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It is known that any function which minimizes a functional of the form $J(y) = \int_a^b f(x, y, y')$ and satisfies prescribed boundary values must be a solution of the corresponding Euler-Lagrange equation: $f_3(x, y, y') - \int_a^x f_2(x, y, y') = c$. Let us call any equation of the form: $g(x, y, y') - \int_a^x h(x, y, y') = c$ a generalized Euler-Lagrange equation.

In this paper we propose a Newton-like method and show that this proposed method is general enough to enable us to construct solutions of the generalized Euler-Lagrange equation.

Let X and Y be Banach spaces, Ω an open subset of X and $P: \Omega \rightarrow Y$. By $[X, Y]$ we mean the Banach space of all bounded linear operators with the usual operator norm, by P' the first derivative of P and by P'' the second derivative of P . The class of all functions defined on Ω which have continuous derivatives up to and including order n at each point of Ω is denoted by $C^n(\Omega)$. Distinct elements of $C^n(\Omega)$ may have totally distinct ranges depending on the application. The distinction between Gateaux and Fréchet is unnecessary since the derivatives will be continuous.

2. The weak Newton sequence. Let H and Y be Banach spaces, Ω a nonempty open subset of H and $P: \Omega \rightarrow Y$.

DEFINITION. Given $x_0 \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n),$$

is called the Newton sequence for x_0 (with respect to P).

DEFINITION. Given $x_0, \bar{x} \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(\bar{x})]^{-1}P(x_n),$$

is called the modified Newton sequence for x_0 at \bar{x} (with respect to P). When $x_0 = \bar{x}$ we say simply the modified Newton sequence for x_0 .

REMARK. The Newton sequences exist if and only if $P'(\bar{x})$ and $P'(x_n)$ exist and are invertible and $x_n \in \Omega$ for all n .

Let X and Y be Banach spaces, H a closed linear subspace of X , Ω a nonempty open subset of H and $P: \Omega \rightarrow Y$.

REMARK. It is easy to verify that Ω is open in X if and only if $H = X$.

Let D be the set of all $x \in \Omega$ for which there exists an operator $\Gamma(x)$ satisfying the following conditions:

- (2.1) (i) $\Gamma(x) \in [B_x, X]$, where B_x is a closed linear subspace of Y containing $P(\Omega)$;
 (ii) $\Gamma(x)P(\Omega) \subset H$;
 (iii) $\Gamma(x)P'(x) = I: H \rightarrow H$.

REMARK. The fact that $\Gamma(x)$ is defined on $P'(x)(H)$ and $\Gamma(x)P'(x)$ is defined from H into H is a consequence of conditions (i) and (ii). The following lemma shows this to be true.

LEMMA 2.1. *If there exists an operator Γ_0 satisfying the following conditions:*

- (a) $\Gamma_0 \in [B_0, X]$, where B_0 is a closed linear subspace of Y containing $P(\Omega)$;
 (b) $\Gamma_0 P(\Omega) \subset H$;

then for all $x \in \Omega$ we have the following:

- (i) $P'(x)(H) \subset B_0$;
 (ii) $\Gamma_0 P'(x)(H) \subset H$.

Proof. Assume $x \in \Omega$ and $h \in H$. It follows from (a) since Ω is open in H that for small t

$$\frac{P(x + th) - P(x)}{t} \in B_0.$$

Therefore $P'(x)(h) \in B_0$ and $P'(x)(H) \subset B_0$. Similarly (a), (b) and the fact that H is closed imply

$$\Gamma_0 P'(x)(h) = \lim_{t \rightarrow 0} \left[\frac{\Gamma_0 P(x + th) - \Gamma_0 P(x)}{t} \right] \in H.$$

DEFINITION. If $x \in D$, then $\Gamma(x)$ is called a left inverse for $P'(x)$.

REMARK. If $H = X$ and $B_x = Y$, then this is the usual notion of a left inverse.

Let $\Gamma(x)$ denote a left inverse for $P'(x)$.

DEFINITION. Given $x_0 \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - \Gamma(x_n)P(x_n),$$

is called the weak Newton sequence for x_0 (with respect to P).

