A NOTE ON A THEOREM OF HILL

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Recently Hill has shown the existence of an abelian $p$-group with the property that each infinite subgroup can be embedded in a direct summand of the same cardinality but the group is not a direct sum of countable groups. Megibben has since observed that this phenomenon occurs in a larger class of abelian groups. In this note we show that such pathology is present in modules for a rather wide class of rings. In fact, the lack of such phenomena for a particular class of modules serves as a characterization for left perfect rings. Our results also yield some facts concerning pure injective modules.

All rings in this paper are associative with identity and all modules are unital.

2. A characterization of left perfect rings. Bass [1] calls a ring $R$ left perfect if each left $R$-module has a projective cover (projective cover is the dual of injective envelope). Among several other characterizations of left perfect rings, Bass proves that $R$ is left perfect if and only if $R$ has the descending chain condition on principal right ideals. Hence, assuming that $R$ is not left perfect, we can obtain a strictly decreasing sequence of principal right ideals of the form

$$a_1R \supset a_2R \supset \cdots \supset a_1a_2 \cdots a_n R \supset \cdots.$$ 

We set $P = \Pi_{n<\omega} Re_n$, where $Re_n \cong R$ for each $n$, and we denote by $S$ the submodule of finitely nonzero sequences in $P$. We shall use the notation $\sum_{i=m}^{\infty} r_i e_i$, for $m \leq n$, to denote a vector in $P$ whose $i$th coordinate is zero for $i > n$ and $i < m$ and whose $i$th coordinate is $r_i e_i$ for $m \leq i \leq n$. We define elements

$$c^{(m)} = \sum_{i \leq m} (a_{m} \cdots a_i) e_i \in P \quad \text{for} \quad m = 1, 2, \cdots.$$ 

Let $A$ be the submodule of $P$ generated by $S$ and the elements $c^{(m)}$ for $m = 1, 2, \cdots$. With this notation established, we prove the following lemma.

**Lemma 2.1.** Let $R$ be a ring that is not left perfect and let $A$ and $S$ be defined as above. Then $A$ is free and $S$ is not a direct summand of $A$.

**Proof.** First we note that if $n < m$, then
and in particular
\[ c^{(n)} = a_n(e_n + c^{(n+1)}) \quad \text{for} \quad n = 1, 2, \ldots. \]

Now suppose that \( A = S \oplus B \). Then \( c^{(n)} = s_n + b_n \) where \( s_n \in S \) and \( b_n \neq 0 \in B \). From property (*) above, we have that, for \( n > 1 \),
\[ c^{(1)} = s_{1_n} + (a_1 \cdots a_{n-1})c^{(n)} \quad \text{where} \quad s_{1_n} \in S. \]

Therefore
\[ s_i + b_i = c^{(1)} = s_{1_n} + (a_i \cdots a_{n-1})c^{(n)} \]
\[ = s_{1_n} + (a_i \cdots a_{n-1})(s_n + b_n) \]
\[ = s_{1_n} + (a_i \cdots a_{n-1})s_n + (a_i \cdots a_{n-1})b_n. \]

Hence \( s_i = s_{1_n} + (a_i \cdots a_{n-1})s_n \) and \( b_i = (a_i \cdots a_{n-1})b_n \) for each \( n > 1 \).

Therefore \( c^{(1)} = s_i + (a_i \cdots a_{n-1})b_n \) for \( n = 2, 3, \ldots \). Since \( s_i \) has only finitely many nonzero coordinates, it follows that there is a positive integer \( r \) such that \( a_i \cdots a_r = a_i \cdots a_r a_{r+1} y \). But this implies that \( a_i \cdots a_r R = a_i \cdots a_{r+1} R \) which is a contradiction. Thus \( S \) is not a summand of \( A \).

