A QUASI-DECOMPOSABLE ABELIAN GROUP WITHOUT PROPER ISOMORPHIC QUOTIENT GROUPS AND PROPER ISOMORPHIC SUBGROUPS

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All of the groups in this paper are abelian $p$-groups without elements of infinite height. A group is said to be quasi-indecomposable if whenever $H$ is a summand of $G$ then either $H$ or $G/H$ is finite. The $p$-socle of $G$ is the subgroup consisting of all the elements $x$ in $G$ such that $px = 0$.

In this paper it is shown that there are conditions that can be imposed on the socle of $G$ which are sufficient for $G$ to (a) have no proper isomorphic subgroups; (b) have no proper isomorphic quotient groups; and (c) be quasi-indecomposable. Furthermore, it is shown that groups which make these results meaningful actually exist.

Let the cardinality of a group $G$ be either $\aleph_0$ or greater than $c = 2^{\aleph_0}$. Then, as is well known, $G$ has a proper isomorphic subgroup and a proper isomorphic quotient group. However P. Crawley [3] showed that the cardinality $c$ is exceptional. He gave an example $G_0$ of cardinality $c$ which has a standard basic subgroup and no proper isomorphic subgroups. After Crawley's example appeared, it was clear that a group, of cardinality $c$ and with a standard basic subgroup, supplies examples of groups with strange but interesting properties. In fact R. S. Pierce [7] gave an example $G_1$ which has no proper isomorphic subgroups and no proper isomorphic quotient groups. And he gave also in [7] an example $G_2$ which is quasi-indecomposable, that is, every direct summand $H$ of $G_2$ is either finite or $G_2/H$ is finite.

The relationship between the above three properties (no proper isomorphic subgroups, no proper isomorphic quotient groups and quasi-indecomposability) of a group $G$ with the cardinality $c$ and a standard basic subgroup seems to authors an interesting problem. In this paper we shall give some results about this problem. In our approach the topological structure of the $p$-socle of the torsion completion of $G$ will be used in an essential way. Theorem 1 tells us that the situation of the $p$-socle of $G$ in the $p$-socle of the torsion completion of $G$ gives us sufficient conditions for these three properties of $G$. In some sense it shows a relationship between the three properties. Theorem 2 shows the existence of a group which has all three properties. Theorem 3 shows the existence of a group which has no proper isomorphic subgroups and no proper isomorphic quotient groups but which is quasi-decomposable.

Now we want to add a simple proof of the following fact which
was mentioned in the opening of this section.

Let $G$ be an infinite reduced $p$-group with $\text{card } G = \aleph_0$ or $\text{card } G > c$. Then $G$ has a proper isomorphic subgroup and a proper isomorphic quotient group.

**Proof.** For simplicity we divide the proof into

**Case 1.** Suppose $G$ is bounded. Then $G = \sum_{k=1}^{\infty} B_k$ where $B_k$ is a direct sum of cyclic groups of order $p^k$, $B_k = \sum C(p^k)$. Now clearly one of these $B_k$'s is infinite and throwing out a cyclic summand of $B_k$ yields the desired subgroup and quotient group.

**Case 2.** Suppose $\text{card } G = \aleph_0$ and $G$ is unbounded. Then $G = H \oplus K$ where $H$ is an unbounded direct sum of cyclic groups (Exercise 19 (a), p. 143 in [4]). It is easy to find a proper subgroup $A$ of $H$ which is isomorphic to $H$ and a non-zero subgroup $B$ of $H$ such that $H/B \cong H$. Whence we obtain our proper isomorphic subgroup $A \oplus K$ and our proper isomorphic quotient group $G/B$.

**Case 3.** Suppose $G$ is unbounded with $\text{card } G > c$, and $B = \sum_{k=1}^{\infty} B_k$ is a basic subgroup where $B_k = \sum C(p^k)$. Then $G = B_1 \oplus B_2 \oplus \cdots \oplus B_n \oplus G_n$ for all $n$ (Theorem 29.3 in [4]). But as is well known (card $B_k \geq \text{card } G > c$ so that some $B_k$ must be infinite. Now throwing out a cyclic summand of $B_k$ yields the result as in Case 1 and the proof is complete.

