bar{\Delta} f = \frac{1}{w} \sum \frac{\partial}{\partial x_i}\left(\frac{1}{w} \sum_j (w^2\delta_{ij} - p_i p_j)\frac{\partial f}{\partial x_j}\right) . \tag{4.4}$$

(Here $\bar{\Delta}$ is the Beltrami operator.) We apply this to $f = x$ and use (3.11) to obtain

$$w(\bar{\Delta} x) = d\left(\frac{1}{w}\boldsymbol{\alpha}^*\right) . \tag{4.5}$$

We also apply (4.4) to $f = u$:

$$\begin{aligned} w(\bar{\Delta} u) &= \sum \frac{\partial}{\partial x_i}\left[\frac{1}{w} \sum_j (w^2\delta_{ij} - p_i p_j) p_j\right] \\ &= \sum \frac{\partial}{\partial x_i}\left[\frac{1}{w}(w^2 p_i - (w^2 - 1)p_i)\right] \\ &= \sum \frac{\partial}{\partial x_i}\left(\frac{p_i}{w}\right) . \end{aligned}$$

These formulas verify the well-known fact that on a minimal hypersurface each of the euclidean coordinate functions $x_1, \cdots, x_n$, u is harmonic.

5. **Equations in component form.** We shall restate the results of § 4 in component form. As in that section we assume $M(u) = 0$. We set

$$G = \frac{1}{w}(\operatorname{cof} B) = \| g_{ij} \| \tag{5.1}$$

so that (4.1) and (3.11) become

$$d\omega_i = g_{ij}*dx_j\ , \tag{5.2}$$

where we use the summation convention as we shall in this section. We write

$$\omega_i = \frac{1}{2}a_{ijk}*(dx_j dx_k)\ , \qquad a_{ijk} + a_{ikj} = 0\ . \tag{5.3}$$

Now (5.2) may be rewritten as

$$\frac{\partial a_{ijk}}{\partial x_k} = g_{ij}\ . \tag{5.4}$$

This is obtained by a direct calculation which hinges on the following readily checked relations:

$$\begin{aligned} dx_k \wedge *(dx_j dx_k) &= *dx_k\ , \\ dx_j \wedge *(dx_j dx_k) &= -\ *dx_j\ . \end{aligned} \tag{5.5}$$

Next we set

$$(-1)^{n-2}\eta_{ij} = \frac{1}{2}b_{ijkl}*(dx_k dx_l)\ , \tag{5.6}$$

where

$$\begin{aligned} b_{ijkl} + b_{jikl} &= 0 \\ b_{ijkl} + b_{ijlk} &= 0\ . \end{aligned} \tag{5.7}$$

In this notation the relations (4.3) become

$$\frac{\partial b_{ijkl}}{\partial x_l} = a_{jik} - a_{ijk}\ . \tag{5.8}$$

Combined with the skew-symmetry of a_{ijk} in the second and third indices, this yields in the usual way

$$a_{ijk} = \frac{\partial c_{ijkl}}{\partial x_l} \tag{5.9}$$

where

$$c_{ijkl} = \frac{1}{2}(-\ b_{ijkl} + b_{jkil} - b_{kijl})\ . \tag{5.10}$$

These relations imply

$$b_{ijkl} = -\ c_{ijkl} - c_{jkil}\ . \tag{5.11}$$

The skew-symmetries in (5.7) thus are equivalent to

(5.12)
$$c_{ijkl} + c_{jkil} + c_{ijlk} + c_{jlik} = 0 \,,$$
$$c_{ijkl} + c_{jkil} + c_{jikl} + c_{ikjl} = 0 \,.$$

Equations (5.9) and (5.4) combine to yield

$$\frac{\partial^2 c_{ijkl}}{\partial x_k \partial x_l} = g_{ij} \,. \tag{5.13}$$

The minimal hypersurface equation $M(u) = 0$ may be interpreted as integrability conditions for (5.13) with the side conditions (5.14).

