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# A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMORPHIC SUBGROUPS

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**All of the group in this paper are abelian $p$-groups without elements of infinite height. A group is said to be quasi-indecomposable if whenever $H$ is a summand of $G$ then either $H$ or $G/H$ is finite. The $p$-socle of $G$ is the sub-group consisting of all the elements $x$ in $G$ such that $px = 0$.**

**In this paper it is shown that there are conditions that can be imposed on the socle of $G$ which are sufficient for $G$ to (a) have no proper isomorphic subgroups; (b) have no proper isomorphic quotient groups; and (c) be quasiindecomposable. Furthermore, it is shown that groups which make these results meaningful actually exist.**

*Let the cardinality of a group $G$ be either $\aleph_0$ or greater than $c = 2^{\aleph_0}$. Then, as is well known, $G$ has a proper isomorphic subgroup and a proper isomorphic quotient group.* However P. Crawley [3] showed that the cardinality $c$ is exceptional. He gave an example $G_0$ of cardinality $c$ which has a standard basic subgroup and *no proper isomorphic subgroups*. After Crawley's example appeared, it was clear that a group, of cardinality $c$ and with a standard basic subgroup, supplies examples of groups with strange but interesting properties. In fact R. S. Pierce [7] gave an example $G_1$ which has no proper isomorphic subgroups and *no proper isomorphic quotient groups*. And he gave also in [7] an example $G_2$ which is *quasi-indecomposable*, that is, every direct summand $H$ of $G_2$ is either finite or $G_2/H$ is finite.

The relationship between the above three properties (no proper isomorphic subgroups, no proper isomorphic quotient groups and quasi-indecomposability) of a group $G$ with the cardinality $c$ and a standard basic subgroup seems to authors an interesting problem. In this paper we shall give some results about this problem. In our approach the topological structure of the $p$-socle of the torsion completion of $G$ will be used in an essential way. Theorem 1 tells us that the situation of the $p$-socle of $G$ in the $p$-socle of the torsion completion of $G$ gives us sufficient conditions for these three properties of $G$. In some sense it shows a relationship between the three properties. Theorem 2 shows the existence of a group which has all three properties. Theorem 3 shows the existence of a group which has no proper isomorphic subgroups and no proper isomorphic quotient groups but which is quasi-decomposable.

Now we want to add a simple proof of the following fact which was mentioned in the opening of this section.

*Let $G$ be an infinite reduced $p$-group with* card $G = \aleph_0$ *or* card $G > c$. *Then $G$ has a proper isomorphic subgroup and a proper isomorphic quotient group.*

*Proof.* For simplicity we divide the proof into

*Case* 1; Suppose $G$ is bounded. Then $G = \sum_{k=1}^{n} B_k$ where $B_k$ is a direct sum of cyclic groups of order $p^k$, $B_k = \sum C(p^k)$. Now clearly one of these $B_k$'s is infinite and throwing out a cyclic summand of $B_k$ yields the desired subgroup and quotient group.

*Case* 2. Suppose card $G = \aleph_0$ and $G$ is unbounded. Then $G = H \oplus K$ where $H$ is an unbounded direct sum of cyclic groups (Exercise 19 ($a$), p. 143 in [4]). It is easy to find a proper subgroup $A$ of $H$ which is isomorphic to $H$ and a non-zero subgroup $B$ of $H$ such that $H/B \cong H$. Whence we obtain our proper isomorphic subgroup $A \oplus K$ and our proper isomorphic quotient group $G/B$.

*Case* 3. Suppose $G$ is unbounded with card $G > c$, and $B = \sum_{k=1}^{\infty} B_k$ is a basic subgroup where $B_k = \sum C(p^k)$. Then $G = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus G_n$ for all $n$ (Theorem 29.3 in [4]). But as is well known (card $B)^{\aleph_0} \geqq$ card $G > c$ so that some $B_n$ must be infinite. Now throwing out a cyclic summand of $B_n$ yields the result as in Case 1 and the proof is complete.