2. **Sufficient conditions for the three properties.** Let $p > 1$ be a fixed prime number, $C(p^n)$ be a cyclic group of order $p^n$, $\Sigma$ be the direct sum of cyclic groups $C(p^n)$, $\Pi$ be the direct product of cyclic groups $C(p^n)$ and $C$ be the torsion group of $\Pi$, that is, $\Sigma$ is the standard basic group and $C$ is the torsion completion of $\Sigma$.

The $p$-socle $C[p]$ of $C$ is a vector space over the prime field of characteristic $p$ and can be topologized as a totally disconnected compact topological group, because $\Pi$ is clearly a totally disconnected compact topological group with respect to the product topology of compact discrete topologies and the $p$-socle $C[p]$ of $C$ is the closed subgroup \{ $x \mid x \in \Pi$, $px = 0$ \} of $\Pi$. Actually $U_n = \{ x \mid x \in C[p] \text{ and } h(x) \geq n \} = (p^n C)[p]$ $(n = 1, 2, \ldots)$ are open compact subgroups of $C[p]$ and $\{ U_n \}$ is a fundamental system of 0-neighborhoods in $C[p]$. These two structures on $C[p]$ which are a vector space and a totally disconnected compact group are used in an essential way in this paper.

Every continuous group homomorphism $T$ on $C[p]$ defines compact subgroups $E_n(T) = \{ x \mid x \in C[p] \text{ and } Tx = qx \} (0 \leq q < p)$ and the compact subgroup $E(T) = E_0(T) \oplus E_1(T) \oplus \cdots \oplus E_{p-1}(T)$. We can define naturally two types of continuous group homomorphism on $C[p]$ as follows. $T$ is a **singular** homomorphism if $E(T)$ is an open compact subgroup of $C[p]$. For instance a continuous projection on $C[p]$ is
singlar. T is a strongly singular homomorphism if for some $q \in E_q(T)$ is an open compact subgroup. If a continuous group homomorphism $T$ on $C[p]$ has a dense subgroup which is invariant under $T$ and on which $T$ is one to one, $T$ is called a semi-isomorphism on $C[p]$.

We have the following theorem which is fundamental to the ideas in what follows.

**Theorem 1.** Let $G$ be a pure subgroup of $C$ which contains $\Sigma$ and $G[p]$ be the $p$-socle of $G$.

1. If $G[p]$ is not invariant under any nonsingular onto homomorphism on $C[p]$, then $G$ has no proper isomorphic quotient groups.

2. If $G[p]$ is not invariant under any nonsingular semi-isomorphism on $C[p]$, then $G$ has no proper isomorphic subgroups.

3. If $G[p]$ is not invariant under any nonstrongly singular projection on $C[p]$, then $G$ is quasi-indecomposable.

**Proof.** Suppose $\varphi$ is a homomorphism of $G$ into $G$. The purity of $G$ in $C$ implies $\varphi(G[p] \cap U_n) \subset U_n$ for all $n = 1, 2, \ldots$. This means that the restriction of $\varphi$ to $G[p]$ is continuous on $G[p]$. Since $G[p] \supset \Sigma[p]$ and $\Sigma[p]$ is dense in $C[p]$, $\varphi |_{G[p]}$ has a unique continuous homomorphism extension $T$ on $C[p]$. Clearly $G[p]$ is invariant under $T$ and $T(U_n) \subset U_n$ for all $n = 1, 2, \ldots$. If this $T$ is singular, then there exists a positive integer $N$ such that

$$T(U)_N \subset U_N \subset E(T).$$

Then we have the following decomposition of $G[p]$,

$$G[p] = (G[p] \cap U_N) \oplus R_N = (E_q(T) \cap G[p] \cap U_N) \oplus (E_{q-1}(T) \cap G[p] \cap U_N) \oplus \cdots \oplus (E_{q-N}(T) \cap G[p] \cap U_N) \oplus R_N,$$

where $R_N$ is a finite subgroup of $G[p]$.