DEFINITION. Given $x_0, \bar{x} \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - \Gamma(\bar{x})P(x_n),$$

is called the weak modified Newton sequence for x_0 at \bar{x} (with respect to P). When $x_0 = \bar{x}$ we say simply the weak modified Newton sequence for x_0 .

REMARK. The weak Newton sequences exist if and only if $\Gamma(\bar{x})$ and $\Gamma(x_n)$ exist and $x_n \in \Omega$ for all n .

LEMMA 2.2. If $x \in \Omega$ and $[P'(x)]^{-1} \in [Y, H]$, then for any $\Gamma(x)$ satisfying (2.1) we have:

- (i) $B_x = Y$;
- (ii) $\Gamma(x)(Y) = H$;
- (iii) $\Gamma(x) = [P'(x)]^{-1}$.

Proof. Since $P'(x)$ is onto, $Y = P'(x)(H)$. By Lemma 2.1 $P'(x)(H) \subset B_x$, therefore $Y = B_x$. Also $\Gamma(x)y = \Gamma(x)P'(x)[P'(x)^{-1}y] = P'(x)^{-1}y$ for all $y \in Y$, therefore $\Gamma(x) = [P'(x)]^{-1}$.

REMARK. In general there are many weak (modified) Newton sequences for a particular point; however, by Lemma 2.2 if the (modified) Newton sequence exists, then any weak (modified) Newton sequence coincides with it.

EXAMPLE. Let $P: R^1 \rightarrow R^2$ be given by $P(x) = (3x - 3/2, 2x - 1)$ hence $P'(x) = (3, 2)$. If $\Gamma_0: R^2 \rightarrow R^1$ is given by

$$\Gamma_0(x_1, x_2) = \left(a, \frac{1 - 3a}{2} \right) \cdot (x_1, x_2)$$

for any real a , then $\Gamma_0 P(x) = x - 1/2$, $\Gamma_0 P'(x) = 1$, the identity map in R^1 , and the weak (modified) Newton sequence for x_0 is given by

$$x_{n+1} = x_n - \Gamma_0 P(x_n) = \frac{1}{2}$$

for all n . Notice that $P(1/2) = (0, 0)$.

REMARK. In this example there are an uncountable number of

left inverses for $P'(x)$; also the (modified) Newton sequence for x_0 does not exist.

3. Convergence theorems. As before let X and Y be Banach spaces, H a closed linear subspace of X , Ω a nonempty open subset of H and $P: \Omega \rightarrow Y$. If for each $x \in \Omega$ there exists an operator $\Gamma(x)$ satisfying conditions 2.1 and we let $B_0 = \bigcap_{x \in \Omega} B_x$, where B_x is the domain of $\Gamma(x)$, then B_0 is a closed linear subspace of Y containing $P(\Omega)$; consequently by restricting $\Gamma(x)$ to B_0 we may consider $\Gamma: \Omega \rightarrow [B_0, X]$.

THEOREM 3.1. *If*

(i) *there exists $x^* \in \Omega$ such that $P(x^*) = 0$,*

(ii) *for each $x \in \Omega$ there exists $\Gamma(x)$ satisfying conditions (2.1) and $\Gamma: \Omega \rightarrow [B_0, X] \in C^0(\Omega)$*

then, if $P \in C^1(\Omega)$, given $0 < \alpha < 1$ there exists a neighborhood of x^ contained in Ω such that for any 2 points x_0 and \bar{x} in this neighborhood the weak modified Newton sequence for x_0 at \bar{x} exists and converges to x^* ; we also have*

$$\|x^* - x_n\| \leq \frac{\alpha^n}{1 - \alpha} \|x_1 - x_0\|.$$

Furthermore, if $P \in C^2(\Omega)$, then there exists a neighborhood of x^ contained in Ω such that if x_0 is any point in this neighborhood a weak Newton sequence for x_0 exists and converges quadratically to x^* , i.e., there exists a constant M such that*

$$\|x^* - x_{n+1}\| \leq M \|x^* - x_n\|^2.$$

Proof. For $x \in \Omega$ let $T(x) = I - \Gamma(x)P'(x^*)$. Given $0 < \alpha < 1$ there exists $\delta > 0$ such that $\{x \mid \|x - x^*\| \leq \delta\} \subset \Omega$ and

$$\|T(x)\| = \|T(x) - T(x^*)\| < \alpha$$

whenever $\|x - x^*\| \leq \delta$. Now if $\|\bar{x} - x^*\| \leq \delta$ and $S = I - \Gamma(\bar{x})P$, then

(i) $S(x^*) = x^*$ and

(ii) $\|S'(x^*)\| < \alpha$.

