A MODULAR TOPOLOGICAL LATTICE

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The purpose of this paper is to present a construction of a compact connected topological lattice which is modular and not distributive. As a special case there will result the example which is a two dimensional subset of \( \mathbb{R}^2 \), not embeddable in \( \mathbb{R}^2 \).

The existence of such an example is related to structure questions in topological lattices considered by Dyer and Shields [3], Anderson [1], and others.

The first step is to present a general method for constructing a class of modular lattices. Let \( D \) denote a distributive lattice which is a chain, \( S \) a nonempty set, and \( L \) the \( S \)-fold product lattice of \( D \). That is, \( L = \{ f : S \to D \} \) and \( f \leq g \) if and only if \( f(s) \leq g(s) \) for every \( s \in S \). It is known that \( (L, \leq) \) is a distributive lattice with its operations \( \lor \) and \( \land \) characterized by \( [f \lor g](s) = f(s) \lor g(s) \) and \( [f \land g](s) = f(s) \land g(s) \) for every \( s \in S \). Define

\[
M = \{ f \in L \mid \text{there exists } r \in S \text{ such that } s, t \in S - \{ r \} \text{ implies } f(s) \leq f(r) \text{ and } f(s) = f(t) \}.
\]

For intuition about \( M \) and the arguments that follow, note that \( M \) simply consists of all of the constant functions of \( L \) and the functions of \( L \) which are essentially constant in the sense that they assume but two values — the larger value at exactly one point.

If the order of \( L, \leq \), is restricted to \( M \), it will be established through a sequence of lemmas that \( (M, \leq) \) is a modular lattice. Recall a lattice \( (M, \lor, \land) \) is modular if and only if for every \( a, b, c \in M \), \( b \leq a \) implies that \( a \land (b \lor c) = b \lor (a \land c) \).

**Lemma 1.** If \( f \in M \) and \( f \) is not constant, there exists a unique \( r \in S \) such that \( s, t \in S - \{ r \} \) implies \( f(s) < f(r) \) and \( f(s) = f(t) \).

The proof of the lemma is immediate from the definition of \( M \), and consequently for \( f \in M \) and not constant, define index \( f \) to be the unique element described in Lemma 1.

**Lemma 2.** \( (M, \leq) \) is a sub \( \land \)-semilattice of \( (L, \leq) \).

It suffices to show that if \( f, g \in M \), then \( f \land g \in M \). If \( f \) and \( g \) are
constant then \( f \land g \) is constant and therefore in \( M \). If \( f \) is constant and \( g \) is not, let \( b = \text{index } g \). Then, \( s, t \in S - \{b\} \) implies

\[
[f \land g](s) = f(s) \land g(s) \leq f(s) \land g(b) = [f \land g](b)
\]

and likewise \([f \land g](s) = [f \land g](t)\) and thus \( f \land g \in M \). If \( f \) and \( g \) are both not constant, let \( a = \text{index } f \) and \( b = \text{index } g \). If \( a = b \), then \( s, t \in S - \{a\} \) implies \([f \land g](s) \leq [f \land g](a) \) and \([f \land g](s) = [f \land g](t)\). If \( a \neq b \), \([f \land g](a) = f(a) \land g(a)\), \([f \land g](b) = f(b) \land g(b)\), and for

\[
 x \in S - \{a, b\} \quad [f \land g](x) = f(b) \land g(a) = [f \land g](a) \land [f \land g](b)
\]

Then since \( D \) is a chain, \([f \land g](x) = [f \land g](a)\) or \([f \land g](x) = [f \land g](b)\) depending upon which is minimal and therefore \( f \land g \in M \).

**Lemma 3.** If \( a, b, c \) are distinct elements of \( S \) and \( f \in M \), then \( f(a) \land f(b) = f(b) \land f(c) = f(c) \land f(a) \).

The facts of the lemma are an immediate consequence of the definition and is stated as a lemma for convenient reference.

**Definition.** For \( f, g \in M \) define \( f \triangledown g : S \to D \) by the following

(i) if \( f \) is constant or \( g \) is constant, or if \( f \) and \( g \) are both not constant and index \( f = \text{index } g \), then \( f \triangledown g = f \triangledown g \),

(ii) if \( f \) and \( g \) are both not constant and index \( f \neq \text{index } g \), let \( a = \text{index } f \) and \( b = \text{index } g \), then

\[
[f \triangledown g](x) = f(x) \triangledown g(x) \quad \text{for} \quad x \in \{a, b\}
\]

\[
[f \triangledown g](x) = [f(a) \triangledown g(a)] \land [f(b) \triangledown g(b)] \quad \text{for} \quad x \in S - \{a, b\}
\]

**Lemma 4.** If \( f, g \in M \), then

(1) \( f \triangledown g \in M \) and \( f \triangledown g \leq f \triangledown g \), and

(2) \( h \in M, f \leq h, \ g \leq h \) implies \( f \triangledown g \leq h \).