Because $C[p]/U_N$ is finite and $G[p]/G[p] \cap U_N$ is isomorphic to a subgroup $C[p]/U_N$, so the dimension of $G[p]/G[p] \cap U_N$ as a vector space over the prime field of characteristic $p$ is finite. Hence there exists a finite subgroup $R_N$ of $G[p]$ such that $G[p] = (G[p] \cap U_N) \oplus R_N$. The decomposition of $G[p] \cap U_N$ can be shown as follows. For each $x$ in $G[p] \cap U_N$, $x$ is the sum of $z_q \in E_q(T) \cap G[p] \cap U_N$ for $0 \leq q \leq p - 1$. Then we have $\varphi^\nu(x) = \sum_{q=0}^{p-1} T^\nu z_q = \sum_{q=0}^{p-1} q^\nu z_q$ for $0 \leq \nu \leq p - 1$. Since the determinant of Vandermonde's matrix is not zero mod $p$, each $z_q$ is a linear combination of $x, \varphi(x), \ldots, \varphi^{p-1}(x)$. This means $z_q \in E_q(T) \cap G[p] \cap U_N$ for $0 \leq q \leq p - 1$.

**Proof of (1).** Suppose $\varphi$ is an onto homomorphism of $G$. Then
the continuous extension $T$ of $\varphi|_{G[p]}$ is clearly an onto homomorphism of $C[p]$ and $G[p]$ is invariant under $T$. By our assumption $T$ must be singular, so we have the above decomposition of $G[p]$. Put $Q_N = (E_1(T) \cap G[p] \cap U_N) \oplus (E_2(T) \cap G[p] \cap U_N) \oplus \cdots \oplus (E_{p-1}(T) \cap G[p] \cap U_N)$, clearly $\varphi(Q_N) = Q_N$ and $\varphi$ is an isomorphism on $Q_N$, and

\[(E_0(T) \cap G[p] \cap U_N) \oplus R_N \cong G[p]/Q_N = \varphi(G[p])/\varphi(Q_N) \cong \varphi(R_N)\]

but $\dim \varphi(R_N) \leq \dim R_N < +\infty$. This implies that $E_0(T) \cap G[p] \cap U_N = \{0\}$ and $R_N$ is isomorphic to $\varphi(R_N)$ by $\varphi$. Therefore $\varphi|_{G[p]}$ is an isomorphism on $G[p]$. Let $0 \neq x \in G$ and the order of $x = p^n > 1$, then $0 \neq \varphi(p^{n-1}x) = p^{n-1}\varphi(x)$, so $\varphi(x) \neq 0$. Thus $\varphi$ must be an isomorphism on $G$.

**Proof of (2).** Suppose $\varphi$ is an isomorphism of $G$ into $G$. We have to show $\varphi(G) = G$. The continuous extension $T$ of $\varphi|_{G[p]}$ is a semigroup and $G[p]$ is invariant under $T$. By our assumption $T$ must be singular, so we have the same decomposition of $G[p]$ as above.

First of all we can see $\varphi(G[p]) = G[p]$. Automatically

\[E_0(T) \cap G[p] \cap U_N = \{0\},\]

because $\varphi$ is one to one, therefore $G[p] = Q_N \oplus R_N \cong \varphi(Q_N) \oplus \varphi(R_N) = Q_N \oplus \varphi(R_N) \subseteq G[p]$ but $\dim R_N = \dim \varphi(R_N) < +\infty$, this implies $\varphi(G[p]) = G[p]$. Next we can see $\varphi(G) \supseteq G[p^2]$. The group $H = \{x \mid x \in G$ and the first $N-1$ coordinates in $x$ are zero\} is a direct summand of $G$ and

\[H[p^2] = G[p] \cap U_N = Q_N\]

\[= (E_1(T) \cap Q_N) \oplus (E_2(T) \cap Q_N) \oplus \cdots \oplus (E_{p-1}(T) \cap Q_N)\].