**2. Sufficient conditions for the three properties.** Let $p > 1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order $p^n$, $\Sigma$ be the direct sum of cyclic groups $C(p^n)$, $\Pi$ be the direct product of cyclic groups $C(p^n)$ and $C$ be the torsion group of $\Pi$, that is, $\Sigma$ is the standard basic group and $C$ is the torsion completion of $\Sigma$.

The $p$-socle $C[p]$ of $C$ is a vector space over the prime field of characteristic $p$ and can be topologized as a totally disconnected compact topological group, because $\Pi$ is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the $p$-socle $C[p]$ of $C$ is the closed subgroup $\{x \mid x \in \Pi, px = 0\}$ of $\Pi$. Actually $U_n = \{x \mid x \in C[p]$ and $h(x) \geqq n\} = (p^n C)[p]$ $(n = 1, 2 \cdots)$ are open compact subgroups of $C[p]$ and $\{U_n\}$ is a fundamental system of 0-neighborhoods in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way in this paper.

Every continuous group homomorphism $T$ on $C[p]$ defines compact subgroups $E_q(T) = \{x \mid x \in C[p]$ and $Tx = qx\}$ $(0 \leqq q < p)$ and the compact subgroup $E(T) = E_0(T) \oplus E_1(T) \oplus \cdots \oplus E_{p-1}(T)$. We can define naturally two types of continuous group homomorphism on $C[p]$ as follows. $T$ is a *singular* homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is singular. $T$ is a *strongly singular* homomorphism if for some $q$ $E_q(T)$ is an open compact subgroup. If a continuous group homomorphism $T$ on $C[p]$ has a dense subgroup which is invariant under $T$ and on which $T$ is one to one, $T$ is called a semi-isomorphism on $C[p]$.

We have the following theorem which is fundamental to the ideas in what follows.

THEOREM 1. *Let $G$ be a pure subgroup of $C$ which contains $\Sigma$ and $G[p]$ be the $p$-socle of $G$.*

(1) *If $G[p]$ is not invariant under any nonsingular onto homomorphism on $C[p]$, then $G$ has no proper isomorphic quotient groups.*

(2) *If $G[p]$ is not invariant under any nonsingular semi-isomorphism on $C[p]$, then $G$ has no proper isomorphic subgroups.*

(3) *If $G[p]$ is not invariant under any nonstrongly singular projection on $C[p]$, then $G$ is quasi-indecomposable.*

*Proof.* Suppose $\varphi$ is a homomorphism of $G$ into $G$. The purity of $G$ in $C$ implies $\varphi(G[p] \cap U_n) \subset U_n$ for all $n = 1, 2, \cdots$. This means that the restriction of $\varphi$ to $G[p]$ is continuous on $G[p]$. since $G[p] \supset \Sigma[p]$ and $\Sigma[p]$ is dense in $C[p]$, $\varphi\,|_{G[p]}$ has a unique continuous homomorphism extension $T$ on $C[p]$. Clearly $G[p]$ is invariant under $T$ and $T(U_n) \subset U_n$ for all $n = 1, 2, \cdots$. If this $T$ is singular, then there exists a positive integer $N$ such that

$$T(U_N) \subset U_N \subset E(T) .$$

Then we have the following decomposition of $G[p]$,

$$\begin{aligned} G[p] &= (G[p] \cap U_N) \oplus R_N = (E_0(T) \cap G[p] \cap U_N) \\ &\quad \oplus (E_1(T) \cap G[p] \cap U_N) \oplus \cdots \oplus (E_{p-1}(T) \cap G[p] \cap U_N) \oplus R_N , \end{aligned}$$

where $R_N$ is a finite subgroup of $G[p]$.