In case (i) of the definition of \( f \triangledown g \), easily \( f \triangledown g \in M \) and the other results are immediate from \( f \triangledown g = f \triangledown g \). In case (ii) let \( a = \text{index } f \) and \( b = \text{index } g \), then since \( D \) is a chain \([f \triangledown g](x) = [f \triangledown g](a)\) or \([f \triangledown g](x) = [f \triangledown g](b)\) for \( x \in S - \{a, b\} \). So in this case also \( f \triangledown g \in M \) and \( f \triangledown g \triangledown g \). Also relative to this case, if \( h \in M, f \leq h, \ g \leq h \), then \( f(x) \triangledown g(x) = [f \triangledown g](x) \) for \( x = a \) or \( x = b \). But from Lemma 3, \( x \in S - \{a, b\} \) implies \( h(a) \land h(b) \leq h(x) \) and thus for

\[
x \in S - \{a, b\} \quad [f \triangledown g](x) = [f \triangledown g](a) \land [f \triangledown](b) \leq h(a) \land h(b) \leq h(x)
\]

Therefore \( f \triangledown g \leq h \).

**Lemma 5.** If \( f, g, h \in M, a, b \in S, a \neq b, [f \triangledown g](x) = h(x) \) for \( x \in \{a, b\} \)
and \([f \lor g](x) \leq h(a) \land h(b) = h(x)\) for \(x \in S - \{a, b\}\), then \(h = f \lor g\).

From the hypothesis \(f \lor g \leq h\) and therefore from Lemma 4 \(f \lor g \leq h\). But \(h(a) = [f \lor g](a) \leq [f \lor g](a)\) and \(h(b) \leq [f \lor g](b)\). Then from Lemma 3, for \(x \neq a\) and \(x \neq b\)

\[h(x) = h(a) \land h(b) \leq [f \lor g](a) \land [f \lor g](b) \leq [f \lor g](x)\]

and \(h \leq f \lor g\).

**Theorem 1.** \((M, \leq)\) is a modular lattice with operations \(\lor\) and \(\land\).

Lemmas 2 and 4 establish that \((M, \leq)\) is a lattice with operations \(\lor\) and \(\land\), it remains to establish that it is modular. Let \(f, g, h \in M\) and \(f \leq g\). It suffices to establish \(g \land (f \lor h) \leq f \lor (g \land h)\) since in any lattice \(f \lor (g \land h) \leq g \land (f \lor h)\). The argument will be a case argument.

If \(f \lor h = f \lor h\), then

\[g \land (f \lor h) = g \land (f \lor h) = f \lor (g \land h) \leq f \lor (g \land h)\]

since \(L\) is itself modular and \(g \land h \in M\) allows Lemma 4 to apply.

If \(h \leq g\), then \(f \lor h \leq g\) and \(g \land (f \lor h) = f \lor h = f \lor (g \land h)\).

If \(f \leq h\), then \(f \leq g \land h\) and \(g \land (f \lor h) = g \land h = f \lor (g \land h)\).

If \(f\) and \(h\) are not constant, \(a = \text{index } f, b = \text{index } g, a \neq b, h \neq g,\) and \(f \leq h\). Then \(f(b) < f(a)\) and \(h(a) < h(b)\). Further, \(f(a) \leq h(a)\) implies \(f \leq h\) and therefore \(h(a) \leq f(a)\). Also \(h(b) \leq g(b)\) and \(h(a) \leq f(a)\) implies \(h \leq g\) and therefore \(h \leq g\) implies \(g(b) < h(b)\). Therefore in this case \(h(a) < f(a) \leq g(a)\) and \(f(b) \leq g(b) < h(b)\). Hence

\[[g \land (f \lor h)](a) = g(a) \land [f(a) \lor h(a)] = f(a)\]

\[= f(a) \lor [g(a) \land h(a)] = [f \lor (g \land h)](a)\]

Likewise

\[[g \land (f \lor h)](b) = g(b) = [f \lor (g \land h)](b)\]

If \(x \in S - \{a, b\}\), then

\[[g \land (f \lor h)](x) = g(x) \land [f(a) \lor h(a)] \land [f(b) \lor h(b)]\]

\[= g(x) \land f(a) \land h(b) = g(x) \land g(a) \land f(a) \land h(b)\]

\[= g(a) \land f(a) \land h(b) = f(a) \land g(b)\]

\[= [g \land (f \lor h)](a) \land [g \land (f \lor h)](b)\]

But \([f \lor (g \land h)](x) \leq [g \land (f \lor h)](x)\) and \(g \land (f \lor h) \in M\), therefore by Lemma 5 \(g \land (f \lor h) = f \lor (g \land h)\).
COROLLARY. If card $S < 3$, $M$ is a distributive lattice. If $3 \leq \text{card } S$ and $2 \leq \text{card } D$, then $M$ is a modular nondistributive lattice.