We can take a finite group $L$ such that $G = H \oplus L$. We have to show first $\varphi(G) \supseteq H[p^2]$. For arbitrary $x$ in $H[p^2]$, $px = \sum_{q=0}^{p-1} z_q$ for some $z_q \in E_q(T) \cap Q_N$ ($1 \leq q \leq p - 1$), then each $z_q$ is a linear combination of $p\varphi(x)$, $p^2\varphi(x)$, $\cdots$, $p^{p-1}\varphi(x)$. This means that there exist $x_q \in G[1 \leq q \leq p - 1]$ such that $z_q = p\varphi(x_q)$ for $1 \leq q \leq p - 1$. Therefore $px = \sum_{q=1}^{p-1} p\varphi(x_q)$, so $x - \varphi(\sum_{q=1}^{p-1} x_q) \in G[p]$, but $G[p] = \varphi(G[p])$ implies $x \in \varphi(G)$. Now $\varphi(G) \supseteq G[p^2]$ can be shown. For $x \in G[p^2]$ there exists a positive integer $M$ and integers $r_i$, $0 \leq r_i \leq p - 1$ (at least one of them is not zero) such that $\sum_{i=0}^{M} r_i p^i \varphi(x) \in Q_N = H[p^2]$, because $G[p]/Q_N$ is finite dimensional. Since $\varphi(Q_N) = Q_N$, we can assume $r_n = 1$ without loss of generality. Then we find $z \in H[p^2]$ such that $p \sum_{i=0}^{M} r_i p^i \varphi(x) = pz$. But $H[p^2] \subseteq \varphi(G)$ has been shown, so $z = \varphi(z')$ for some $z' \in G$, therefore $x + \sum_{i=0}^{M} r_i p^i \varphi(x) - \varphi(z') \in G[p] = \varphi(G[p])$, this implies $x \in \varphi(G)$. Now we can see $\varphi(G) \supseteq G[p^n]$ for all $n = 1, 2 \cdots$ by induction. Namely in general $\varphi(G) \supseteq G[p^n]$ and the special form of $\varphi$ on $Q_N$ imply $\varphi(G) \supseteq H[p^{n+1}]$. And $\varphi(G) \supseteq H[p^{n+1}]$ and the finiteness of $L$ imply $\varphi(G) \supseteq G[p^{n+1}]$.\]
Proof of (3). Suppose $G$ is the direct sum of two subgroups $G_1$ and $G_2$ and $\varphi$ is the projection onto $G_1$. The continuous extension $T$ of $\varphi |_{G_1[p]}$ is also a projection defined on $C[p]$, therefore $C[p] = E_\alpha(T) \oplus E_\gamma(T)$ and $G[p] = (E_\alpha(T) \cap G[p]) \oplus (E_\gamma(T) \cap G[p])$. Since $G[p]$ is invariant under $T$, $T$ must be strongly singular by our assumption about $G[p]$. Suppose $E_\alpha(T)$ is open, then $E_\alpha(T)$ is finite, hence $G[p] = E_\alpha(T) \cap G[p]$ is finite. The finiteness of $G[p]$ implies the finiteness of $G_2$.

The following is a direct corollary of Theorem 1.

Corollary. Let $G$ be a pure subgroup of $C$ which contains $\Sigma$. If $G[p]$ is not invariant under any nonstrongly singular homomorphism on $C[p]$, then $G$ has the three properties stated in (1), (2) and (3) in Theorem 1. Namely $G$ has no proper isomorphic quotient group and no proper isomorphic subgroup, and $G$ is quasi-indecomposable.

3. Existence theorem

Theorem 2. There exists a pure subgroup $G$ of $C$ which contains $\Sigma$ and satisfies three properties;

1. $G$ has no proper isomorphic quotient groups,
2. $G$ has no proper isomorphic subgroups,
3. $G$ is quasi-indecomposable.

And an arbitrary pure subgroup $H$ of $C$ such that $H$ contains $\Sigma$ and $H[p] = G[p]$ satisfies above three properties.

This theorem comes from the corollary of Theorem 1 and following two lemmas. Lemma 1 is known as the purification property, so we omit the proof of Lemma 1 (see more general form in [6]).

Lemma 1. For an arbitrary subgroup $Q$ between $\Sigma[p]$ and $C[p]$ there exists a pure subgroup $G$ of $C$ such that $G$ contains $\Sigma$ and $G[p] = Q$.

Lemma 2. For any family $\{T_\lambda | \lambda \in \Lambda\}$ of nonstrongly singular homomorphisms on $C[p]$ there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any $T_\lambda (\lambda \in \Lambda)$.

The existence of such $Q$ can be shown by transfinite induction which is Crawley's idea in [3]. We need following lemma which is also essentially Crawley's.

Lemma 3. Suppose $T$ is a nonstrongly singular homomorphism on $C[p]$. Then there exists a one-parameter family $\Delta(T) = \{x_t | 0 \leq t \leq 1\}$ of elements in $C[p]$ such that four elements $x_0, x_1, Tx$, and $Tx$, are
linearly independent for arbitrary \( s \neq t \).