Let $\|S'(x^*)\| = \alpha_0$, then there exists $0 < \delta_0 \leq \delta$ such that

$$\|S'(x) - S'(x^*)\| \leq \alpha - \alpha_0$$

whenever $x \in \Omega_0 = \{x \mid \|x - x^*\| \leq \delta_0\}$. Clearly Ω_0 is closed, convex and contained in Ω . Also

(i)' $\|S'(x)\| \leq \alpha$ whenever $x \in \Omega_0$ and

(ii)' $S(\Omega_0) \subset \Omega_0$;

since if $x \in \Omega_0$,

$$\begin{aligned} \|S(x) - x^*\| &= \|S(x) - S(x^*)\| \\ &\leq \|S'(x + \theta(x^* - x))\| \|x - x^*\| \quad 0 < \theta < 1 \\ &\leq \alpha \|x - x^*\| \leq \delta_0. \end{aligned}$$

The first part of this theorem now follows from a well-known fixed point theorem [4, 661]. The latter part of the theorem is a consequence of the following Banach space inequality [3, pp. 162-163].

(3.1) $\|P(x + \Delta x) - P(x) - P'(x)(\Delta x)\| \leq \frac{1}{2} \|P''(x + \theta \Delta x)\| \|\Delta x\|^2,$

$0 < \theta < 1$. There clearly exists K and $\delta > 0$ such that $\|P''(x)\| \leq K$ and $\|\Gamma(x)\| \leq K$, whenever $\|x^* - x\| \leq \delta$. If $\|x^* - x\| \leq \delta$, then from (3.1) with $\Delta x = x^* - x$ we obtain

(3.2) $\|x^* - [x - \Gamma(x)P(x)]\| \leq M \|x^* - x\|^2$

where $M = \frac{1}{2}K^2$. Choose $0 < \delta_0 < \min(M^{-1}, \delta)$ and such that

$$\Omega_0 = \{x \mid \|x^* - x\| \leq \delta_0\} \subset \Omega.$$

We now show if $X_n \in \Omega_0$, then

(3.3)
$$\begin{aligned} \|x^* - x_{n+1}\| &\leq M \|x^* - x_n\|^2 \\ &\leq \|x^* - x_0\| (M \|x^* - x_0\|)^{2^{n+1}-1} \leq \delta_0; \end{aligned}$$

consequently $x_{n+1} \in \Omega_0$. For $n = 0$ inequality (3.3) is just (3.2) with $x = x_0$. If we assume (3.3) holds for $n < k$, then a direct application of (3.2) shows (3.3) holds for $n = k$, and consequently for all n . This proves the theorem.

Consider $\Gamma_0: B_0 \rightarrow X$, where B_0 is a closed linear subspace of Y containing $P(\Omega)$, satisfying:

(3.4)
$$\begin{aligned} (i) \quad &\Gamma_0 \in [B_0, X]; \\ (ii) \quad &\Gamma_0 P(\Omega) \subset H. \end{aligned}$$

COROLLARY 3.1. *Given $x_0 \in \Omega$, if there exists*

- (i) $x^* \in \Omega$ such that $P(x^*) = 0$, and
- (ii) Γ_0 satisfying (3.4) and such that $\Gamma_0 P'(x_0): H \rightarrow H$ is invertible,

then if x_0 is sufficiently near x^* and $P \in C^2(\Omega)$ a weak (modified) Newton sequence for x_0 exists and converges to x^* . Furthermore a left inverse for $P'(x)$ is given by $\Gamma(x) = [\Gamma_0 P'(x)]^{-1} \Gamma_0$.