Because $C[p]/U_N$ is finite and $G[p]/G[p] \cap U_N$ is isomorphic to a subgroup $C[p]/U_N$, so the dimension of $G[p]/G[p] \cap U_N$ as a vector space over the prime field of characteristic $p$ is finite. Hence there exists a finite subgroup $R_N$ of $G[p]$ such that $G[p] = (G[p] \cap U_N) \oplus R_N$. The decomposition of $G[p] \cap U_N$ can be shown as follows. For each $x$ in $G[p] \cap U_N$ $x$ is the sum of $z_q \in E_q(T)$ $(0 \leqq q < p)$; $x = \sum_{1=0}^{p-1} z_q$. Then we have $\varphi^\nu(x) = \sum_{q=0}^{p-1} T^\nu z_q = \sum_{q=0}^{p-1} q^\nu z_q$ for $0 \leqq \nu \leqq p-1$. Since the determinant of Vandermonde's matrix is not zero mod $p$, each $z_q$ $(0 \leqq q \leqq p-1)$ is a linear combination of $x, \varphi(x), \cdots, \varphi^{p-1}(x)$. This means $z_q \in E_q(T) \cap G[p] \cap U_N$ for $0 \leqq q \leqq p-1$.

*Proof of* (1). Suppose $\varphi$ is an onto homomorphism of $G$. Then the continuous extension $T$ of $\varphi|_{G[p]}$ is clearly an onto homomorphism of $C[p]$ and $G[p]$ is invariant under $T$. By our assumption $T$ must be singular, so we have the above decomposition of $G[p]$. Put $Q_N = (E_1(T) \cap G[p] \cap U_N) \oplus (E_2(T) \cap G[p] \cap U_N) \oplus \cdots \oplus (E_{p-1}(T) \cap G[p] \cap U_N)$, clearly $\varphi(Q_N) = Q_N$ and $\varphi$ is an isomorphism on $Q_N$, and

$$(E_0(T) \cap G[p] \cap U_N) \oplus R_N \cong G[p]/Q_N = \varphi(G[p])/\varphi(Q_N) \cong \varphi(R_N)$$

but $\dim \varphi(R_N) \leqq \dim R_N < +\infty$. This implies that $E_0(T) \cap G[p] \cap U_N = \{0\}$ and $R_N$ is isomorphic to $\varphi(R_N)$ by $\varphi$. Therefore $\varphi|_{G[p]}$ is an isomorphism on $G[p]$. Let $0 \neq x \in G$ and the order of $x = p^n > 1$, then $0 \neq \varphi(p^{n-1}x) = p^{n-1}\varphi(x)$, so $\varphi(x) \neq 0$. Thus $\varphi$ must be an isomorphism on $G$.

*Proof of* (2). Suppose $\varphi$ is an isomorphism of $G$ into $G$. We have to show $\varphi(G) = G$. The continuous extension $T$ of $\varphi|_{G[p]}$ is a semi-isomorphism and $G[p]$ is invariant under $T$. By our assumption $T$ must be singular, so we have the same decomposition of $G[p]$ as above. First of all we can see $\varphi(G[p]) = G[p]$. Automatically

$$E_0(T) \cap G[p] \cap U_N = \{0\},$$

because $\varphi$ is one to one, therefore $G[p] = Q_N \oplus R_N \cong \varphi(Q_N) \oplus \varphi(R_N) = Q_N \oplus \varphi(R_N) \subset G[p]$ but $\dim R_N = \dim \varphi(R_N) < +\infty$, this implies $\varphi(G[p]) = G[p]$. Next we can see $\varphi(G) \supset G[p^2]$. The group $H = \{x \mid x \in G$ and the first $N-1$ coordinates in $\Pi$ are zero$\}$ is a direct summand of $G$ and

$$\begin{aligned} H[p] &= G[p] \cap U_N = Q_N \\ &= (E_1(T) \cap Q_N) \oplus (E_2(T) \cap Q_N) \oplus \cdots \oplus (E_{p-1}(T) \cap Q_N). \end{aligned}$$

