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The main result is

THEOREM. If A is a simple Lie-admissible power-associative ring with characteristic prime to six, and if A has an idempotent e relative to which A has a Peirce decomposition such that $A_{00}=0$, then either e is a unity element of A or $A \cong B$, where B is a three-dimensional algebra having a basis $\{e, x, y\}$ such that $e^2=e$, $ex=x$, $ye=y$, $xy=-yx=e$ and $xe=ey=x^2=y^2=0$.

If A is a simple Lie-admissible power-associative ring then A belongs to a class of rings which includes associative rings, Lie rings, commutative power-associative rings, Jordan rings, anti-flexible rings, rings of type (γ, δ) and others. Lie rings do not have idempotent elements, and simple (γ, δ) rings with an idempotent $e \neq 1$ have been shown [2, 3, 4, 5, 6, 8] to be associative. Thus if A has an idempotent element $e \neq 1$ then A belongs to a class which includes rings of the associative, commutative power-associative, and antiflexible types. Assuming that A has an idempotent e satisfying,

$$(1) \qquad (e, e, x) = (e, x, e) = (x, e, e) = 0 ,$$

suffices to establish a Peirce decomposition,

$$A = A_{11} + A_{10} + A_{01} + A_{00} ,$$

where $A_{ij} = \{x \in A \mid ex = ix, xe = jx\}$ for $i, j = 0, 1$. This assumption eliminates the possibility that A is commutative, for then $A_{10} = A_{01} = 0$, so [2] $A = A_{11} \oplus A_{00}$ and simplicity implies that $A = A_{11}$, hence e is a unity element for A.

The class of rings under consideration does contain members which are not associative. Kosier [7] has given examples of simple Lie-admissible power-associative finite-dimensional algebras, the so-called anti-flexible algebras. These have the property that $A = A_{11} + A_{00}$ in every Peirce decomposition.

There are no rings with unity element, 1, which possess a Peirce decomposition with respect to an idempotent $e \neq 1$ in which $A_{00} = 0$. This is because $1 - e \in A_{00}$.

The algebra B of our theorem was introduced in [9]. It has the property that $B^{(-)}$ is a simple Lie algebra.

The associator, $(x, y, z) = (xy)z - x(yz)$, and the commutator, $[x, y] = xy - yx$, are functions which, defined on any ring, are linear in each variable and related by the identity,

$$(2)\qquad [xy, z] + [yz, x] + [zx, y] = (x, y, z) + (y, z, x) + (z, x, y)\ .$$

A Lie-admissible ring satisfies,

$$(3)\qquad [xy - yx, z] + [yz - zy, x] + [zx - xz, y] = 0\ ,$$

and a power-associative ring whose characteristic is prime to two satisfies

$$(4)\qquad [xy + yx, z] + [yz + zy, x] + [zx + xz, y] = 0\ ,$$

hence in the ring A the function

$$\begin{aligned} H(x, y, z) &= (x, y, z) + (y, z, x) + (z, x, y) \\ &= [xy, z] + [yz, x] + [zx, y] \end{aligned}$$

is identically zero. Also, the fourth-power-associativity identities $(x^2, x, x) = 0$ and $(x, x, x^2) = 0$ may be linearized to yield functions $P(a, b, x, y) = \sum (ab, x, y)$ and $Q(a, b, x, y) = \sum (a, b, xy)$ which are identically zero. The $\sum$ here in both cases indicates a sum to be taken over the twenty-four permutations of a, b, x and y.

We will use $\cdot$ as well as juxtaposition in denoting products, with juxtaposition taking precedence. Thus $a\cdot bc = a(bc)$.

LEMMA. *Let A be a ring whose characteristic is prime to six and in which the functions H, P and Q vanish identically. Suppose A contains an idempotent e relative to which A has a Peirce decomposition. If a_{mn} denotes the component of an element a in the module A_{mn} then*

$$(5)\qquad x_{ii}y_{jj} = 0\ ,$$

$$(6)\qquad x_{ii}y_{ii} + y_{ii}x_{ii} \in A_{ii}\ ,$$

$$(7)\qquad x_{ii}^2 \in A_{ii}\ ,$$

$$(8)\qquad x_{ii}y_{ij} = (x_{ii}y_{ij})_{ij} + (x_{ii}y_{ij})_{jj} \in A_{ij} + A_{jj}\ ,$$