Proof. Use Theorem 3.1 on $Q = \Gamma_0 P$.

4. A general problem. Given an interval $[a, b]$ let

$$\begin{aligned}
 X &= \{y \in C^0[a, b] \mid y: [a, b] \rightarrow R^1\}, \\
 H &= \left\{h \in X \mid \int_a^b h = 0\right\}, \\
 Y &= \{y \in C^1[a, b] \mid y: [a, b] \rightarrow R^1\}, \\
 (4.1) \quad K &= \{y \in Y \mid y(a) = \mu_1 \text{ and } y(b) = \mu_2\}, \\
 B &= [a, b] \times R^2, \\
 \|y\|_X &= \sup \{|y(t)| \mid t \in [a, b]\} \text{ for } y \in X, \text{ and} \\
 \|y\|_Y &= \sup \{|y(t)|, |y'(t)| \mid t \in [a, b]\} \text{ for } y \in Y.
 \end{aligned}$$

REMARK. If X, Y and H are given by (4.1), then X and Y are Banach spaces and H is a closed linear subspace of X .

Given continuous $\varphi: B \rightarrow R^1$ and $y_0 \in K$ define

$$\bar{Q}: Y \rightarrow X, \quad \text{and} \quad Q: X \rightarrow X$$

as follows

$$(4.2) \quad \bar{Q}(y)(x) = \varphi(x, y(x), y'(x)) \quad \text{for } y \in Y;$$

$$(4.3) \quad Q(y) = \bar{Q}\left(y_0 + \int_a^x y\right) \quad \text{for } y \in X.$$

It is clear that if $y \in X$, then $y_0 + \int_a^x y \in Y$. Assume Q satisfies the following two conditions:

$$\begin{aligned}
 (4.4) \quad & \text{(i)} \quad Q \in C^2[X]; \\
 & \text{(ii)} \quad [Q'(0)]^{-1} \in [X, X].
 \end{aligned}$$

Define $P: H \rightarrow X$ by

$$(4.5) \quad P(h) = Q(h) - \frac{\int_a^b \Gamma_0 Q(h)}{\int_a^b \Gamma_0(1)} \quad \text{for } h \in H,$$

where $\Gamma_0 = [Q'(0)]^{-1}$ and 1 is the constant function 1.

LEMMA 4.1. *If \bar{Q} is given by (4.2) and P by (4.5), then the following two problems are equivalent:*

$$\begin{aligned}
 (4.6) \quad & \text{Problem I. Find } y \in K \text{ such that } \bar{Q}(y) \text{ is constant;} \\
 & \text{Problem II. Find } h \in H \text{ such that } P(h) = 0.
 \end{aligned}$$

Proof. If $y \in K$ and $\bar{Q}(y)$ is constant, then $h = y' - y'_0 \in H$ and

$Q(h) = \bar{Q}\left(y_0 + \int_a^x (y' - y'_0)\right)$ is constant; therefore $P(h) = 0$. Now if $h \in H$ and $P(h) = 0$, then $Q(h) = \bar{Q}\left(y_0 + \int_a^x h\right)$ is constant; therefore $y = y_0 + \int_a^x h \in K$ and $\bar{Q}(y)$ is constant.

LEMMA 4.2. *If Γ_0 and P are given by (4.5), then Γ_0 is a left inverse for $P'(0)$.*

Proof. We show $\Gamma_0 P'(0) = I$ satisfies (2.1) for $x = 0$. If $B_0 = X$ and $\Omega = H$, then (i) of (2.1) holds. If $h \in H$, then from (4.5) we obtain,

$$\int_a^b \Gamma_0 P(h) = 0$$

and (ii) of (2.1) holds. By differentiating (4.5) at the origin we see that

$$\Gamma_0 P'(0)(h) = h \quad \text{for } h \in H.$$

Therefore $\Gamma_0 P'(0) = I: H \rightarrow H$ and the lemma is proved.

Let $y^* \in Y$ be a solution of Problem I and $h^* \in X$ the corresponding solution of Problem II.