We can take a finite group $L$ such that $G = H \oplus L$. We have to show first $\varphi(G) \supset H[p^2]$. For arbitrary $x$ in $H[p^2]$ $px = \sum_{q=0}^{p-1} z_q$ for some $z_q \in E_q(T) \cap Q_N$ $(1 \leqq q \leqq p-1)$, then each $z_q$ is a linear combination of $p\varphi(x), p\varphi^2(x), \cdots, p\varphi^{p-1}(x)$. This means that there exist $x_q \in G$ $(1 \leqq q \leqq p-1)$ such that $z_q = p\varphi(x_q)$ for $1 \leqq q \leqq p-1$. Therefore $px = \sum_{q=1}^{p-1} p\varphi(x_q)$, so $x - \varphi(\sum_{q=1}^{p-1} x_q) \in G[p]$, but $G[p] = \varphi(G[p])$ implies $x \in \varphi(G)$. Now $\varphi(G) \supset G[p^2]$ can be shown. For $x \in G[p^2]$ there exists a positive integer $M$ and integers $r_i$, $0 \leqq r_i \leqq p-1$ (at least one of them is not zero) such that $\sum_{i=0}^{M} r_i p\varphi^i(x) \in Q_N = H[p]$, because $G[p]/Q_N$ is finite dimensional. Since $\varphi(Q_N) = Q_N$, we can assume $r_0 = 1$ without loss of generality. Then we find $z \in H[p^2]$ such that $p\sum_{i=0}^{M} r_i\varphi^i(x) = pz$. But $H[p^2] \subset \varphi(G)$ has been shown, so $z = \varphi(z')$ for some $z' \in G$, therefore $x + \sum_{i=1}^{M} r_i\varphi^i(x) - \varphi(z') \in G[p] = \varphi(G[p])$, this implies $x \in \varphi(G)$. Now we can see $\varphi(G) \supset G[p^n]$ for all $n = 1, 2 \cdots$ by induction. Namely in general $\varphi(G) \supset G[p^n]$ and the special form of $\varphi$ on $Q_N$ imply $\varphi(G) \supset H[p^{n+1}]$. And $\varphi(G) \supset H[p^{n+1}]$ and the finiteness of $L$ imply $\varphi(G) \supset G[p^{n+1}]$.

*Proof of* (3). Suppose $G$ is the direct sum of two subgroups $G_1$ and $G_2$ and $\varphi$ is the projection onto $G_1$. The continuous extension $T$ of $\varphi|_{G[p]}$ is also a projection defined on $C[p]$, therefore $C[p] = E_0(T) \oplus E_1(T)$ and $G[p] = (E_0(T) \cap G[p]) \oplus (E_1(T) \cap G[p])$. Since $G[p]$ is invariant under $T$, $T$ must be strongly singular by our assumption about $G[p]$. Suppose $E_1(T)$ is open, then $E_0(T)$ is finite, hence $G_2[p] = E_0(T) \cap G[p]$ is finite. The finiteness of $G_2[p]$ implies the finiteness of $G_2$.

The following is a direct corollary of Theorem 1.

COROLLARY. *Let $G$ be a pure subgroup of $C$ which contains $\Sigma$. If $G[p]$ is not invariant under any nonstrongly singular homomorphism on $C[p]$, then $G$ has the three properties stated in* (1), (2) *and* (3) *in Theorem* 1. *Namely $G$ has no proper isomorphic quotient group and no proper isomorphic subgroup, and $G$ is quasi-indecomposable.*

## 3. Existence theorem

THEOREM 2. *There exists a pure subgroup $G$ of $C$ which contains $\Sigma$ and satisfies three properties;*

(1) *$G$ has no proper isomorphic quotient groups,*

(2) *$G$ nas no proper isomorphic subgroups,*

(3) *$G$ is quasi-indecomposable.*

*And an arbitrary pure subgroup $H$ of $C$ such that $H$ contains $\Sigma$ and $H[p] = G[p]$ satisfies above three properties.*

This theorem comes from the corollary of Theorem 1 and following two lemmas. Lemma 1 is known as the purification property, so we omit the proof of Lemma 1 (see more general form in [6]).

LEMMA 1. *For an arbitrary subgroup $Q$ between $\Sigma[p]$ and $C[p]$ there exists a pure subgroup $G$ of $C$ such that $G$ contains $\Sigma$ and $G[p] = Q$.*

LEMMA 2. *For any family $\{T_\lambda \mid \lambda \in \Lambda\}$ of nonstrongly singular homomorphisms on $C[p]$ there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any $T_\lambda (\lambda \in \Lambda)$.*

The existence of such $Q$ can be shown by transfinite induction which is Crawley's idea in [3]. We need following lemma which is also essentially Crawley's.