$$(9)\qquad y_{ij}x_{ii} = (x_{ii}y_{ij})_{jj} \in A_{jj}\ ,$$

$$(10)\qquad y_{ji}x_{ii} = (y_{ji}x_{ii})_{ji} + (y_{ji}x_{ii})_{jj} \in A_{ji} + A_{jj}\ ,$$

$$(11)\qquad x_{ii}y_{ji} = (y_{ji}x_{ii})_{jj} \in A_{jj}\ ,$$

$$(12)\qquad x_{ij}y_{ij} = y_{ij}x_{ij} \in A_{ii} + A_{jj}\ ,$$

$$(13)\qquad x_{ij}y_{ji} \in A_{ii} + A_{jj}\ ,$$

$$(14)\qquad x_{ii}y_{ii} \in A_{ii} + A_{jj}\ ,$$

(15) $$[A_{ij}^2, A_{ii} + A_{ji} + A_{jj}] = 0 ,$$

(16) $$[A_{ij}A_{ji}, A_{jj}] = 0 ,$$

(17) $$[A_{ji}A_{jj}, A_{ij}] = 0 ,$$

(18) $$[A_{jj}A_{ij}, A_{ji}] = 0 ,$$

(19) $$a_{ii}\cdot x_{ij}y_{ij} = x_{ij}y_{ij}\cdot a_{ii} = (1/2)((a_{ii}x_{ij}\cdot y_{ij})_{ii} + (a_{ii}y_{ij}\cdot x_{ij})_{ii}) ,$$

(20) $$a_{ii}\cdot x_{ji}y_{ji} = x_{ij}y_{ij}\cdot a_{ii} = (1/2)((x_{ji}a_{ii}\cdot y_{ji})_{ii} + (y_{ji}a_{ii}\cdot x_{ji})_{ii}) ,$$

(21) $$(a_{ii}(x_{ij}y_{ji} + y_{ji}x_{ij}))_{ii} = (a_{ii}x_{ij}\cdot y_{ji})_{ii} + (y_{ji}a_{ii}\cdot x_{ij})_{ii} ,$$

and

(22) $$((x_{ij}y_{ji} + y_{ji}x_{ij})a_{ii})_{ii} = (y_{ji}\cdot a_{ii}x_{ij})_{ii} + (x_{ij}\cdot y_{ji}a_{ii})_{ii} .$$

Proof of the lemma. Identities (5), (6) and (7) are derived in [1] using only the fact that the functions H, P and Q vanish identically. All of the identities are obtained by relatively straightforward substitution of elements into H, P and Q. Due to the excessive length of many of the computations involved we leave the proofs to the reader.

Since our theorem hypothesizes that $A_{00} = 0$ the multiplicative properties stated in (8) through (14) of the preceding lemma can be more compactly exhibited in our case by the module multiplication table:

(23)

	A_{11}	A_{10}	A_{01}
A_{11}	A_{11}	A_{10}	0
A_{10}	0	A_{11}	A_{11}
A_{01}	A_{01}	A_{11}	A_{11} .

We will henceforth make free use of the properties shown in this table. Note also that (12) can be written

(24) $$[A_{10}, A_{10}] = [A_{01}, A_{01}] = 0 ,$$

and (15) can be written

(25) $$[A_{10}^2, A_{11} + A_{01}] = [A_{01}^2, A_{11} + A_{10}] = 0 .$$

From (16) we have

(26) $$[A_{01}A_{10}, A_{11}] = 0 ,$$

and (19) through (22) specialize to

(27) $$a_{11}\cdot x_{10}y_{10} = x_{10}y_{10}\cdot a_{11} = (1/2)(a_{11}x_{10}\cdot y_{10} + a_{11}y_{10}\cdot x_{10}) ,$$

(28) $$a_{11}\cdot x_{01}y_{01} = x_{01}y_{01}\cdot a_{11} = (1/2)(x_{01}a_{11}\cdot y_{01} + y_{01}a_{11}\cdot x_{01}) ,$$

$$(29) \qquad a_{11}(x_{10}y_{01} + y_{01}x_{10}) = a_{11}x_{10}\cdot y_{01} + y_{01}a_{11}\cdot x_{10}$$

and

$$(30) \qquad (x_{10}y_{01} + y_{01}x_{10})a_{11} = y_{01}\cdot a_{11}x_{10} + x_{10}\cdot y_{01}a_{11} \ ,$$

respectively.