If card $S < 3$, then $M = L$ and $M$ is distributive. If $3 \leq \text{card } S$ and $2 \leq \text{card } D$, let $s_1, s_2, s_3$ be three distinct elements of $S$ and $c < d$ be two elements of $D$. Define $f_1, f_2, f_3$ by $f_i(x) = d$ if $x = s_i$ and $f(x) = c$ for $x \in S - \{s_i\}$. Also define $g$ and $k$ by $g(s) = d$ for every $s \in S$ and $k(s) = c$ for every $s \in S$. Then $f_1 \wedge f_2 = f_2 \wedge f_3 \wedge f_1 = k$ and $f_1 \lor f_2 = f_2 \lor f_3 = f_3 \lor f_1 = g$ and $\{f_1, f_2, f_3, g, k\}$ is a modular five sublattice of $M$. Therefore $M$ is not distributive [2].

At this stage the algebraic nature of $M$ has been established, in the section that follows the topological nature of $M$ will be studied. It will be assumed in the following that $D$ is topological chain, that is $D$ is a Hausdorff topological space with the operations $\lor$ and $\land$ continuous [3]. If $L$ is considered with the product topology, it is as usual a topological lattice and $M$ may be considered as a topological space in the relative topology that it inherits from $L$. In this context, the following theorem results.

**Theorem 2.** If $D$ is a topological chain, then

1. $M$ is a closed subset of $L$,
2. $M$ is compact if $D$ is compact, and
3. $M$ is connected if $D$ is connected.

Since with card $S \leq 2$, $M = L$, it suffices to consider $3 \leq \text{card } S$ and to establish (1) and (3).

(1) $L - M$ is open for if $f \in M$, then $f$ is not constant and there exist distinct $a, b, c \in S$ such that $f(b) < f(a)$ and $f(b) < f(c)$. Then since $D$ is a chain $f(b) < f(a) \land f(c)$. If there exists $z \in D$ such that $f(b) < z < f(a) \land f(c)$, define $W = \{g \in L \mid z < g(a), z < g(c), \text{ and } (g(b) < z)\}$ and define $W = \{g \in L \mid f(b) < g(a), f(b) < g(c), \text{ and } (g(b) < f(a) \land f(c))\}$ if no such $z$ exists. In either case, $f \in W$, $W$ is open, and $W \cap M = \emptyset$.

(3) If $D$ is connected, consider the map $T: D \rightarrow M$ where for each $d \in D$ $T(d) = k_d$ and $k_d$ is the constant function generated by $d$. Clearly $T$ is continuous and $K$ the set of all constant functions is a connected subset of $M$. If $f \in M - K$, let $a = \text{index } f$, $m = \text{max } f$, and $r = \text{min } f$ define the map $H$ from $[r, m] = \{x \in D \mid r \leq x \leq m\}$ into $M$ by $H(x) = f_x$ where $f_x(a) = x$ and $f_x(s) = r$ for $s \in S - \{a\}$. Again $H$ is continuous and since $[r, m]$ is connected then the range of $H$ is a connected subset of $M$ containing $f$ and intersecting $K$. Therefore $M$ is connected.

*Note.* It is clear that $\land$ will be continuous as an operation on
since it is continuous on $L$. Thus when $D$ is a topological chain $M$ is a closed topological sub-$\wedge$-semilattice of $L$. In order to study the operation $\triangledown$ relative to continuity, it is necessary to restrict $S$ to being finite, in view of the following lemma.

**Lemma 6.** If $D$ is a topological chain and $2 \leq \text{card } D$ and $S$ is infinite, then $\triangledown$ is not continuous.

Let $c < d$ in $D$ and define $k: S \to D$ by $k(s) = c$ for every $s \in S$. Then $k \triangledown k = k$. Let $r \in S$ and define $W_r = \{ f \in M \mid f(r) < d \}$; then $W_r$ is an open subset of $M$ containing $k$. Let $U$ be any open set of $M$ containing $k$, then there exist $s_1, s_2, \ldots, s_n$ distinct elements of $S$ and $U_1, U_2, \ldots, U_n$ open sets of $D$ such that if $W = \{ f \in M \mid f(s_i) \in U_i \}$ for $i = 1, 2, \ldots, n$, $k \in W \subset D$. Now $k \in W$ implies

$$c \in \cap \{ U_i \mid i = 1, 2, \ldots, n \}.$$

Since $S$ is infinite there exist $a, b \in S - \{ s_1, s_2, \ldots, s_n \}$ such that $a \neq b$. Define $h$ and $g$ by $h(a) = d$ and $h(x) = c$ if $x \in S - \{ a \}$, and $g(b) = d$ and $g(x) = c$ if $x \in S - \{ b \}$. Therefore $h, g \in W$ and $h \triangledown g \in W_r$ since $h \triangledown g$ is the constant function defined by $d$. Therefore $U \triangleleft U \not\subset W_r$ and $\triangledown$ is not continuous.

