ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT

EARL F. ECKLUND JR.
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E. F. ECKLUND, JR.

A classical theorem discovered independently by J. Sylvester and I. Schur states that in a set of \( k \) consecutive integers, each of which is greater than \( k \), there is a number having a prime divisor greater than \( k \). In giving an elementary proof, P. Erdős expressed the theorem in the following form:

If \( n \geq 2k \), then \( \binom{n}{k} \) has a prime divisor \( p > k \).

Recently, P. Erdős suggested a problem of a complementary nature:

If \( n \geq 2k \), then \( \binom{n}{k} \) has a prime divisor \( p \leq \frac{n}{2} \).

The problem is solved by the following theorem.

**Theorem.** If \( n \geq 2k \), then \( \binom{n}{k} \) has a prime divisor

\[ p \leq \max \left\{ \frac{n}{k}, \frac{n}{2} \right\}, \text{ with the exception } \binom{7}{3}. \]

Throughout the paper, \( p \) denotes a prime. J. Rosser and L. Schoenfeld [2] have obtained fairly precise estimates for \( \theta(x) = \sum_{p \leq x} \log(p) \) and \( \pi(x) = \sum_{p \leq x} 1 \).

1. \[ \frac{x}{\log x} \left( 1 + \frac{1}{2 \log x} \right) < \pi(x) \quad \text{for } x \geq 59. \]
2. \[ \pi(x) < \frac{x}{\log x} \left( 1 + \frac{3}{2 \log x} \right) \quad \text{for } x > 1. \]
3. \[ \pi(x) < \frac{1.25506x}{\log x} \quad \text{for } x > 1. \]
4. \[ \theta(x) < 1.01624x \quad \text{for } x > 0. \]
5. \[ x - 2.05282 \sqrt{x} < \theta(x) < x \quad \text{for } 0 < x \leq 10^8. \]

Using these results, we are able to prove the theorem.

First we establish the following lemmas.

**Lemma 1.** If \( \binom{n}{k} \) has no prime divisors \( p \leq n/2 \), then

\[ \binom{n}{k} \leq e^{\theta(n) - \theta(n-k)} \leq n^{\pi(n) - \pi(n-k)}. \]

**Lemma 2.** For \( k \geq 59 \),
LEMMA 3.

\[ \binom{n}{k} \leq \prod_{n-k < p \leq n} p \leq \prod_{n-k < p \leq n} n. \]
Hence

\[ \binom{n}{k} \leq \binom{n}{n-k} \leq n^{\pi(n) - \pi(n-k)}. \]

Proof of Lemma 1. \( (\binom{n}{k}) \leq \prod_{n-k < p \leq n} p \leq \prod_{n-k < p \leq n} n. \) Hence

\[ \binom{n}{k} \leq e^{\theta(n) - \theta(n-k)} \leq n^{\pi(n) - \pi(n-k)}. \]

Proof of Lemma 2. From (1) and (2), we have

\[ n^{\pi(n) - \pi(n-k)} < n^{\log n [1 + 1/(2\log n)]} - (n-k) / \log(n-k) \leq n^{\log n [1 + 1/(2\log n)]} - \frac{n-k}{\log(n-k)} \]
\[ < e^{\log n [1 + 1/(2\log n)]} - (n-k) / \log(n-k) \]
\[ < e^{\log n + k + 2/(2\log n)}. \]

Lemma 3 is proved by induction on \( n \) for all values of \( k \).

The proof of the theorem is by contradiction. Three cases are considered. The general case is a Sylvester-Schur type argument. The other cases involve deducing contradictions from appropriate upper and lower bounds on the inequalities, (6), of Lemma 1.