We assume throughout that e is not a unity element for A. We will show

$$(31) \qquad (A, A_{11}, A_{11}) = (A_{11}, A, A_{11}) = (A_{11}, A_{11}, A) = 0 \ .$$

The submodule A_{11} is a subring of A, and for $i \neq j$, two of the associators in $H(x_{11}, y_{11}, a_{ij}) = (x_{11}, y_{11}, a_{ij}) + (y_{11}, a_{ij}, x_{11}) + (a_{ij}, x_{11}, y_{11}) = 0$, are equal to zero, hence all three are equal to zero. Thus it suffices to show that A_{11} is associative.

We assert that the submodule $I = (A_{11}, A_{11}, A_{11}) + (A_{11}, A_{11}, A_{11})A_{11}$ is an ideal of A. We will use the fact that the function $T(a, x, y, b) = (ax, y, b) - (a, xy, b) + (a, x, yb) - a(x, y, b) - (a, x, y)b$ is identically zero in any nonassociative ring. Thus $0 = T(a_{mn}, x_{11}, y_{11}, b_{ij})$, with $m + n = i + j = 2$ implies that $A_{11}(A_{11}, A_{11}, A_{11}) \subseteqq I$, and with $m + n = 2$, $i + j = 1$, implies that $(A_{11}, A_{11}, A_{11}) A_{ij} = 0$, using the fact that $(A_{11}, A_{11}, A_{ij}) = 0$. If $m + n = 1$ and $i + j = 2$ then we get $A_{mn}(A_{11}, A_{11}, A_{11}) = 0$. Thus $(A_{11}, A_{11}, A_{11})A + A(A_{11}, A_{11}, A_{11})$ is in I. Furthermore,

$$(A_{11}, A_{11}, A_{11})A_{11}\cdot A \subseteqq ((A_{11}, A_{11}, A_{11}), A_{11}, A) + (A_{11}, A_{11}, A_{11})A \subseteqq I \ ,$$

so $IA \subseteqq I$. Finally,

$$\begin{aligned} A\cdot(A_{11}, A_{11}, A_{11})A_{11} &\subseteqq (A, (A_{11}, A_{11}, A_{11}), A_{11}) \\ &\quad + A(A_{11}, A_{11}, A_{11})\cdot A_{11} \subseteqq I + IA_{11} \subseteqq I \ , \end{aligned}$$

and it follows that $AI \subseteqq I$. Hence I is an ideal of A. If $A = I$ then e is a unity element for A, which contradicts our assumption. Therefore $I = 0$ and in particular $(A_{11}, A_{11}, A_{11}) = 0$, which proves (31).

We assert next that $A_{10}^2 = A_{01}^2 = 0$. First we prove that $J = A_{10}^2 + A_{10}^2A_{10}$ is an ideal of A. We have

$$A_{10}A_{10}^2 \subseteqq A_{10}A_{11} = 0 \ , \qquad A_{01}A_{10}^2 = A_{10}^2A_{01} \subseteqq A_{11}A_{01} = 0$$

by using (25), and $A_{11}A_{10}^2 = A_{10}^2A_{11} \subseteqq A_{10}^2$ by (27). Thus $A_{10}^2A + AA_{10}^2 \subseteqq J$. Moreover, $(A_{10}^2A_{10})A_{11} \subseteqq A_{10}A_{11} = 0$, and, by using (31), $A_{11}(A_{10}^2A_{10}) = (A_{11}A_{10}^2)A_{10} \subseteqq A_{10}^2A_{10} \subseteqq J$. Letting $a_{11} = u_{10}v_{10}$ in (29) we have

$$u_{10}v_{10}(x_{10}y_{01} + y_{01}x_{10}) = (u_{10}v_{10}\cdot x_{10})y_{01} + (y_{01}\cdot u_{10}v_{10})x_{10} \ .$$

But $y_{01}\cdot u_{10}v_{10} = u_{10}v_{10}\cdot y_{01} = 0$ by using (25), so

$$(u_{10}v_{10}\cdot x_{10})y_{01} = u_{10}v_{10}(x_{10}y_{01} + y_{01}x_{10}) \in A_{10}^2A_{11} \subseteqq J$$

and therefore $A_{10}^2A_{10}\cdot A_{01} \subseteqq J$. Finally, $H(u_{10}v_{10}, x_{10}, y_{01}) = 0$ implies

$$y_{01}(u_{10}v_{10}\cdot x_{10}) = (u_{10}v_{10}\cdot x_{10})y_{01} \in A_{10}^2A_{10}\cdot A_{01} \subseteqq J\ .$$

Thus J is an ideal of A.