We may cut down the number of variables by introducing

(5.14)
$$h_{ijkl} = \frac{1}{4}(c_{ijkl} + c_{ijlk} + c_{jikl} + c_{jilk})$$
$$= \frac{1}{4}(b_{ikjl} + b_{jlik} + b_{jkil} + b_{iljk}) \,.$$

Then we have

(5.15)
$$h_{ijkl} = h_{jikl}$$
$$h_{ijkl} = h_{ijlk}$$

while (5.13) implies

$$\frac{\partial^2 h_{ijkl}}{\partial x_k \partial x_l} = g_{ij} \,. \tag{5.16}$$

In addition to the symmetries in (5.15) the quantities h satisfy

$$h_{ijkl} = h_{klij} = 0 \tag{5.17}$$

and

$$h_{ijkl} + h_{jkil} + h_{kijl} = 0 \,. \tag{5.18}$$

These are easy consequences of (5.14) and (5.7). The relations (5.15), (5.17), (5.18) span all relations in the h's. To see this we must count dimensions. The space of tensors (b) subject to (5.7) has dimension $n^2(n-1)^2/4$. The nullity of the mapping $(b) \rightarrow (h)$ given by (5.14) is determined by finding independent solutions of

$$(ijkl) + (jlik) + (jkil) + (iljk) = 0 \tag{5.19}$$

where we abbreviate $(ijkl) = b_{ijkl}$. We need consider only $(ijkl)$ where $i < j$ and $k < l$, using (5.7) to determine the others. By (5.19),

$$4(1212) = 0 \,, \qquad (1212) = 0 \,.$$

The $(ijkl)$ with three distinct indices are represented by (say) indices

1, 1, 2, 3 and this gives us (1213) and (1312). But by (5.19),

$$2(1213) + 2(1312) = 0 ,$$

hence we are free to choose only one of these. We thus have $3\binom{n}{3}$ degrees of freedom in choosing $(ijkl)$ with three distinct indices. If there are four distinct indices, say 1, 2, 3, 4, the quantities we consider are these six:

$$(1234) , \quad (1324) , \quad (1423) , \quad (2314) , \quad (2413) , \quad (3412) .$$

The relations (5.19) are seen to yield two independent relations amongst these:

$$(1234) + (3412) - (2314) - (1423) = 0 ,$$
$$(1234) + (3412) + (1234) + (2413) = 0 .$$

This means that with all indices distinct we have $4\binom{n}{4}$ degrees of freedom. Thus the desired nullity is

$$3\binom{n}{3} + 4\binom{n}{4}$$

and the rank equals dimension of the (h) space is

$$\frac{n^2(n-1)^2}{4} - 3\binom{n}{3} - 4\binom{n}{4} = \frac{n^2(n^2-1)}{12} .$$

On the other hand, the space of (h) tensors subject to (5.15), (5.17), and (5.18) has precisely the same dimensions. To see this we use (5.15) and (5.17) to limit the parameter to those $(ijkl)$ for which $i \leqq j$, $k \leqq l$, and $(ij) \leqq (kl)$ in lexicographic order. (Now $(ijkl)$ denotes h_{ijkl}.) By (5.17), (1111) = 0 and (1112) = 0. With two distinct indices we need only consider (1212) and (1122). By (5.17) these are related by

$$(1122) + 2(1212) = 0 .$$

Thus with only two distinct indices we have $\binom{n}{2}$ degrees of freedom. With three distinct indices, say 1, 1, 2, 3, the only possibilities, (1123) and (1213), are again related by

$$(1123) + 2(1213) = 0 .$$

We thus have $3\binom{n}{3}$ degrees of freedom in this case. Finally with four distinct indices, say 1, 2, 3, 4, the three possibilities, (1234), (1324), and (1423), are related by

$$(1234) + (1324) + (1423) = 0$$

so we have $2\binom{n}{4}$ degrees of freedom in this case. In total the space of (h) we are considering has dimension

$$\binom{n}{2} + 3\binom{n}{3} + 2\binom{n}{4} = \frac{n^2(n^2-1)}{12}\,.$$

This completes our proof that the relations (5.15), (5.17), and (5.18) span all relations between the h's. In the course of the proof we have obtained a set of independent parameters for the (h) space:

$$\begin{array}{ll} h_{ijij} & (i < j)\,, \\ h_{ijik} & (i < j < k)\,, \\ h_{ijkl}\,,\quad h_{ikjl} & (i < j < k < l)\,. \end{array}$$

This result $n^2(n^2 - 1)/12$ is certainly better than the number of b's (or c's), namely $n^2(n - 1)^2/4$. When $n = 2$, both numbers are one so that equations (5.16) only involve a single unknown function $h = h_{1212}$. This is what makes a proof of Bernstein's Theorem along the lines discussed in the introduction work.

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