**Proof.** The proof can be divided into two cases (a) and (b).

(a) \( T \) is singular but not strongly singular. In this case, by Baire's category theorem (\( C[p] \) is a complete metric space) there are at least two \( q \) and \( q' \) such that both \( E_q(T) \) and \( E_{q'}(T) \) are infinite compact groups, so card \( E_q(T) = \text{card} \ E_{q'}(T) = c \) (for instance, see [5], p. 31). Therefore \( \dim E_q(T) = \dim E_{q'}(T) = c \). Let \( \{y_t | 0 \leq t \leq 1\} \) be a basis of \( E_q(T) \) and \( \{y'_t | 0 \leq t \leq 1\} \) be a basis of \( E_{q'}(T) \). Then \( \mathcal{A}(T) = \{y_t + y'_t | 0 \leq t \leq 1\} \) is the desired family.

(b) \( T \) is not singular. In this case, by Baire's category theorem \( U_n/E(T) \cap U_n \) are infinite compact groups for all \( n = 1, 2, \ldots \), so as above \( \dim U_n/E(T) \cap U_n = c \). Hence \( U_n = (E(T) \cap U_n) \oplus D_n \) with \( \dim D_n = c \) for all \( n = 1, 2, \ldots \). Take \( 0 \neq x_0 \in D_1 \), then \( x_0 \) and \( Tx_0 \) are linearly independent. Let \( \{z_0, z_1, \ldots, z_{2^\lambda - 1}\} \) be the group generated by \( x_0 \) and \( Tx_0 \), then by the continuity of \( T \) we can find \( U_M \) such that \( z_i + U_M + T(U_M) \) (where \( i \leq 2^\lambda - 1 \)) are mutually disjoint. For this \( M \) we take a basis \( \{y_t | 0 \leq t \leq 1\} \) of \( D_M \). Then \( \mathcal{A}(T) = \{x_0 + y_t | 0 \leq t \leq 1\} \) is the desired system. Because, suppose \( n_i(x_0 + y_t) + n'_i(Tx_0 + Ty_t) = n_i'(x_0 + y_t) + n'_i(Tx_0 + Ty_t) \) for \( i \neq t \) when \( n_i, n_i', n'_i \) are integers, then \( n_i x_0 + n_i Tx_0 + n_i y_t + n'_i T y_t = n_i' x_0 + n'_i T x_0 + n_i' y_t + n'_i T y_t \), and \( n_i x_0 + n_i Tx_0 \) must be some \( z_i \) and also \( n'_i x_0 + n'_i T x_0 \) must be some \( z_i \), but \( z_i = z_j \) by our choice of \( U_M \). This implies \( n_i = n_i' \mod p \) and \( n_i = n_i' \mod p \), therefore we have \( n_i y_t + n_i T y_t = n_i y_t + n_i T y_t \), whence \( n_i (y_t - y_t) = -n_i T(y_t - y_t) \). However \( 0 \neq y_t - y_t \in D_M \) and \( D_M \cap E(T) = 0 \), hence \( n_i = n_i' = 0 \mod p \).

**Proof of Lemma 2.** \( \{T_\lambda | \lambda \in \Lambda\} \) is given, then card \( \Lambda \) is at most \( c \) (note that the cardinality of the set of all continuous homomorphisms on \( C[p] \) is at most \( c \), because \( C[p] \) is a separable compact group). We assume that \( \Lambda \) is a well ordered set of ordinal numbers which are less than \( \Omega \), where \( \Omega \) is the first ordinal number whose cardinality is \( c \). Choose \( e \in C[p] \) but \( e \in \Sigma[p] \), then we can construct a family of subgroups \( R_\lambda (\lambda \in \Lambda) \) by transfinite induction as follows:

(a) \( \Sigma[p] = R_0 \supseteq R_1 \supseteq R_2 \supseteq C[p] \) if \( 0 \leq \lambda < \mu \) (\( \lambda, \mu \in \Lambda \)),

(b) \( \text{card} R_\lambda \leq \text{card} \lambda \cdot \aleph_0 < c \) for all \( \lambda \in \Lambda \),

(c) \( e \in R_\lambda \) but there exists \( \lambda \in \Lambda \) such that \( e - T_\lambda x_\lambda \in R_\lambda \).