Proof of the theorem. Assume \( \binom{n}{k} \) has no prime divisors

\[ p \leq \max \left\{ \frac{n}{k}, \frac{n}{2} \right\}. \]

1. \( k < n^{2/3} \).

\[ \binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} \geq \left( \frac{n}{k} \right)^k. \]

By sieving all multiples of 2, and 3, we have

\[ \pi(n) - \pi(n-k) \leq \frac{k}{2} \quad \text{for } k \geq 4. \]

Therefore from (6), we have \( (n/k)^k \leq n^{k/2} \). Thus the assumption is false if \( 4 \leq k < n^{1/2} \). By sieving all multiples of 2, 3, and 5, we have

\[ \pi(n) - \pi(n-k) \leq \frac{k}{3} \quad \text{for } k \geq 60. \]

Thus from (6), we have \( (n/k)^k \leq n^{k/13} \). Hence the assumption is false if \( 60 \leq k < n^{2/3} \).
2. \( n^{3/3} \leq k \leq n/16 \). Let \( \tilde{n} = [n/2] \), and \( \tilde{k} = [k/2] \); where \([x]\) denotes the integral part of \( x \). If \( p > k \) and \( p \) divides \( \left( \frac{\tilde{n}}{\tilde{k}} \right) \), then \( p \) divides \( \left( \frac{n}{k} \right) \) and \( p \leq n/2 \). By assumption, there are no such primes. Therefore, \( \left( \frac{\tilde{n}}{\tilde{k}} \right) \) has no prime divisors \( p > 2\tilde{k} + 1 \). Thus \( \left( \frac{\tilde{n}}{\tilde{k}} \right) < \tilde{n}^{\log \tilde{k}} \cdot e^{\log^{2} \tilde{k} + 1} \) (see paper of M. Faulkner [1]). From (3), (4), and (8), we have

\[
\frac{2^{2\tilde{k} - 1}}{\sqrt{\tilde{k}}} < \tilde{n}^{\log \tilde{n} / \log \tilde{k}} \cdot e^{\log^{2} \tilde{k} + 1}.
\]

Taking logarithms, we obtain

\[
3.45\tilde{k} - 0.70 - \frac{1}{2} \log (\tilde{k}) < 2.52\sqrt{\tilde{n}} + 1.02(2\tilde{k} + 1),
\]

which is a contradiction for \( \tilde{k} > 32 \). Therefore the assumption is false if \( n^{3/3} \leq k \leq n/16 \) when \( k \geq 65 \).

3. \( n/16 < k \leq n/2 \). Consider \( n/16 < k \leq n/8 \). By (6), (7) and (8), we have

\[
\frac{2^{4k - 1}}{\sqrt{k}} < e^{n/\log n + k \log k}.
\]

Taking logarithms, we obtain

\[
2.76k - 0.70 - \frac{1}{2} \log (k) < \frac{n}{\log n} + \frac{k}{2 \log n};
\]

which is false for \( k \geq 1901 \). By (5), (6), and (8), we have

\[
\frac{2^{4k - 1}}{\sqrt{k}} < e^{k + 2 \log \sqrt{k}}.
\]

Taking logarithms, we obtain

\[
2.76k - 0.70 - \frac{1}{2} \log (k) < k + 2.6\sqrt{15k};
\]

which is false for \( k \geq 25 \). Thus the assumption is false if \( n/16 < k \leq n/8 \) when \( k \geq 25 \). By similar arguments, we show the assumption is false is \( n/8 < k \leq n/4 \) when \( k \geq 32 \); and if \( n/4 < k \leq n/2 \) when \( k > 105 \).

We have proved the theorem for \( k \geq 4 \) with the exception of a finite number of cases. The cases \( k = 1, 2, \) and \( 3, \) are easily resolved; and the remaining cases have been checked with the aid of an IBM 1620 computer in the following manner:

The values which were checked are \( 4 \leq k \leq 60 \) with \( 2k \leq n \leq k^{2} \), and \( 61 \leq k \leq 105 \) with \( 2k \leq n \leq 4k \).

For the \( i \)-th prime, \( p_{i} \), the exponent to which \( p_{i} \) occurred in the “numerator”, \( n(n - 1) \cdots (n - k + 1) \), and in the “denominator”, \( k! \),
of \( \binom{n}{k} \), \( \alpha_i \) and \( \beta_i \) respectively, were determined; and the values of \( p_i, n, \) and \( k \), were reported if the difference, \( \alpha_i - \beta_i \), was positive. Cross-checking was done manually. The first ten primes proved sufficient to verify the theorem in these cases.

This concludes the proof of the theorem.

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REFERENCES


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