LEMMA 3. *Suppose $T$ is a nonstrongly singular homomorphism on $C[p]$. Then there exists a one-parameter family $\Delta(T) = \{x_t \mid 0 \leqq t \leqq 1\}$ of elements in $C[p]$ such that four elements $x_s$, $x_t$, $Tx_s$ and $Tx_t$ are* *linearly independent for arbitrary* $s \neq t$.

*Proof*. The proof can be divided into two cases (*a*) and (*b*).

(*a*) $T$ is singular but not strongly singular. In this case, by Baire's category theorem ($C[p]$ is a complete metric space) there are at least two $q$ and $q'$ such that both $E_q(T)$ and $E_{q'}(T)$ are infinite compact groups, so card $E_q(T) = \operatorname{card} E_{q'}(T) = c$ (for instance, see [5], p. 31). Therefore $\dim E_q(T) = \dim E_{q'}(T) = c$. Let $\{y_t \mid 0 \leqq t \leqq 1\}$ be a basis of $E_q(T)$ and $\{y'_t \mid 0 \leqq t \leqq 1\}$ be a basis of $E_{q'}(T)$. Then $\varDelta(T) = \{y_t + y'_t \mid 0 \leqq t \leqq 1\}$ is the desired family.

(*b*) $T$ is not singular. In this case, by Baire's category theorem $U_n/E(T) \cap U_n$ are infinite compact groups for all $n = 1, 2 \cdots$, so as above $\dim U_n/E(T) \cap U_n = c$. Hence $U_n = (E(T) \cap U_n) \oplus D_n$ with $\dim D_n = c$ for all $n = 1, 2, \cdots$. Take $0 \neq x_0 \in D_1$, then $x_0$ and $Tx_0$ are linearly independent. Let $\{z_0, z_1, \cdots, z_{p^2-1}\}$ be the group generated by $x_0$ and $Tx_0$, then by the continuity of $T$ we can find $U_M$ such that $z_i + U_M + T(U_M)$ $(0 \leqq i \leqq p^2 - 1)$ are mutually disjoint. For this $M$ we take a basis $\{y_t \mid 0 \leqq t \leqq 1\}$ of $D_M$. Then $\varDelta(T) = \{x_0 + y_t \mid 0 \leqq t \leqq 1\}$ is the desired system. Because, suppose $n_1(x_0 + y_t) + n_2(Tx_0 + Ty_t) = n'_1(x_0 + y_s) + n'_2(Tx_0 + Ty_s)$ for $s \neq t$ where $n_1, n_2, n'_1$ and $n'_2$ are integers, then $n_1x_0 + n_2Tx_0 + n_1y_t + n_2Ty_t = n'_1x_0 + n'_2Tx_0 + n'_1y_s + n'_2Ty_s$, and $n_1x_0 + n_2Tx_0$ must be some $z_i$ and also $n'_1x_0 + n'_2Tx_0$ must be some $z_j$, but $z_i = z_j$ by our choice of $U_M$. This implies $n_1 = n'_1 \bmod p$ and $n_2 = n'_2 \bmod p$, therefore we have $n_1y_t + n_2Ty_t = n_1y_s + n_2Ty_s$, whence $n_1(y_t - y_s) = -n_2T(y_t - y_s)$. However $0 \neq y_t - y_s \in D_M$ and $D_M \cap E(T) = \{0\}$, hence $n_1 = n_2 = 0 \bmod p$.

*Proof of Lemma* 2. $\{T_\lambda \mid \lambda \in \varLambda\}$ is given, then card $\varLambda$ is at most $c$ (note that the cardinality of the set of all continuous homomorphisms on $C[p]$ is at most $c$, because $C[p]$ is a separable compact group). We assume that $\varLambda$ is a well ordered set of ordinal numbers which are less than $\varOmega$, where $\varOmega$ is the first ordinal number whose cardinality is $c$. Choose $e \in C[p]$ but $e \notin \varSigma[p]$, then we can construct a family of subgroups $R_\lambda (\lambda \in \varLambda)$ by transfinite induction as follows:

(a) $\varSigma[p] = R_0 \subset R_\lambda \subset R_\mu \subset C[p]$ if $0 \leqq \lambda < \mu$ $(\lambda, \mu \in \varLambda)$,

(b) $\operatorname{card} R_\lambda \leqq \operatorname{card} \lambda \cdot \aleph_0 < c$ for all $\lambda \in \varLambda$,

(c) $e \notin R_\lambda$ but there exists $x_\lambda \in R_\lambda \cap \varDelta(T_\lambda)$ such that $e - T_\lambda x_\lambda \in R_\lambda$.