Suppose \( R_\lambda \) has been constructed for all \( \lambda < \mu \in \Lambda \). Let \( R'_\mu = \bigcup_{\lambda < \mu} R_\lambda \). Then card \( (\text{card} (\text{card} R'_\mu) \leq \text{card} \mu \cdot \aleph_0 < c \), where \( [e] \) is the group generated by \( e \). The property of \( \mathcal{A}(T_\mu) \) in Lemma 3 guarantees the existence of \( x_0 \in \mathcal{A}(T_\mu) \) such that \( ([e] + R'_\mu) \cap ([x_\lambda] + \{T_\mu x_\lambda\}) = 0 \). Then \( R_\mu = R'_\mu + \{x_\lambda\} + [e - T_\mu x_\lambda] \) is the desired subgroup. Let \( Q = \bigcup_{\lambda \in \Lambda} R_\lambda \), then by (a) \( Q \) is a subgroup of \( C[p] \) which contains \( \Sigma[p] \) and by (c) \( Q \) is not invariant under any \( T_\lambda (\lambda \in \Lambda) \).
4. A quasi-decomposable group without proper isomorphic quotient groups and proper isomorphic subgroups.

**Theorem 3.** There exists a pure subgroup $G$ of $C$ which contains $\Sigma$ and satisfies properties:

(1) $G$ has no proper isomorphic quotient groups,
(2) $G$ has no proper isomorphic subgroups,
(3) $G$ has a decomposition $G_1 \oplus G_2$ such that $G_1$ and $G_2$ are not bounded.

The following lemma is essential for our proof of this theorem.

**Lemma 4.** For any family $\{T_\lambda | \lambda \in \Lambda\}$ of nonsingular homomorphisms on $C[p]$ there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any $T_\lambda (\lambda \in \Lambda)$ but invariant under the canonical projection $P_e$ onto even coordinates.

The outline of the proof of this lemma will be given later.

**Proof of Theorem 3.** Every element of $C$ has countable coordinates as an element of the product space $\prod_{n=1}^\infty C(p^n)$; $x \in C$ is called an even (odd) element if all odd (even) coordinates are zero. For a subset $A$ of $C$ $A'(A^o)$ means the set of all even (odd) elements in $A$. Then clearly $C = C^e \oplus C^o$ and $\Sigma = \Sigma^e \oplus \Sigma^o$. By Lemma 4 there exists a subgroup $Q$ between $\Sigma[p]$ and $C[p]$ such that $Q$ is not invariant under any nonsingular homomorphisms on $C[p]$ but is invariant under $P_e$, therefore $\Sigma'[p] = \Sigma[p]^e \subset Q^e \subset C[p]^e = C'[p]$, $\Sigma^o[p] = \Sigma[p]^o \subset Q^o \subset C[p]^o = C^o[p]$ and $Q = Q^e \oplus Q^o$. With exactly the same proof as that of Lemma 1 we can show that there exists a pure subgroup $G_i(G_2)$ of $C^e(C^o)$ which contains $\Sigma'(\Sigma^o)$ and $G_i[p] = Q'(G_i[p] = Q^o)$. Clearly $G_i$ and $G_2$ are not bounded. Let $G = G_1 \oplus G_2$, then $G$ is a pure subgroup of $C$ which contains $\Sigma$ and $G[p] = G_i[p] \oplus G_2[p] = Q^e \oplus Q^o = Q$. By Theorem 1 $G$ has the properties (1) and (2) in Theorem 3.

The outline of the proof of Lemma 4. In order to prove Lemma 4 we can apply a similar method to the construction of $Q$ in Lemma 2. However before doing it we have to prepare some reformation of Lemma 3. Precisely our reformation is as follows, hereafter we shall use notations $A^e = P_e(A)(A^o = (I - P_e)(A))$ for a subset $A$ of $C[p]$ and $x^e = P_e x (x^o = x - P_e x)$ for an element $x$ in $C[p]$.

For an arbitrary nonsingular homomorphism $T$ we can find a one-parameter family $\Delta(T) = \{x_t | 0 \leq t \leq 1\}$ of elements in $C[p]$ which has one of the following six properties: $1^e, 2^e, 3^e, 1^o, 2^o$ and $3^o$. 