LEMMA 4.3. *For given y_0 in (4.3)*

$$\|h^*\|_X \leq \|y_0 - y^*\|_Y.$$

Proof. The proof follows from (4.1) and Lemma 4.1.

REMARK. If Problem I has a solution and the given y_0 in (4.3) is sufficiently near this solution, then by Lemmas 4.1, 4.2, 4.3 and Corollary 3.1 with $\Gamma_0 = [Q'(0)]^{-1}$, $x_0 = 0$, $\Omega = H$, $Y = X$ and H , X and P given by (4.1) and (4.5) both the weak Newton and weak modified Newton sequences for x_0 exist and can be used to obtain this solution. In addition the weak modified Newton sequence can still be used if we only have $Q \in C^1[H]$.

5. A variation of the weak Newton sequence. If Γ_0 and P are given by (4.5) and h_n is the n^{th} term in the weak Newton sequence for $h_0 = 0$, then by Corollary 3.1 a left inverse for $P'(h_n)$ is given by

$$\Gamma(h_n) = [\Gamma_0 P'(h_n)]^{-1} \Gamma_0.$$

This operator may be difficult to evaluate; we therefore consider the

following variation of the weak Newton sequence for Problem II.

For Q given by (4.3), assume the following conditions hold:

$$(5.1) \quad \begin{aligned} & \text{(i) } Q \in C^2[X]; \\ & \text{(ii) } [Q'(x)]^{-1} \in [X, X] \text{ for all } x \in X. \end{aligned}$$

Inherent in the above assumption is the requirement that a procedure for evaluating $[Q'(x)]^{-1}$ is known. Let

$$(5.2) \quad \Gamma(x) = [Q'(x)]^{-1} \text{ for } x \in X$$

and define $h_n \in H$ and $P_n: H \rightarrow X$ recursively by

$$(5.3) \quad \begin{aligned} & h_0 = 0 \\ & P_0(h) = Q(h) - \frac{\int_a^b \Gamma(h_0)Q(h)}{\int_a^b \Gamma(h_0)(1)} \text{ for } h \in H, \text{ and} \\ & h_{n+1} = h_n - \Gamma(h_n)P_n(h_n), \\ & P_{n+1}(h) = Q(h) - \frac{\int_a^b \Gamma(h_{n+1})Q(h)}{\int_a^b \Gamma(h_{n+1})(1)}, \text{ for } n = 0, 1, \dots. \end{aligned}$$

LEMMA 5.1. *If $\Gamma(h_n)$ and $P_n(h_n)$ are given by (5.2) and (5.3), then $\Gamma(h_n)$ is a left inverse for $P'_n(h_n)$.*

Proof. The proof of this lemma is the same as the proof of Lemma 4.2.

THEOREM 5.1. *If (5.1) holds and the given y_0 in (4.3) is sufficiently close to a solution of Problem I, then the sequence $\{h_n\}_0^\infty$ given by (5.3) will converge quadratically to the corresponding solution of Problem II.*

Proof. The proof of this theorem is essentially the same as the latter part of the proof of Theorem 3.1.

LEMMA 5.2. *For y_0 used in (4.3) and $\{h_n\}_0^\infty$ given by (5.3), Problem I is equivalent to the following problem:*

Problem III. *Find $y_0 \in K$ such that the sequence $\{h_n\}_0^\infty$ converges.*

Proof. If y is a solution to Problem I then by Theorem 5.1 we need only pick $y_0 \in K$ near y . Now assume $\{h_n\}_0^\infty$ converges to h , then $\Gamma(h_n)P_n(h_n) \rightarrow 0$ and since $\Gamma^{-1}(h) = Q'(h)$ exists we have $P_n(h_n) \rightarrow 0$ or $Q(h)$ is constant. Also, since H is closed, $h \in H$; by Lemma 4.1 $y =$

$y_0 + \int_a^x h$ solves Problem 1.

REMARK. We have shown that whenever the sequence $\{h_n\}_0^\infty$ given by (5.3) converges, it converges quadratically to a solution of Problem II and consequently gives a solution to Problem I.

