

Pacific Journal of Mathematics

ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT

EARL F. ECKLUND JR.

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E. F. ECKLUND, JR.

A classical theorem discovered independently by J. Sylvester and I. Schur states that in a set of k consecutive integers, each of which is greater than k , there is a number having a prime divisor greater than k . In giving an elementary proof, P. Erdős expressed the theorem in the following form:

If $n \geq 2k$, then $\binom{n}{k}$ has a prime divisor $p > k$.

Recently, P. Erdős suggested a problem of a complementary nature:

If $n \geq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leq \frac{n}{2}$

The problem is solved by the following

THEOREM. If $n \geq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leq \max \left\{ \frac{n}{k}, \frac{n}{2} \right\}$, with the exception $\binom{7}{3}$.

Throughout the paper, p denotes a prime. J. Rosser and L. Schoenfeld [2] have obtained fairly precise estimates for $\theta(x) = \sum_{p \leq x} \log(p)$, and $\pi(x) = \sum_{p \leq x} 1$.

$$(1) \quad \frac{x}{\log x} \left(1 + \frac{1}{2 \log x} \right) < \pi(x) \quad \text{for } x \geq 59.$$

$$(2) \quad \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2 \log x} \right) \quad \text{for } x > 1.$$

$$(3) \quad \pi(x) < \frac{1.25506x}{\log x} \quad \text{for } x > 1.$$

$$(4) \quad \theta(x) < 1.01624x \quad \text{for } x > 0.$$

$$(5) \quad x - 2.05282\sqrt{x} < \theta(x) < x \quad \text{for } 0 < x \leq 10^8.$$

Using these results, we are able to prove the theorem.

First we establish the following lemmas.

LEMMA 1. If $\binom{n}{k}$ has no prime divisors $p \leq n/2$, then

$$(6) \quad \binom{n}{k} \leq e^{\theta(n) - \theta(n-k)} \leq n^{\pi(n) - \pi(n-k)}.$$

LEMMA 2. For $k \geq 59$,

$$(7) \quad n^{\pi(n)-\pi(n-k)} < e^{(n/\log n + k + k/2\log n)} .$$

LEMMA 3.

$$(8) \quad \frac{2^{(n+1)k-1}}{\sqrt{k}} \leq \binom{2^n k}{k} .$$

Proof of Lemma 1. $\binom{n}{k} \leq \prod_{n-k < p \leq n} p \leq \prod_{n-k < p \leq n} n$. Hence

$$\binom{n}{k} \leq e^{\theta(n)-\theta(n-k)} \leq n^{\pi(n)-\pi(n-k)} .$$

Proof of Lemma 2. From (1) and (2), we have

$$\begin{aligned} n^{\pi(n)-\pi(n-k)} &< n^{\lfloor n/\log n \rfloor [1+3/(2\log n)] - (n-k)/\log(n-k) \lfloor 1+1/(2\log(n-k)) \rfloor} \\ &< n^{\lfloor n/\log n \rfloor [1+3/2\log n] - (n-k)/\log n \lfloor 1+1/2\log n \rfloor} \\ &< e^{\lfloor n[1+3/2\log n] \rfloor - (n-k) \lfloor 1+1/2\log n \rfloor} \\ &< e^{(n/\log n + k + k/2\log n)} . \end{aligned}$$

Lemma 3 is proved by induction on n for all values of k .

The proof of the theorem is by contradiction. Three cases are considered. The general case is a Sylvester-Schur type argument. The other cases involve deducing contradictions from appropriate upper and lower bounds on the inequalities, (6), of Lemma 1.

Proof of the theorem. Assume $\binom{n}{k}$ has no prime divisors

$$p \leq \max \left\{ \frac{n}{k}, \frac{n}{2} \right\} .$$

$$1. \quad k < n^{2/3}. \quad \binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} \geq \left(\frac{n}{k} \right)^k .$$

By sieving all multiples of 2, and 3, we have

$$\pi(n) - \pi(n-k) \leq \frac{k}{2} \quad \text{for } k \geq 4 .$$

Therefore from (6), we have $(n/k)^k \leq n^{k/2}$. Thus the assumption is false if $4 \leq k < n^{1/2}$. By sieving all multiples of 2, 3, and 5, we have

$$\pi(n) - \pi(n-k) \leq \frac{k}{3} \quad \text{for } k \geq 60 .$$

Thus from (6), we have $(n/k)^k \leq n^{k/3}$. Hence the assumption is false if $60 \leq k < n^{2/3}$.

2. $n^{2/3} \leq k \leq n/16$. Let $\tilde{n} = [n/2]$, and $\tilde{k} = [k/2]$; where $[x]$ denotes the integral part of x . If $p > k$ and p divides $\binom{\tilde{n}}{\tilde{k}}$, then p divides $\binom{n}{k}$ and $p \leq n/2$. By assumption, there are no such primes. Therefore, $\binom{\tilde{n}}{\tilde{k}}$ has no prime divisors $p > 2\tilde{k} + 1$. Thus $\binom{\tilde{n}}{\tilde{k}} < \tilde{n}^{\pi(\sqrt{\tilde{n}})} \cdot e^{\theta(2\tilde{k}+1)}$ (see paper of M. Faulkner [1]). From (3), (4), and (8), we have

$$\frac{2^{5\tilde{k}-1}}{\sqrt{\tilde{k}}} < \tilde{n}^{(1.26\sqrt{\tilde{n}}/\log\sqrt{\tilde{n}})} \cdot e^{1.02(2\tilde{k}+1)}.$$

Taking logarithms, we obtain

$$3.45\tilde{k} - 0.70 - \frac{1}{2} \log(\tilde{k}) < 2.52\sqrt{\tilde{n}} + 1.02(2\tilde{k} + 1),$$

which is a contradiction for $\tilde{k} > 32$. Therefore the assumption is false if $n^{2/3} \leq k \leq n/16$ when $k \geq 65$.

3. $n/16 < k \leq n/2$. Consider $n/16 < k \leq n/8$. By (6), (7) and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(n/\log n + k + k/2 \log n)}.$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2} \log(k) < \frac{n}{\log n} + k + \frac{k}{2 \log n};$$

which is false for $k \geq 1901$. By (5), (6), and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(k+2.06\sqrt{15k})}.$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2} \log(k) < k + 2.6\sqrt{15k};$$

which is false for $k \geq 25$. Thus the assumption is false if $n/16 < k \leq n/8$ when $k \geq 25$. By similar arguments, we show the assumption is false if $n/8 < k \leq n/4$ when $k \geq 32$; and if $n/4 < k \leq n/2$ when $k > 105$.

We have proved the theorem for $k \geq 4$ with the exception of a finite number of cases. The cases $k = 1, 2$, and 3 , are easily resolved; and the remaining cases have been checked with the aid of an IBM 1620 computer in the following manner:

The values which were checked are $4 \leq k \leq 60$ with $2k \leq n \leq k^2$, and $61 \leq k \leq 105$ with $2k \leq n \leq 4k$.

For the i -th prime, p_i , the exponent to which p_i occurred in the "numerator", $n(n-1) \cdots (n-k+1)$, and in the "denominator", $k!$,

of $\binom{n}{k}$, α_i and β_i respectively, were determined; and the values of p_i , n , and k , were reported if the difference, $\alpha_i - \beta_i$, was positive. Cross-checking was done manually. The first ten primes proved sufficient to verify the theorem in these cases.

This concludes the proof of the theorem.

In closing, I would like to thank Professor M. Faulkner for her gracious assistance.

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