Since A is simple either $J=0$ or $J=A$. If $J=A$ then $A_{10}^2=A_{11}$, $A_{11}A_{10} = A_{10}$ and $A_{01} = 0$. Thus we may write

$$e = \sum_{i=1}^{t} x_{10}^{(i)}y_{10}^{(i)}\ .$$

But

$$\begin{aligned} 0 = H(x_{10}, x_{10}, y_{10}) &= [x_{10}^2, y_{10}] + [x_{10}y_{10}, x_{10}] + [y_{10}x_{10}, x_{10}] \\ &= x_{10}^2y_{10} + 2x_{10}y_{10}\cdot x_{10} \end{aligned}$$

and

$$\begin{aligned} 0 &= (1/4)P(x_{10}, x_{10}, y_{10}, y_{10}) \\ &= (x_{10}^2, y_{10}, y_{10}) + 2(x_{10}y_{10}, x_{10}, y_{10}) + 2(x_{10}y_{10}, y_{10}, x_{10}) + (y_{10}^2, x_{10}, x_{10}) \\ &= x_{10}^2y_{10}\cdot y_{10} - x_{10}^2y_{10}^2 + 2(x_{10}y_{10}\cdot x_{10})y_{10} - 2x_{10}y_{10}\cdot x_{10}y_{10} + 2(x_{10}y_{10}\cdot y_{10})x_{10} \\ &\quad - 2x_{10}y_{10}\cdot y_{10}x_{10} + y_{10}^2x_{10}\cdot x_{10} - y_{10}^2x_{10}^2\ , \end{aligned}$$

so using the fact that A_{10}^2 is in the center of the associative subring A_{11}, we obtain $x_{10}^2y_{10}^2 = -2(x_{10}y_{10})^2$. It follows that

$$(x_{10}y_{10})^4 = (1/4)(x_{10}^2y_{10}^2)^2 = (1/4)x_{10}^4y_{10}^4 = 0$$

since $x_{10}^3 = x_{10}\cdot x_{10}^2 = 0$. But then we have

$$e = e^{3t+1} = \left(\sum_{i=1}^{t} x_{10}^{(i)}y_{10}^{(i)}\right)^{3t+1} = 0\ ,$$

since every term in the multinomial expansion must contain, for some j, a factor $(x_{10}^{(i)}y_{10}^{(i)})^4 = 0$. From this contradiction we conclude that $J=0$, hence $A_{10}^2 = 0$. Then also $A_{01}^2 = (A_{01}^\#)^2 = 0$, where A is a ring which is anti-isomorphic to A.

We may now replace (23) with the table,

(32)

	A_{11}	A_{10}	A_{01}
A_{11}	A_{11}	A_{10}	0
A_{10}	0	0	A_{11}
A_{01}	A_{01}	A_{11}	0

.

We will continue to make free use of these multiplicative properties in the sequel. Of special interest are the identities,

$$y_{01}\cdot z_{01}x_{10} = -z_{01}\cdot x_{10}y_{01} \tag{33}$$

and

$$(34) \qquad y_{10}z_{01}\cdot x_{10} = -z_{01}x_{10}\cdot y_{10} ,$$

obtained by using the function H and (32).

We show next that the subring A_{11} is itself a simple ring.

Let B_{11} be any nonzero ideal of A_{11} and consider the submodule,

$$\begin{aligned} L = B_{11} + B_{11}A_{10} + A_{01}B_{11} + A_{01}\cdot B_{11}A_{10} \\ + A_{01}B_{11}\cdot A_{10} + (A_{01}\cdot B_{11}A_{10})A_{11} + A_{11}(A_{01}B_{11}\cdot A_{10}) .\end{aligned}$$

We will show that L is an ideal of A.