Suppose $R_\lambda$ has been constructed for all $\lambda < \mu \in \varLambda$. Let $R'_\mu = \bigcup_{\lambda<\mu} R_\lambda$. Then card $([e] + R'_\mu) \leqq \operatorname{card} \mu \cdot \aleph_0 < c$, where $[e]$ is the group generated by $e$. The property of $\varDelta(T_\mu)$ in Lemma 3 guarantees the existence of $x_{t0} \in \varDelta(T_\mu)$ such that $([e] + R'_\mu) \cap ([x_{t0}] + [T_\mu x_{t0}]) = \{0\}$. Then $R_\mu = R'_\mu + [x_{t0}] + [e - T_\mu x_{t0}]$ is the desired subgroup. Let $Q = \bigcup_{\lambda \in \varLambda} R_\lambda$, then by (a) $Q$ is a subgroup of $C[p]$ which contains $\varSigma[p]$ and by (c) $Q$ is not invariant under any $T_\lambda (\lambda \in \varLambda)$.

## 4. A quasi-decomposable group without proper isomorphic quotient groups and proper isomorphic subgroups.

THEOREM 3. *There exists a pure subgroup $G$ of $C$ which contains $\Sigma$ and satisfies properties;*

(1) *$G$ has no proper isomorphic quotient groups,*

(2) *$G$ has no proper isomorphic subgroups,*

(3) *$G$ has a decomposition $G_1 \oplus G_2$ such that $G_1$ and $G_2$ are not bounded.*

The following lemma is essential for our proof of this theorem.

LEMMA 4. *For any family $\{T_\lambda \mid \lambda \in \Lambda\}$ of nonsingular homomorphisms on $C[p]$ there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under the canonical projection $P_e$ onto even coordinates.*

The outline of the proof of this lemma will be given later.

*Proof of Theorem* 3. Every element of $C$ has countable coordinates as an element of the product space $\prod_{n=1}^{\infty} C(p^n)$; $x \in C$ is called an even (odd) element if all odd (even) coordinates are zero. For a subset $A$ of $C$ $A^e(A^0)$ means the set of all even (odd) elements in $A$. Then clearly $C = C^e \oplus C^0$ and $\Sigma = \Sigma^e \oplus \Sigma^0$. By Lemma 4 there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any nonsingular homomorphisms on $C[p]$ but is invariant under $P_e$, therefore $\Sigma^e[p] = \Sigma[p]^e \subset Q^e \subset C[p]^e = C^e[p]$, $\Sigma^0[p] = \Sigma[p]^0 \subset Q^0 \subset C[p]^0 = C^0[p]$ and $Q = Q^e \oplus Q^0$. With exactly the same proof as that of Lemma 1 we can show that there exists a pure subgroup $G_1(G_2)$ of $C^e(C^0)$ which contains $\Sigma^e(\Sigma^0)$ and $G_1[p] = Q^e(G_2[p] = Q^0)$. Clearly $G_1$ and $G_2$ are not bounded. Let $G = G_1 \oplus G_2$, then $G$ is a pure subgroup of $C$ which contains $\Sigma$ and $G[p] = G_1[p] \oplus G_2[p] = Q^e \oplus Q^0 = Q$. By Theorem 1 $G$ has the properties (1) and (2) in Theorem 3.

*The outline of the proof of Lemma* 4. In order to prove Lemma 4 we can apply a similar method to the construction of $Q$ in Lemma 2. However before doing it we have to prepare some reformation of Lemma 3. Precisely our reformation is as follows, hereafter we shall use notations $A^e = P_e(A)(A^0 = (I - P_e)(A))$ for a subset $A$ of $C[p]$ and $x^e = P_e x(x^0 = x - P_e x)$ for an element $x$ in $C[p]$.