To show that \( A \) is free, let \( y_1 = e_1 + c^{(1)} \) for \( n = 1, 2, \ldots \). Since \( c^{(n)} = a_n y_n \) by property (*) above, it follows that \( A \) is generated by \( \{y_n\}_{n<\omega} \). Suppose that \( r_1 y_1 + \cdots + r_n y_n = 0 \) where \( r_i \in R \). Then
\[ r_1 c^{(1)} + r_2 c^{(2)} + \cdots + r_n c^{(n+1)} = -r_1 e_i - r_2 e_2 - \cdots - r_n e_n. \]

Since the first coordinate of the left hand side is zero, it follows that \( r_1 = 0 \). A repetition of the preceding argument shows that \( r_1 = r_2 = \cdots = r_n = 0 \). This implies that \( A \) is free with \( \{y_n\}_{n<\omega} \) for a basis.

We observe from [1] that a left \( R \)-module is torsionless if and only if it can be embedded as a submodule of a direct product of copies of \( R \). We shall call a left \( R \)-module \( \mathfrak{S}_1 \)-separable provided \( G \) is flat, torsionless and that each countably generated submodule of \( G \) is contained in a countably generated direct summand of \( G \) (this definition parallels the definition given by L. Fuchs [4] in the context of \( \mathfrak{S}_1 \)-free groups). We now prove the main result of this section. The proof is modeled after that of Hill's [5].

**Theorem 2.2.** A ring \( R \) is left perfect if and only if each \( \mathfrak{S}_1 \)-separable left \( R \)-module is a direct sum of countably generated modules.
Proof. If $R$ is left perfect, then by Theorem 3.2 [2] any flat left module is projective. Since an $\mathfrak{R}_i$-separable left module is flat, it follows from Kaplansky's theorem [6] that each $\mathfrak{R}_i$-separable left $R$-module is a direct sum of countably generated modules.

Now suppose that $R$ is not left perfect. This implies by Theorem 1 [1] that $R$ has a strictly decreasing sequence

$$a_1 R \supset a_2 a_1 R \supset \cdots \supset a_1 \cdots a_n R \supset \cdots$$

of principal right ideals. Set $P^* = \bigcup_{\alpha < \Omega} P_{\alpha}$ where $P_{\alpha} = Re_{\alpha}$ for each $\alpha < \Omega$ ($\Omega$ denotes the first uncountable ordinal). We construct a left submodule $G$ of $P^*$ such that $G = \bigcup_{\alpha < \Omega} G_{\alpha}$ where $\{G_{\alpha}\}_{\alpha < \Omega}$ is a monotone increasing chain defined as follows:

(i) If $\alpha$ is a limit ordinal, $\alpha < \beta$, $G_{\alpha} = \bigcup_{\alpha < \beta} G_{\alpha}$.

(ii) If $\alpha - 1$ and $\alpha - 2$ exist, $G_{\alpha} = G_{\alpha - 1} \oplus Re_{\alpha - 1}$.

(iii) If $\alpha - 1$ exists and is a limit, there is a monotone increasing sequence $\sigma_{\alpha}(n)$ of ordinals less than $\alpha - 1$ such that $\sigma_{\alpha}(n) - \alpha - 1$ is defined for each $n$ and such that $\sigma_{\alpha}(n)$ converges to $\alpha - 1$. Then $c^{(m)}_{\alpha} = \sum_{i \in S_m} \alpha_i e_{\sigma_{\alpha}(i)}$ for $m = 1, 2, \cdots$ and $G_{\alpha}$ is generated by $G_{\alpha - 1}$ and $\{c^{(n)}_{\alpha}\}_{n < \omega}$.

(iv) If $\rho_{\gamma + 1}$ denotes the natural projection of $P^*$ onto $\bigcup_{\gamma < \alpha} Re_{\gamma}$, and if $\gamma + 1 < \alpha < \beta$, then $\rho_{\gamma + 1}(G_{\alpha}) = G_{\gamma + 1}$.

(v) $G_{\alpha}$ is not a direct summand of $G_{\alpha + 1}$ if $\alpha$ is a limit ordinal.

(vi) $G_{\alpha}$ is flat for $\alpha < \beta$.