6. The generalized Euler-Lagrange equation. Let X, H, Y, K and B be given by (4.1). In the calculus of variations one is interested in finding $y \in K$ such that for all $x \in [a, b]$

$$(6.1) \quad f_3(x, y(x), y'(x)) - \int_a^x f_2(x, y(x), y'(x)) = c,$$

where $f: B \rightarrow R^1, f_i$ denotes the i^{th} partial derivative of f and c is an unknown constant. Historically equation (6.1) has been called the Euler-Lagrange equation.

For $g, h: B \rightarrow R^1$, we would like to find $y \in K$ such that for all $x \in [a, b]$

$$(6.2) \quad g(x, y(x), y'(x)) - \int_a^x h(x, y(x), y'(x)) = c.$$

We are therefore interested in solving Problem I (4.6) with $\bar{Q}: Y \rightarrow X$ given by:

$$(6.3) \quad \bar{Q}(y)(x) = g(x, y(x), y'(x)) - \int_a^x h(x, y(x), y'(x)).$$

Since equation (6.1) is a special case of equation (6.2), we call (6.2) the generalized Euler-Lagrange equation.

Given $f_{i,j} \in X, i = 1, 2, \dots, n$ and $j = 1, 2, \dots, 5$, define the integral operator $L: X \rightarrow X$ as follows: for $u \in X$ let

$$(6.4) \quad L(u)(x) = \sum_{i=1}^n \left[f_{i,1}(x) \int_a^x f_{i,2}(y) u(y) dy + f_{i,3}(x) \int_a^x \left(f_{i,4}(y) \int_a^y f_{i,5}(t) u(t) dt \right) dy \right].$$

THEOREM 6.1. If L is given by (6.4) and λ is any constant, then

- (i) $L \in [X, X]$,
- (ii) for all $u \in X$ the series

$$\|u\| + \|\lambda L(u)\| + \dots + \|\lambda^n L^n(u)\| + \dots$$

is convergent.

Furthermore,

- (iii) for all $\varphi \in X$ the equation

$$(6.5) \quad f = \varphi + \lambda L(f),$$

has a unique solution in X which can be obtained by iterating (6.5) beginning with an arbitrary element of X ; finally

$$(6.6) \quad \text{(iv) the operator } T = I - \lambda L \text{ has a continuous inverse which can be evaluated by iteration.}$$

Proof. The proof of (i) is immediate. Since (iii) proceeds directly from (ii) and (iv) follows from (i) and (iii) we will only prove (ii).

For $u \in X$ let $K(u)(x) = \int_a^x u$. Then there exists a constant B such that for $x \in [a, b]$

$$|L(u)(x)| \leq B(K + K^2)(|u|)(x) \leq B \|u\| (K + K^2)(1)(x).$$

Assume for $1 \leq n \leq k$

$$(6.7) \quad |L^n(u)(x)| \leq B^n (K + K^2)^n (|u|)(x) \leq B^n \|u\| (K + K^2)^n (1)(x),$$

then

$$\begin{aligned} |L^{k+1}(u)(x)| &= |L^k(L(u))(x)| \leq B^k (K + K^2)^k (|L(u)|)(x) \\ &\leq B^{k+1} (K + K^2)^{k+1} (|u|)(x) \leq B^{k+1} \|u\| (K + K^2)^{k+1} (1)(x) \end{aligned}$$

and by induction (6.7) holds for all n .

If we let $M = \max(1, \|u\|, |\lambda| B)$ and denote $K^n(1)(x)$ by K^n , then

$$(6.8) \quad \begin{aligned} \sum_{n=0}^{\infty} |\lambda^n L^n(u)(x)| &\leq \sum_{n=0}^{\infty} M^{n+1} (K + K^2)^n \leq \sum_{n=0}^{\infty} \sum_{j=0}^n M^{n+j+1} \binom{n}{j} K^{n+j} \\ &= \sum_{m=0}^{\infty} A_m M^{m+1} \frac{(x-a)^m}{m!}, \end{aligned}$$

where $A_m = \sum_{i=0}^{[m/2]} \binom{m-i}{i}$ and $[x]$ denotes the largest integer less than or equal to x . The first inequality in (6.8) follows from (6.7) and the second inequality is a direct application of the binomial theorem. The equality in (6.8) is justified by observing that the coefficients of K^m for arbitrary but fixed m are of the form $M^{n+j+1} \binom{n}{j}$ where $j \leq n$ and $n+j=m$ and also that $K^m = (x-a)^m/m!$. It is not difficult to show that A_m satisfies the difference equation $A_{m+1} = A_m + A_{m-1}$; consequently the radius of convergence of the last series in (6.8) is infinite. This proves (ii) and the theorem follows.