*For an arbitrary nonsingular homomorphism $T$ we can find a one-parameter family $\Delta(T) = \{x_t \mid 0 \leqq t \leqq 1\}$ of elements in $C[p]$ which has one of the following six properties;* $1^0$, $2^0$, $3^0$, $1^e$, $2^e$ *and* $3^e$,

$1^0$ $x_t, Tx_t \in C[p]^0$ *for all* $0 \leqq t \leqq 1$ *and four elements* $x_s, x_t, Tx_s$ *and* $Tx_t$ *are linearly independent for arbitrary* $s \neq t$,

$2^0$ *there exists* $q, 0 \leqq q \leqq p-1$ *such that* $x_t \in C[p]^0$ *and*

$$Tx_t - qx_t \in C[p]^e$$

*for all* $0 \leqq t \leqq 1$ *and four elements* $x_s, x_t, Tx_s - qx_s$ *and* $Tx_t - qx_t$ *are linearly independent for arbitrary* $s \neq t$,

$3^0$ $x_t \in C[p]^0$ *for all* $0 \leqq t \leqq 1$ *and six elements* $x_s, x_t, (Tx_s)^0, (Tx_s)^e, (Tx_t)^0$ *and* $(Tx_t)^e$ *are linearly independent for arbitrary* $s \neq t$.

$1^e, 2^e$ *and* $3^e$ *are dual properties* $1^0, 2^0$ *and* $3^0$ *by exchanging odd for even.*

In the proof of this we have some difficulty coming from non-commutativity of nonsingular homomorphism and $P_e$. The proof in our original manuscript needs a long computation, in this paper we omit our detailed computation according to referee's suggestion but authors can supply the detailed proof to interested readers.

Using above $\Delta(T)$ the existence of $Q$ in Lemma 4 can be shown as follows. Let $\{T_\lambda | \lambda \in \Lambda\}$ be a given family of nonsingular homomorphisms on $C[p]$. We assume that $\Lambda$ is a well ordered set of ordinal numbers which are less than the first ordinal number whose cardinality is $c$. Choose $c \in C[p]$ but $c^0, c^e \notin \Sigma[p]$. By transfinite induction we can construct the following family of subgroups $R_\lambda(\lambda \in \Lambda)$;

(a) $\Sigma[p] = R_0 \subset R_\lambda \subset R_\mu \subset C[p]$ if $0 \leqq \lambda < \mu(\lambda, \mu \in \Lambda)$,

(b) $\operatorname{card} R_\lambda \leqq \operatorname{card} \lambda \cdot \aleph_0 < c$ for all $\lambda \in \Lambda$,

(c) $R_\lambda$ is invariant under $P_e$ for all $\lambda \in \Lambda$,

(d) $c^0$ and $c^e \notin R_\lambda$ but there exists $x_\lambda \in R_\lambda \cap \Delta(T_\lambda)$ such that $c^0 - T_\lambda x_\lambda$ or $c^e - T_\lambda x_\lambda$ or $c - T_\lambda x_\lambda \in R_\lambda$ for all $\lambda \in \Lambda$.

Suppose $R_\lambda$ has been constructed for all $\lambda < \mu \in \Lambda$. Let $R'_\mu = \bigcup_{\lambda<\mu} R_\lambda$. Then $\operatorname{card} R'_\lambda \leqq \operatorname{card} \lambda \cdot \aleph_0 < c$ and $R'_\lambda$ is invariant under $P_e$ and $c^0$ and $c^e \notin R'_\lambda$. Let $\Delta(T_\mu)$ be one having one of properties $1^0 \sim 3^0$ and $1^e \sim 3^e$. Suppose $\Delta(T_\mu)$ has property $1^0$, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^0] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu]) = \{0\}$. Let