If $\beta$ is a limit ordinal we set $G_{\beta} = \bigcup_{\alpha < \beta} G_{\alpha}$ and if both $\beta - 1$ and $\beta - 2$ exist we set $G_{\beta} = G_{\beta - 1} \oplus Re_{\beta - 1}$. It is straightforward in either of the above two cases to show that (i)-(vi) hold for the collection $\{G_{\alpha}\}_{\alpha < \beta}$. Now suppose that $\beta - 1$ is a limit ordinal. Define $\sigma_{\beta}(n)$ and $c^{(m)}_{\beta}$ so that (iii) is satisfied and define $G_{\beta}$ to be the submodule generated by $G_{\beta - 1}$ and $\{c^{(n)}_{\beta}\}_{n < \omega}$. Suppose that $\gamma + 1 < \beta$ and consider $\rho_{\gamma + 1}(G_{\beta})$. To show that (iv) is satisfied, it clearly suffices to show that $\rho_{\gamma + 1}(c^{(n)}_{\beta}) \in G_{\gamma + 1}$. But this is a direct consequence of the fact that $c^{(n)}_{\beta} = \sum_{i \in S_m} (a_m \cdots a_{i - 1}) e_{\sigma_{\beta}(i)}$ and that $\sigma_{\beta}(i) > \gamma + 1$ for all $i$ larger than some integer $i_0$. To see that (v) holds, let $A_{\beta}$ be the set of ordinals $\{\sigma_{\beta}(1), \sigma_{\beta}(2), \cdots\}$ and let $I_{\beta}$ be the ordinals less than $\beta$ that are not in $A_{\beta}$. Let $B = G_{\beta} \cap \bigcup_{\gamma < \beta} Re_{\gamma}$, $A = G_{\beta} \cap \bigcup_{\gamma < \beta} Re_{\gamma}$ and let $S$ denote the finite sequences in $\bigcup_{\gamma < \beta} Re_{\gamma}$. It is routine to show that $G_{\beta} = B \oplus A$ and that $G_{\beta - 1} = B \oplus S$.

We observe that (up to isomorphism) our $A$ and $S$ here are the same as the $A$ and $S$, respectively, in Lemma 2.1. It follows that $G_{\beta - 1}$ is not a direct summand of $G_{\beta}$. We also see that $G_{\beta - 1}$ is flat since $B$ is necessarily flat and since $A$ is free. Thus the collection $\{G_{\alpha}\}_{\alpha \leq \beta}$ satisfies (i)-(vi) and hence we obtain $G = \bigcup_{\alpha < \Omega} G_{\alpha}$ where $\{G_{\alpha}\}_{\alpha \leq \beta}$ satisfies (i)-(vi). Note that $G$ is torsionless since $G$ is a submodule of $P^*$. $G$ is flat.
from (vi) since a direct limit of flat modules is flat. Property (v) implies that $G$ is not a direct sum of countably generated modules. Finally, property (iv) guarantees that $\rho_{r+1}$, when restricted to $G$, is a projection of $G$ onto $G_{r+1}$. Thus $G$ is $R_i$-separable.

From the above proof, we obtain the following corollary.

**Corollary 2.3.** A ring $R$ is left perfect if and only if each $R_i$-separable left $R$-module is projective.

3. Some remarks on pure injective modules over artinian rings. An interesting consequence of our Lemma 2.1 is that the direct sum of $\mathbb{N}_0$ copies of a ring $R$ (as a left $R$-module) is not a direct summand of the corresponding direct product of $\mathbb{N}_0$ copies of $R$ if $R$ is not left perfect. In this section we wish to consider in part the question of when the direct sum of infinitely many copies of $R$ (as a left $R$-module) is a direct summand of the corresponding direct product of copies of $R$. More generally, we consider the problem of determining when projective modules are pure injective modules in the sense of Warfield [7]. For commutative Noetherian rings we obtain a complete answer to both of the above questions. A submodule $A$ of a left $R$-module $B$ is called a pure submodule provided, for any right module $M$, the natural homomorphism $M \otimes A \rightarrow M \otimes B$ is injective. A module $Q$ is called pure injective, if for every module $B$ and pure submodule $A$, each homomorphism of $A$ into $Q$ extends to a homomorphism of $B$ into $Q$. Hence, if a pure injective module $Q$ is a pure submodule of a module $B$, then $Q$ is a direct summand of $B$. Our main theorems of this section follow the next lemma.