Evidently, $AB_{11} + B_{11}A \subseteqq L$. Also $B_{11}A_{10}\cdot A_{11} \subseteqq A_{10}A_{11} = 0$; and by (31), $A_{11}\cdot B_{11}A_{10} = A_{11}B_{11}\cdot A_{10} \subseteqq B_{11}A_{10} \subseteqq L$. By (24),

$$A_{10}\cdot B_{11}A_{10} = B_{11}A_{10}\cdot A_{10} \subseteqq A_{10}^2 = 0 .$$

Noting that $B_{11}A_{10}\cdot A_{01} \subseteqq A_{01}B_{11}\cdot A_{10} + B_{11} \subseteqq L$ by (29), and $A_{01}\cdot B_{11}A_{10} \subseteqq L$ by the definition of L, we see that $A\cdot B_{11}A_{10} + B_{11}A_{10}\cdot A \subseteqq L$. Moreover, $A\cdot A_{01}B_{11} + A_{01}B_{11}\cdot A \subseteqq L$ from the left-right symmetry of our identities. Similarly, verification that the fourth and sixth terms in in the definition of L yield elements of L when multiplied on the left or right by an element of A implies the same result for the fifth and seventh terms.

By (26), $[A_{01}\cdot B_{11}A_{10}, A_{11}] \subseteqq [A_{01}A_{10}, A_{11}] = 0$. Since $(A_{01}\cdot B_{11}A_{10})A_{11} \subseteqq L$ by definition of L, it follows that $A_{11}(A_{01}\cdot B_{11}A_{10}) \subseteqq L$ also. Clearly, $A_{10}(A_{01}\cdot B_{11}A_{10}) \subseteqq A_{10}A_{11} = 0$, and by (34), $(A_{01}\cdot B_{11}A_{10})A_{10} \subseteqq A_{10}A_{01}\cdot B_{11}A_{10} \subseteqq L$. Also $(A_{01}\cdot B_{11}A_{10})A_{01} \subseteqq A_{11}A_{01} = 0$, and by (30) and (33),

$$\begin{aligned} A_{01}(A_{01}\cdot B_{11}A_{10}) \subseteqq A_{01}(B_{11} + A_{10}\cdot A_{01}B_{11}) \subseteqq L \\ + A_{01}(A_{10}\cdot A_{01}B_{11}) \subseteqq L + A_{01}B_{11}\cdot A_{01}A_{10} \subseteqq L .\end{aligned}$$

Thus $A(A_{01}\cdot B_{11}A_{10}) + (A_{01}\cdot B_{11}A_{10})A \subseteqq L$.

Since $[A_{01}\cdot B_{11}A_{10}, A_{11}] \subseteqq [A_{01}A_{10}, A_{11}] = 0$ it suffices to show that $(A_{01}\cdot B_{11}A_{10})A_{11}\cdot A$ and $A\cdot A_{11}(A_{01}\cdot B_{11}A_{10})$ are in L. By (31),

$$(A_{01}\cdot B_{11}A_{10})A_{11}\cdot A = (A_{01}\cdot B_{11}A_{10})\cdot A_{11}A \subseteqq (A_{01}\cdot B_{11}A_{10})A \subseteqq L$$

and $A\cdot A_{11}(A_{01}\cdot B_{11}A_{10}) = AA_{11}\cdot(A_{01}\cdot B_{11}A_{10}) \subseteqq A(A_{01}\cdot B_{11}A_{10}) \subseteqq L$. This completes the verification that L is an ideal of A.

Since A is simple and $0 \neq B_{11} \subseteqq L$ we must have $L = A$, hence $B_{11}A_{10} = A_{10}$ and $A_{01}B_{11} = A_{01}$.

If $b_{11} \in B_{11}$ then $b_{11}(a_{11}x_{10})\cdot y_{01} + y_{01}b_{11}\cdot a_{11}x_{10} \in B_{11}$ and

$$(b_{11}a_{11})x_{10}\cdot y_{01} + y_{01}(b_{11}a_{11})\cdot x_{10} \in B_{11}$$

by (29). Taking the difference of these two elements and using (31) gives $(y_{01}b_{11})a_{11}\cdot x_{10} - y_{01}b_{11}\cdot a_{11}x_{10} \in B_{11}$. Since $A_{01}B_{11} = A_{01}$ it follows that $(A_{01}, A_{11}, A_{10}) \subseteqq B_{11}$.

If the intersection of all proper ideals of A_{11} is the zero ideal, then $(A_{01}, A_{11}, A_{10}) = 0$. Hence, by (31) and (33),

$$z_{01}(a_{11} \cdot x_{10}y_{01}) = z_{01}a_{11} \cdot x_{10}y_{01} = -y_{01}(z_{01}a_{11} \cdot x_{10})$$
$$= -y_{01}(z_{01} \cdot a_{11}x_{10}) = z_{01}(a_{11}x_{10} \cdot y_{01}) ;$$

i.e., $z_{01}(a_{11}, x_{10}, y_{01}) = 0$.