1° $x_t, Tx_t \in C[p]^*$ for all $0 \leq t \leq 1$ and four elements $x_s, x_t, Tx_s$, and $Tx_t$ are linearly independent for arbitrary $s \neq t$.

2° there exists $q, 0 \leq q \leq p - 1$ such that $x_t \in C[p]^0$ and

$$Tx_t - qx_t \in C[p]^*$$

for all $0 \leq t \leq 1$ and four elements $x_s, x_t, Tx_s - qx_s$, and $Tx_t - qx_t$ are linearly independent for arbitrary $s \neq t$.

3° $x_t \in C[p]^e$ for all $0 \leq t \leq 1$ and six elements $x_s, x_t, (Tx_s)^0, (Tx_s)^e$, $(Tx_t)^0$ and $(Tx_t)^e$ are linearly independent for arbitrary $s \neq t$.

1°, 2° and 3° are dual properties 1°, 2° and 3° by exchanging odd for even.

In the proof of this we have some difficulty coming from non-commutativity of nonsingular homomorphism and $P_\ast$. The proof in our original manuscript needs a long computation, in this paper we omit our detailed computation according to referee's suggestion but authors can supply the detailed proof to interested readers.

Using above $\mathcal{A}(T)$ the existence of $Q$ in Lemma 4 can be shown as follows. Let $\{T_\lambda | \lambda \in A\}$ be a given family of nonsingular homomorphisms on $C[p]$. We assume that $A$ is a well ordered set of ordinal numbers which are less than the first ordinal number whose cardinality is $c$. Choose $c \in C[p]$ but $c^0, c^e \notin \Sigma[p]$. By transfinite induction we can construct the following family of subgroups $R_(\lambda \in A)$:

(a) $\Sigma[p] = R_0 \subset R_1 \subset R_\lambda \subset C[p]$ if $0 \leq \lambda < \mu (\lambda, \mu \in A)$,

(b) card $R_\lambda \leq $ card $\lambda \cdot \kappa_c < c$ for all $\lambda \in A$,

(c) $R_\lambda$ is invariant under $P_\lambda$ for all $\lambda \in A$,

(d) $c^0$ and $c^e \in R_\lambda$ but there exists $x_\lambda \in R_\lambda \cap \mathcal{A}(T_\lambda)$ such that $c^0 - T_\lambda x_\lambda$ or $c^e - T_\lambda x_\lambda$ or $c - T_\lambda x_\lambda \in R_\lambda$ for all $\lambda \in A$.

Suppose $R_\lambda$ has been constructed for all $\lambda < \mu \in A$. Let $R_\mu = \bigcup_{\lambda < \mu} R_\lambda$. Then $\text{card } R_\mu \leq \text{card } \lambda \cdot \kappa_c < c$ and $R_\mu$ is invariant under $P_\lambda$ and $c^0$ and $c^e \in R_\mu$. Let $\mathcal{A}(T_\mu)$ be one having one of properties $1° \sim 3°$ and $1° \sim 3°$. Suppose $\mathcal{A}(T_\mu)$ has property $1°$, then we can find $x_\mu \in \mathcal{A}(T_\mu)$ such that $(R_\mu' + [c^e] + [c^e]) \cap (\{x_\mu\} \bigoplus [T_\mu x_\mu]) = \{0\}$. Let