**Lemma 3.1.** If $R$ is a left artinian ring, then any pure submodule of a left projective $R$-module is a direct summand.

**Proof.** Suppose that $A$ is a pure submodule of a left projective module $P$ and suppose that $M$ is an arbitrary right $R$-module. From the exact sequence

$$0 = \text{Tor}^R_n(M, P) \rightarrow \text{Tor}^R_n(M, P/A) \rightarrow M \otimes A \rightarrow M \otimes P,$$

we obtain that $\text{Tor}^R_n(M, P/A) = 0$ since the homomorphism $M \otimes A \rightarrow M \otimes P$ is injective. Hence $P/A$ is a flat left $R$-module. By Theorem $P$ [1], $P/A$ is projective and thus $A$ is a direct summand of $P$.

In what follows, $\sum A_i$ will denote the finitely nonzero vectors in the direct product $\Pi A_i$. 

Theorem 3.2. If $R$ is a commutative artinian ring, then each projective $R$-module is pure injective. Moreover, if $R$ is a commutative Noetherian ring and if each projective $R$-module is pure injective, then $R$ is artinian.

Proof. First suppose that $R$ is a commutative artinian ring. It suffices to show that each free $R$-module is pure projective. By Proposition 9 [7], $R$ is pure injective as a module over itself. Let $F = \sum_{\alpha} R$ be an arbitrary free $R$-module and let $P$ denote the direct product $P = \Pi_{\alpha} R$ containing $F$. It is elementary to see that $F$ is a pure submodule of $P$ and that $P$ is pure injective since $R$ is pure injective. By Theorem 3.4 [2], $P$ is also a projective $R$-module. Hence, by Lemma 3.1, $F$ is a direct summand of $P$ and therefore is pure injective.

Now suppose that $R$ is a commutative Noetherian ring for which each projective module is pure injective. Let $S$ and $A$ be as in Lemma 2.1. Note that $S = \sum_{\alpha} R$ and that $S \subseteq A \subseteq \Pi_{\alpha} R$. Therefore $S$ is pure in $A$ and is therefore a direct summand of $A$. Hence Lemma 2.1 yields that $R$ is a perfect ring. Since $R$ is also Noetherian, we have that $R$ is artinian.

Corollary 3.3. If $R$ is a commutative artinian ring, then the direct sum $\sum_{\alpha} R$ is a direct summand of the direct product $\Pi_{\alpha} R$ for each cardinal number $\alpha$. Moreover, if $R$ is a commutative Noetherian ring and if $\sum_{\alpha} R$ is a direct summand of $\Pi_{\alpha} R$, then $R$ is artinian.

We conclude our consideration of pure injective modules with an answer to the converse problem answered in Theorem 3.2, that is, we classify those rings for which every pure injective $R$-module is projective. Our solution here needs no initial assumptions on the ring.

Theorem 3.4. A ring $R$ has the property that each pure injective left $R$-module is projective if and only if $R$ is semi-simple and artinian.

Proof. The sufficiency is clear. Hence suppose that $R$ has the property that each pure injective left $R$-module is projective. Since each injective left module is pure injective, it follows that each injective left $R$-module is also projective. By Theorem 5.3 [3] of Faith and Walker, we have that $R$ is quasi-Frobenius. Since each left $R$-module can be embedded as a pure submodule of a pure injective left $R$-module by Corollary 6 [7], we have that any left $R$-module is isomorphic to a pure submodule of a projective module. Since a quasi-Frobenius ring is left artinian, it follows by Lemma 3.1 that each
left $R$-module is projective. It is well-known that such a ring is a semi-simple artinian ring.

REFERENCES


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