LEMMA 6.1. For $g, h \in C^0(B)$ let Q be given by (4.3) and (6.3). If $g, h \in C^n(B)$ then $Q \in C^n(X)$, for $n = 0, 1, 2$.

Proof. The proof is straightforward although somewhat lengthy [6].

If X, K and B are given by (4.1) and $y_0 \in K$, then for continuous $g: B \rightarrow R^1$ define $\bar{g}: X \rightarrow X$ as follows:

$$\bar{g}(y)(x) = g\left(x, y_0(x) + \int_a^x y(t), y'_0(x) + y(x)\right)$$

for $y \in X$ and $x \in [a, b]$.

THEOREM 6.2. For $g, h: B \rightarrow R^1$, if

- (a) $g, h \in C^2(B)$, and
- (b) $\bar{g}_3(y) \neq 0$ for $y \in X$

and if $Q: X \rightarrow X$ is given by (4.3) and (6.3) then

- (i) $Q \in C^2[X]$,
- (ii) $[Q'(y)]^{-1} \in [X, X]$ for all $y \in X$ and can be evaluated by iteration.

Proof. Part (i) follows from Lemma 6.1. A direct calculation shows that

$$Q'(y)(\eta) = \bar{g}_3(y)\eta + \bar{g}_2(y) \int_a^x \eta - \int_a^x (\bar{h}_2(y) \int_a^x \eta) - \int_a^x \bar{h}_3(y)\eta ,$$

for $y, \eta \in X$. The subscripts on \bar{g} and \bar{h} denote partial derivatives. If we let

$$(6.9) \quad \begin{aligned} f &= \frac{1}{\bar{g}_3(y)} \\ f_{11} &= f \cdot \bar{g}_2(y), f_{12} = 1, f_{13} = -f, f_{14} = \bar{h}_2(y), f_{15} = 1, \\ f_{21} &= -f, f_{22} = \bar{h}_3(y) \quad \text{and} \quad f_{23} = f_{24} = f_{25} = 0, \quad \text{then} \end{aligned}$$

$$T(\eta) = \eta + L(\eta) = f \cdot Q'(y)(\eta)$$

where $y, \eta \in X$ and $L: X \rightarrow X$ is given by (6.4) and (6.9) with $n = 2$ and $T: X \rightarrow X$ is given by (6.6) with $\lambda = -1$. Given $\varphi, y \in X$ constructing $\eta = [Q'(y)]^{-1}(\varphi)$ is equivalent to solving

$$T(\eta) = f\varphi \quad \text{for} \quad \eta \in X ,$$

hence $\eta = T^{-1}(f\varphi)$ and according to Theorem 6.1, η is given by

$$\eta = f\varphi - L(f\varphi) + L^2(f\varphi) - L^3(f\varphi) + \dots .$$

REMARKS. It follows that the theory developed in §1 through §5 can be used to solve the generalized Euler-Lagrange equation. In

the calculus of variations condition (b) above is called the strengthened Legendre condition.

Theorem 6.2 includes the problem of finding $y \in K$ such that

$$(6.10) \quad y'' = h(x, y, y') \quad \text{where} \quad h \in C^2(B).$$

If $g: B \rightarrow R^1$ is defined by

$$g(x, y, z) = z, \quad \text{then} \quad \bar{g}_s(y) = 1 \neq 0$$

and $g \in C^2(B)$, therefore Theorem 6.2 holds. Now if

$$(6.11) \quad y' - \int_a^x h(x, y, y') = \text{constant},$$

then since two of the three terms in (6.11) are differentiable, the third must also be, giving (6.10).

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