$$R_\mu = R'_\mu + [x_\mu] + [c^0 - T_\mu x_\mu] ,$$

then clearly $R_\mu$ satisfies above (a), (b) and (c). And $c^0$ and $c^e \notin R_\mu$ also holds. Suppose $c^0 \in R_\mu$, then $c^0 = x + nx_\mu + m(c^0 - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers $n$ and $m$, so $-x + (1-m)c^0 = nx_\mu - mT_\mu x_\mu$, but by our choice of $x_\mu$, $nx_\mu - mT_\mu x_\mu = 0$ and $x + (m-1)c^0 = 0$. This implies $n = m = 0 \bmod p$ and $c^0 = x \in R'_\mu$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^0 - T_\mu x_\mu)$ for some $x \in R'_\mu$ and some integers $n$ and $m$, but $x_\mu$ and $T_\mu x_\mu \in C[p]^0$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property $2^0$, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^0] + [c^e]) \cap ([x_\mu] \oplus [T_\mu x_\mu - qx_\mu]) = \{0\}$. Let $R_\mu = R'_\mu + [x_\mu] + [c^e - T_\mu x_\mu + qx_\mu]$, then clearly $R_\mu$ satisfies above (a), (b) and (c). And $c^0$ and $c^e \notin R_\mu$ also holds. Suppose $c^0 \in R_\mu$, then $c^0 = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers $n$ and $m$, but $x_\mu \in C[p]^0$ and $T_\mu x_\mu - qx_\mu \in C[p]^e$, hence we have $c^0 = x^0 + nx_\mu$, that is, $-x^0 + c^0 = nx_\mu$. Our choice of $x_\mu$ implies $nx_\mu = 0 = -x^0 + c^0$, so we have $c^0 = x^0 \in R'_\mu$ which is a contradiction. Suppose $c^e \in S_\mu$, then $c^e = x + nx_\mu + m(c^e - T_\mu x_\mu + qx_\mu)$ for some $x \in R'_\mu$ and some integers $n$ and $m$. Hence $-x + (1-m)c^e = nx_\mu - m(T_\mu x_\mu - qx_\mu)$, but by our choice of $x_\mu$ we see $-x + (1-m)c^e = 0 = nx_\mu - m(T_\mu x_\mu - qx_\mu)$. This implies $n = m = 0 \bmod p$, so $c^e = x \in R'_\mu$ which is also a contradiction. Suppose $\Delta(T_\mu)$ has property $3^0$, then we can find $x_\mu \in \Delta(T_\mu)$ such that $(R'_\mu + [c^0] + [c^e]) \cap ([x_\mu] \oplus [(T_\mu x_\mu)^0] \oplus [(T_\mu x_\mu)^e]) = \{0\}$. Let

$$R_\mu = R'_\mu + [x_\mu] + [c^0 - (T_\mu x_\mu)^0] + [c^e - (T_\mu x_\mu)^e] .$$

Then $R_\mu$ clearly satisfies (a), (b) and (c). And $c^0$ and $c^e \notin R_\mu$ can be seen as follows. Suppose $c^0 = x + nx_\mu + m(c^0 - (T_\mu x_\mu)^0) + m'(c^e - (T_\mu x_\mu)^e)$ for some $x \in R'_\mu$ and integers $n$, $m$ and $m'$, then

$$c^0 = x^0 + nx_\mu + m(c^0 - (T_\mu x_\mu)^0),$$

so $-x^0 + (1-m)c^0 = nx_\mu - m(T_\mu x_\mu)^0$. This implies $nx_\mu - m(T_\mu x_\mu)^0 = 0 = -x^0 + (1-m)c^0$ by our choice of $x_\mu$. Hence $m = 0$ and $c^0 = x^0 \in R'_\mu$ which is a contradiction. We can see also $c^e \notin R_\mu$ for same reason. And $x_\mu$ and $c - T_\mu x_\mu \in R_\mu$ is clear. The construction of $R_\mu$ for $\Delta(T_\mu)$ having one of properties $1^e \sim 3^e$ is exactly similar by exchanging odd for even.

Let $Q = \bigcup_{\lambda \in \Lambda} R_\lambda$. Then the above properties (a) $\sim$ (d) for all $R_\lambda$ guarantee that $Q$ is a subgroup between $\Sigma[p]$ and $C[p]$ not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under $P_e$.

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