$$R_\mu = R_\mu' + [x_\mu] + [c^0 - T_\mu x_\mu] ,$$

then clearly $R_\mu$ satisfies above (a), (b) and (c). And $c^0$ and $c^e \in R_\mu$ also holds. Suppose $c^e \in R_\mu$, then $c^0 = x + nx_\mu + m(c^0 - T_\mu x_\mu)$ for some $x \in R_\mu'$ and some integers $n$ and $m$, so $x + (1 - m)c^0 = nx_\mu - mT_\mu x_\mu$, but by our choice of $x_\mu, nx_\mu - mT_\mu x_\mu = 0$ and $x + (m - 1)c^0 = 0$. This implies $n = m = 0 \text{ mod } p$ and $c^e = x \in R_\mu'$ which is a contradiction. Suppose $c^e \in R_\mu$, then $c^e = x + nx_\mu + m(c^0 - T_\mu x_\mu)$ for some $x \in R_\mu'$ and some integers $n$ and $m$, but $x_\mu$ and $T_\mu x_\mu \in C[p]^e$, so $c^e = x \in R_\mu'$ which
is also a contradiction. Suppose \( \triangle(T_\mu) \) has property 2°, then we can find \( x_\mu \in \triangle(T_\mu) \) such that \( (R'_\mu + [c^\circ] + [c^\circ]) \cap ([x_\mu] \oplus [T_\mu x_\mu - qx_\mu]) = \{0\} \). Let \( R_\mu = R'_\mu + [x_\mu] + [c^\circ - T_\mu x_\mu + qx_\mu] \), then clearly \( R_\mu \) satisfies above (a), (b) and (c). And \( c^\circ \) and \( c^\circ \in R_\mu \) also holds. Suppose \( c^\circ \in R_\mu \), then \( c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu + qx_\mu) \) for some \( x \in R'_\mu \) and some integers \( n \) and \( m \), but \( x_\mu \in C[p]^0 \) and \( T_\mu x_\mu - qx_\mu \in C[p]^* \), hence we have \( c^\circ = x^\circ + nx_\mu \), that is, \( -x^\circ + c^\circ = nx_\mu \). Our choice of \( x_\mu \) implies \( nx_\mu = 0 = -x^\circ + c^\circ \), so we have \( c^\circ = x^\circ \in R'_\mu \) which is a contradiction. Suppose \( c^\circ \in S_\mu \), then \( c^\circ = x + nx_\mu + m(c^\circ - T_\mu x_\mu + qx_\mu) \) for some \( x \in R'_\mu \) and some integers \( n \) and \( m \). Hence \( -x + (1 - m)c^\circ = nx_\mu - m(T_\mu x_\mu - qx_\mu) \), but by our choice of \( x_\mu \) we see \( -x + (1 - m)c^\circ = 0 = nx_\mu - m(T_\mu x_\mu - qx_\mu) \). This implies \( n = m = 0 \mod p \), so \( c^\circ = x \in R'_\mu \) which is also a contradiction. Suppose \( \triangle(T_\mu) \) has property 3°, then we can find \( x_\mu \in \triangle(T_\mu) \) such that \( (R'_\mu + [c^\circ] + [c^\circ]) \cap ([x_\mu] \oplus [(T_\mu x_\mu)^0] \oplus [(T_\mu x_\mu)^*]) = \{0\} \). Let

\[
R_\mu = R'_\mu + [x_\mu] + [c^\circ - (T_\mu x_\mu)^0] + [c^\circ - (T_\mu x_\mu)^*].
\]

Then \( R_\mu \) clearly satisfies (a), (b) and (c). (a) and (c) \( c^\circ \) and \( c^\circ \in R_\mu \) can be seen as follows. Suppose \( c^\circ = x + nx_\mu + m(c^\circ - (T_\mu x_\mu)^0) + m'(c^\circ - (T_\mu x_\mu)^*) \) for some \( x \in R'_\mu \) and integers \( n \), \( m \) and \( m' \), then

\[
c^\circ = x^\circ + nx_\mu + m(c^\circ - (T_\mu x_\mu)^0),
\]

so \( -x^\circ + (1 - m)c^\circ = nx_\mu - m(T_\mu x_\mu)^0 = 0 = -x^\circ + (1 - m)c^\circ \) by our choice of \( x_\mu \). Hence \( m = 0 \) and \( c^\circ = x^\circ \in R'_\mu \) which is a contradiction. We can see also \( c^\circ \in R_\mu \) for same reason. And \( x_\mu \) and \( c - T_\mu x_\mu \in R_\mu \) is clear. The construction of \( R_\mu \) for \( \triangle(T_\mu) \) having one of properties 1° ~ 3° is exactly similar by exchanging odd for even.

Let \( Q = \bigcup_{i \in I} R_i \). Then the above properties (a) ~ (d) for all \( R_i \) guarantee that \( Q \) is a subgroup between \( \Sigma[p] \) and \( C[p] \) not invariant under any \( T_i(\lambda \in A) \) but invariant under \( P_i \).

**References**


Received August 22, 1967.

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