

# Pacific Journal of Mathematics

**ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT**

EARL F. ECKLUND JR.

# ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT

E. F. ECKLUND, JR.

A classical theorem discovered independently by J. Sylvester and I. Schur states that in a set of  $k$  consecutive integers, each of which is greater than  $k$ , there is a number having a prime divisor greater than  $k$ . In giving an elementary proof, P. Erdős expressed the theorem in the following form:

If  $n \geq 2k$ , then  $\binom{n}{k}$  has a prime divisor  $p > k$ .

Recently, P. Erdős suggested a problem of a complementary nature:

If  $n \geq 2k$ , then  $\binom{n}{k}$  has a prime divisor  $p \leq \frac{n}{2}$

The problem is solved by the following

**THEOREM.** If  $n \geq 2k$ , then  $\binom{n}{k}$  has a prime divisor  $p \leq \max\left\{\frac{n}{k}, \frac{n}{2}\right\}$ , with the exception  $\binom{7}{3}$ .

Throughout the paper,  $p$  denotes a prime. J. Rosser and L. Schoenfeld [2] have obtained fairly precise estimates for  $\theta(x) = \sum_{p \leq x} \log(p)$ , and  $\pi(x) = \sum_{p \leq x} 1$ .

$$(1) \quad \frac{x}{\log x} \left(1 + \frac{1}{2 \log x}\right) < \pi(x) \quad \text{for } x \geq 59.$$

$$(2) \quad \pi(x) < \frac{x}{\log x} \left(1 + \frac{3}{2 \log x}\right) \quad \text{for } x > 1.$$

$$(3) \quad \pi(x) < \frac{1.25506x}{\log x} \quad \text{for } x > 1.$$

$$(4) \quad \theta(x) < 1.01624x \quad \text{for } x > 0.$$

$$(5) \quad x - 2.05282\sqrt{x} < \theta(x) < x \quad \text{for } 0 < x \leq 10^8.$$

Using these results, we are able to prove the theorem.

First we establish the following lemmas.

**LEMMA 1.** If  $\binom{n}{k}$  has no prime divisors  $p \leq n/2$ , then

$$(6) \quad \binom{n}{k} \leq e^{\theta(n) - \theta(n-k)} \leq n^{\pi(n) - \pi(n-k)}.$$

**LEMMA 2.** For  $k \geq 59$ ,

$$(7) \quad n^{\pi(n) - \pi(n-k)} < e^{(n/\log n + k + k/2\log n)} .$$

LEMMA 3.

$$(8) \quad \frac{2^{(n+1)k-1}}{\sqrt{k}} \leq \binom{2n}{k} .$$

*Proof of Lemma 1.*  $\binom{n}{k} \leq \prod_{n-k < p \leq n} p \leq \prod_{n-k < p \leq n} n$ . Hence

$$\binom{n}{k} \leq e^{\theta(n) - \theta(n-k)} \leq n^{\pi(n) - \pi(n-k)} .$$

*Proof of Lemma 2.* From (1) and (2), we have

$$\begin{aligned} n^{\pi(n) - \pi(n-k)} &< n^{\{n/\log n [1+3/(2\log n)] - (n-k)/\log(n-k) [1+1/(2\log(n-k))]\}} \\ &< n^{\{n/\log n [1+3/2\log n] - (n-k)/\log n [1+1/2\log n]\}} \\ &< e^{\{n[1+3/2\log n] - (n-k)[1+1/2\log n]\}} \\ &< e^{(n/\log n + k + k/2\log n)} . \end{aligned}$$

Lemma 3 is proved by induction on  $n$  for all values of  $k$ .

The proof of the theorem is by contradiction. Three cases are considered. The general case is a Sylvester-Schur type argument. The other cases involve deducing contradictions from appropriate upper and lower bounds on the inequalities, (6), of Lemma 1.