**Definition.** For $S$ finite and $2 \leq \text{card } S$, denote max

$$f = \max \{ f(s) \mid s \in S \}, \quad I(f) = \{ s \in S \mid f(s) = \max f \}.$$  

Then define $f^-: S \to D$ by

1. if $I(f)$ is not a unit set, $f^-(s) = \max f$ for every $s \in S$, and
2. if $I(f)$ is a unit set, $f^-(s) = \max f$ for $s \in I(f)$, and

$$f^-(s) = \max \{ f(t) \mid t \in S - I(f) \} \quad \text{for } s \in S - I(f).$$

**Lemma 7.** If $S$ is finite and $2 \leq \text{card } S$, then

1. $f \in L$ implies $f^- M$ and $f \in M$ if and only if $f = f^-$,
2. $f \leq f^-$, $f^-=f^-$, and $f \leq g$ implies $f^- \leq g$,
3. $f, g \in M$ implies $f \triangledown g = (f \triangledown g)^-.$

The lemma is a straight forward catalog of the properties following from the definition directly.

**Lemma 8.** If $S$ is finite, $2 \leq \text{card } S$ and $D$ is a topological chain, then the function $J: L \to M$ defined by $J(f) = f^-$ is a retraction of $L$ onto $M$. 


From Lemma 7 it suffices to show that $J$ is continuous. This is done by letting $U$ be an open set in $D$ and $r \in S$, defining $W = \{ f \in M : f(r) \in U \}$ and showing that $J^{-1}(W)$ is open. It is shown to be open by case argument. Let $g \in L$ and $g^- \in W$. If $I(g)$ is not a unit set, then $g^-$ is constant and $\max g \in U$. Define $V_1 = \{ f \in L : f(s) \in U \}$ for every $s \in S$. Since $S$ is finite, $V_1$ is open and contains $g$. Further $h \in V_1$ implies $h^-(r) \in U$. If $I(g) = \{ r \}$, let $b$ be an element of $S - \{ r \}$. Define $V_2 = \{ f \in L : f(r) \in U \}$ and $f(s) \in U_2$ for $s \in S - \{ r \}$ where $U_1 = U \cap \{ x \in D : z < x \}$ and $U_2 = \{ x \in D : x < z \}$ if there exist $z \in D$ such that $g^-(b) < z < g^-(r)$, and if there does not exist such an element $z$, $U_1 = U \cap \{ x \in D : g^-(b) < x \}$ and $U_2 = \{ x \in D : x < g^-(r) \}$. Then $g \in V_1$, $V_1$ is open and $f \in V_2$ implies $f^-(r) \in U$. The other case is handled in a similar fashion.

**Theorem 3.** If $D$ is a topological chain and $S$ is finite, then $(M, \leq)$ is a modular topological lattice which is nondistributive if $\text{card } S > 2$ and $\text{card } D > 1$.

If $\text{card } S = 1$, $M = L$ and therefore $(M, \leq)$ is a topological distributive lattice. If $\text{card } S \geq 2$, then Lemma 7 and 8 establish that $\bigvee$ is continuous since it is the composition of continuous maps. Therefore $(M, \leq)$ is a topological lattice since $\land$ is continuous for every $S$. Theorem 1 establishes the modularity of $M$ while its corollary the nondistributive nature of $M$ when $3 \leq \text{card } S$ and $2 \leq \text{card } D$.

**Definition.** Let $n$ be a positive integer and $3 \leq n$, then let $M_n$ denote the lattice constructed as the $M$ above in the case where $S = \{ 1, 2, \ldots, n \}$ and $D = \{ x \in R : 0 \leq x \leq 1 \}$ with its usual order and the operations of $D$ being $x \lor y = \max \{ x, y \}$ and $x \land y = \min \{ x, y \}$. When $S$ is the set of positive integers and $D$ as previously described let $M_\infty$ denote the lattice $M$ constructed. Then the following results are immediate.

**Theorem 4.** For each positive integer $n \geq 3$, $M_n$ is a compact connected topological lattice which is modular and not distributive.

**Corollary 1.** $M_3$ is a compact connected topological lattice, modular and not distributive, which is a two dimensional subset of $R^3$ that cannot be embedded in $R^2$.

**Corollary 2.** $M_\infty$ is a compact connected topological semilattice, which is a modular lattice, and not a topological lattice.
REFERENCES


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