Since the set N_{11} of elements of A_{11} which annihilate A_{01} is an ideal of A_{11}, and since $0 = A_{01}N_{11} \neq A_{01}$, it follows that $N_{11} = 0$, hence $(A_{11}, A_{10}, A_{01}) = 0$. Thus $A_{10}A_{01} = (B_{11}A_{10})A_{01} = B_{11}(A_{10}A_{01}) \subseteqq B_{11}$ and, by using (29), $A_{01}A_{10} = (A_{01}B_{11})A_{10} \subseteqq B_{11}A_{10} \cdot A_{01} + B_{11} \subseteqq B_{11}$. This implies that the ideal L is given by $L = B_{11} + B_{11}A_{10} + A_{01}B_{11}$. Since A is simple, $B_{11} = A_{11}$ and A_{11} is simple.

The other possibility is that A_{11} contains a unique minimal ideal, M_{11}. If $(A_{01}, A_{11}, A_{10}) = 0$ we may proceed as above. Thus assume that there exists a nonzero element b_{11} of the form (y_{01}, a_{11}, x_{10}). Since $(A_{01}, A_{11}, A_{10}) \subseteqq B_{11}$ for every nonzero ideal B_{11} of A_{11}, we see that $b_{11} \in M_{11}$. Moreover b_{11} is in the center of A_{11} by (26). Since M_{11} is minimal, $M_{11} = b_{11}A_{11}$. If $b_{11}c_{11} = 0$ then, since $A_{01}M_{11} = A_{01}$, $A_{01}c_{11} = A_{01}A_{11}b_{11}c_{11} = 0$. Thus $c_{11} \in N_{11} = 0$; i.e., no nonzero element of A_{11} annihilates b_{11}. Hence $b_{11}^2 \neq 0$ and $M_{11} = b_{11}^2A_{11}$. Then there exists $b_{11} \in A_{11}$ such that $b_{11} = b_{11}^2d_{11}$, or $b_{11}(e - b_{11}d_{11}) = 0$. It follows that $e = b_{11}d_{11} \in M_{11}$ hence $M_{11} = A_{11}$ is simple in this case also.

By (31), (33), and (26), $z_{01}(x_{10}y_{01} \cdot a_{11}) = (z_{01} \cdot x_{10}y_{01})a_{11} = -(y_{01} \cdot z_{01}x_{10})a_{11} = -y_{01}(z_{01}x_{10} \cdot a_{11}) = -y_{01}(a_{11} \cdot z_{01}x_{10}) = -y_{01}a_{11} \cdot z_{01}x_{10} = z_{01}(x_{10} \cdot y_{01}a_{11})$; i.e., $z_{01}(x_{10}, y_{01}, a_{11}) = 0$, or $(A_{10}, A_{01}, A_{11}) \subseteqq N_{11} = 0$. Then (30) reduces to $y_{01}x_{10} \cdot a_{11} = y_{01} \cdot a_{11}x_{10}$, which, in view of (26), implies that $A_{01}A_{10}$ is an ideal of A_{11}. If $A_{01}A_{10} = 0$ then (34) implies that $A_{10}A_{01}$ annihilates A_{10}, hence $A_{10}A_{01} = 0$. But then we easily see from (32) that both A_{10} and A_{01} are ideals of A, hence $A_{10} = A_{01} = 0$, which implies that e is an unity element for $A = A_{11}$. From this contradiction we conclude that $A_{01}A_{10} = A_{11}$, hence by (26), A_{11} is commutative and therefore a field.

Let $A_{11} = \Phi e$. To prove that A_{01} is one-dimensional over Φ, choose $0 \neq z_{01} \in A_{01}$ such that $z_{01}A_{10} = A_{11} = \Phi e$. Suppose $z_{01}x_{10} = e$. Then for every $y_{01} \in A_{01}$ we have, by (33), $y_{01} = -z_{01} \cdot x_{10}y_{01} = \alpha z_{01}$ for $\alpha \in \Phi$. Also $A_{10} = A_{01}^{\#}$ is one-dimensional over Φ.

We now have $A_{11} = \Phi e$, $A_{10} = \Phi x$ and $A_{01} = \Phi y$. Since (34) gives $(xy + yx)x = 0$ and $xy + yx \in \Phi e$, we must have $xy + yx = 0$. Without loss of generality we may take $xy = -yx = e$, which completes the proof of the theorem.

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