*Proof of the theorem.* Assume  $\binom{n}{k}$  has no prime divisors

$$p \leq \max \left\{ \frac{n}{k}, \frac{n}{2} \right\} .$$

$$1. \quad k < n^{2/3}. \quad \binom{n}{k} = \frac{n \cdot (n-1) \cdots (n-k+1)}{k \cdot (k-1) \cdots 1} \geq \left(\frac{n}{k}\right)^k .$$

By sieving all multiples of 2, and 3, we have

$$\pi(n) - \pi(n-k) \leq \frac{k}{2} \quad \text{for } k \geq 4 .$$

Therefore from (6), we have  $(n/k)^k \leq n^{k/2}$ . Thus the assumption is false if  $4 \leq k < n^{1/2}$ . By sieving all multiples of 2, 3, and 5, we have

$$\pi(n) - \pi(n-k) \leq \frac{k}{3} \quad \text{for } k \geq 60 .$$

Thus from (6), we have  $(n/k)^k \leq n^{k/3}$ . Hence the assumption is false if  $60 \leq k < n^{2/3}$ .

2.  $n^{2/3} \leq k \leq n/16$ . Let  $\tilde{n} = [n/2]$ , and  $\tilde{k} = [k/2]$ ; where  $[x]$  denotes the integral part of  $x$ . If  $p > k$  and  $p$  divides  $\binom{\tilde{n}}{\tilde{k}}$ , then  $p$  divides  $\binom{n}{k}$  and  $p \leq n/2$ . By assumption, there are no such primes. Therefore,  $\binom{\tilde{n}}{\tilde{k}}$  has no prime divisors  $p > 2\tilde{k} + 1$ . Thus  $\binom{\tilde{n}}{\tilde{k}} < \tilde{n}^{\pi(\sqrt{\tilde{n}})} \cdot e^{\theta(2\tilde{k}+1)}$  (see paper of M. Faulkner [1]). From (3), (4), and (8), we have

$$\frac{2^{5\tilde{k}-1}}{\sqrt{\tilde{k}}} < \tilde{n}^{(1.26\sqrt{\tilde{n}}/\log\sqrt{\tilde{n}})} \cdot e^{1.02(2\tilde{k}+1)}.$$

Taking logarithms, we obtain

$$3.45\tilde{k} - 0.70 - \frac{1}{2} \log(\tilde{k}) < 2.52\sqrt{\tilde{n}} + 1.02(2\tilde{k} + 1),$$

which is a contradiction for  $\tilde{k} > 32$ . Therefore the assumption is false if  $n^{2/3} \leq k \leq n/16$  when  $k \geq 65$ .

3.  $n/16 < k \leq n/2$ . Consider  $n/16 < k \leq n/8$ . By (6), (7) and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(n/\log n + k + k/2 \log n)}.$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2} \log(k) < \frac{n}{\log n} + k + \frac{k}{2 \log n};$$

which is false for  $k \geq 1901$ . By (5), (6), and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(k+2.06\sqrt{15k})}.$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2} \log(k) < k + 2.6\sqrt{15k};$$

which is false for  $k \geq 25$ . Thus the assumption is false if  $n/16 < k \leq n/8$  when  $k \geq 25$ . By similar arguments, we show the assumption is false is  $n/8 < k \leq n/4$  when  $k \geq 32$ ; and if  $n/4 < k \leq n/2$  when  $k > 105$ .

We have proved the theorem for  $k \geq 4$  with the exception of a finite number of cases. The cases  $k = 1, 2$ , and  $3$ , are easily resolved; and the remaining cases have been checked with the aid of an IBM 1620 computer in the following manner:

The values which were checked are  $4 \leq k \leq 60$  with  $2k \leq n \leq k^2$ , and  $61 \leq k \leq 105$  with  $2k \leq n \leq 4k$ .

For the  $i$ -th prime,  $p_i$ , the exponent to which  $p_i$  occurred in the "numerator",  $n(n-1) \cdots (n-k+1)$ , and in the "denominator",  $k!$ ,

of  $\binom{n}{k}$ ,  $\alpha_i$  and  $\beta_i$  respectively, were determined; and the values of  $p_i$ ,  $n$ , and  $k$ , were reported if the difference,  $\alpha_i - \beta_i$ , was positive. Cross-checking was done manually. The first ten primes proved sufficient to verify the theorem in these cases.

This concludes the proof of the theorem.

In closing, I would like to thank Professor M. Faulkner for her gracious assistance.

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2. J. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. **6** (1962), 64-94.

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