

Pacific Journal of Mathematics

A NOTE ON A THEOREM OF HILL

PHILLIP ALAN GRIFFITH

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Recently Hill has shown the existence of an abelian p -group with the property that each infinite subgroup can be embedded in a direct summand of the same cardinality but the group is not a direct sum of countable groups. Megibben has since observed that this phenomenon occurs in a larger class of abelian groups. In this note we show that such pathology is present in modules for a rather wide class of rings. In fact, the lack of such phenomena for a particular class of modules serves as a characterization for left perfect rings. Our results also yield some facts concerning pure injective modules.

All rings in this paper are associative with identity and all modules are unital.

2. A characterization of left perfect rings. Bass [1] calls a ring R left perfect if each left R -module has a projective cover (projective cover is the dual of injective envelope). Among several other characterizations of left perfect rings, Bass proves that R is left perfect if and only if R has the descending chain condition on principal right ideals. Hence, assuming that R is not left perfect, we can obtain a strictly decreasing sequence of principal right ideals of the form

$$a_1R \supset a_1a_2R \supset \cdots \supset a_1 \cdots a_nR \supset \cdots .$$

We set $P = \prod_{n < \omega} Re_n$, where $Re_n \cong R$ for each n , and we denote by S the submodule of finitely nonzero sequences in P . We shall use the notation $\sum_{i=m}^n r_i e_i$, for $m \leq n$, to denote a vector in P whose i th coordinate is zero for $i > n$ and $i < m$ and whose i th coordinate is $r_i e_i$ for $m \leq i \leq n$. We define elements

$$c^{(m)} = \sum_{i \geq m} (a_m \cdots a_i) e_i \in P \quad \text{for } m = 1, 2, \dots .$$

Let A be the submodule of P generated by S and the elements $c^{(m)}$ for $m = 1, 2, \dots$. With this notation established, we prove the following lemma.

LEMMA 2.1. *Let R be a ring that is not left perfect and let A and S be defined as above. Then A is free and S is not a direct summand of A .*

Proof. First we note that if $n < m$, then

$$(*) \quad c^{(n)} = \sum_{i \geq n}^{m-1} (a_n \cdots a_i) e_i + (a_n \cdots a_{m-1}) c^{(m)}$$

and in particular

$$c^{(n)} = a_n(e_n + c^{(n+1)}) \quad \text{for } n = 1, 2, \dots$$

Now suppose that $A = S \oplus B$. Then $c^{(n)} = s_n + b_n$ where $s_n \in S$ and $b_n \in B$. From property (*) above, we have that, for $n > 1$,

$$c^{(1)} = s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \quad \text{where } s_{1n} \in S.$$

Therefore

$$\begin{aligned} s_1 + b_1 &= c^{(1)} = s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \\ &= s_{1n} + (a_1 \cdots a_{n-1})(s_n + b_n) \\ &= s_{1n} + (a_1 \cdots a_{n-1})s_n + (a_1 \cdots a_{n-1})b_n. \end{aligned}$$

Hence $s_1 = s_{1n} + (a_1 \cdots a_{n-1})s_n$ and $b_1 = (a_1 \cdots a_{n-1})b_n$ for each $n > 1$. Therefore $c^{(1)} = s_1 + (a_1 \cdots a_{n-1})b_n$ for $n = 2, 3, \dots$. Since s_1 has only finitely many nonzero coordinates, it follows that there is a positive integer r such that $a_1 \cdots a_r = a_1 \cdots a_r a_{r+1} y$. But this implies that $a_1 \cdots a_r R = a_1 \cdots a_{r+1} R$ which is a contradiction. Thus S is not a summand of A .

To show that A is free, let $y_n = e_n + c^{(n+1)}$ for $n = 1, 2, \dots$. Since $c^{(n)} = a_n y_n$ by property (*) above, it follows that A is generated by $\{y_n\}_{n < \omega}$. Suppose that $r_1 y_1 + \cdots + r_n y_n = 0$ where $r_i \in R$. Then

$$r_1 c^{(2)} + r_2 c^{(3)} + \cdots + r_n c^{(n+1)} = -r_1 e_1 - r_2 e_2 - \cdots - r_n e_n.$$

Since the first coordinate of the left hand side is zero, it follows that $r_1 = 0$. A repetition of the preceding argument shows that $r_1 = r_2 = \cdots = r_n = 0$. This implies that A is free with $\{y_n\}_{n < \omega}$ for a basis.

We observe from [1] that a left R -module is torsionless if and only if it can be embedded as a submodule of a direct product of copies of R . We shall call a left R -module G \aleph_1 -separable provided G is flat, torsionless and that each countably generated submodule of G is contained in a countably generated direct summand of G (this definition parallels the definition given by L. Fuchs [4] in the context of \aleph_1 -free groups). We now prove the main result of this section. The proof is modeled after that of Hill's [5].

THEOREM 2.2. *A ring R is left perfect if and only if each \aleph_1 -separable left R -module is a direct sum of countably generated modules.*

Proof. If R is left perfect, then by Theorem 3.2 [2] any flat left module is projective. Since an \aleph_1 -separable left module is flat, it follows from Kaplansky's theorem [6] that each \aleph_1 -separable left R -module is a direct sum of countably generated modules.

Now suppose that R is not left perfect. This implies by Theorem P [1] that R has a strictly decreasing sequence

$$a_1R \supset a_1a_2R \supset \dots \supset a_1 \dots a_nR \supset \dots$$

of principal right ideals. Set $P^* = \prod_{\alpha < \Omega} Re_\alpha$ where $Re_\alpha \cong R$ for each $\alpha < \Omega$ (Ω denotes the first uncountable ordinal). We construct a left submodule G of P^* such that $G = \bigcup_{\alpha < \Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha < \Omega}$ is a monotone increasing chain defined as follows: $G_0 = 0, G_1 = Re_1$ and suppose that G_α has been defined for each $\alpha < \beta$ such that the following conditions hold:

(i) If α is a limit ordinal, $\alpha < \beta, G_\alpha = \bigcup_{\gamma < \alpha} G_\gamma$.

(ii) If $\alpha - 1$ and $\alpha - 2$ exist, $G_\alpha = G_{\alpha-1} \oplus Re_{\alpha-1}$.

(iii) If $\alpha - 1$ exists and is a limit, there is a monotone increasing sequence $\sigma_\alpha(n)$ of ordinals less than $\alpha - 1$ such that $\sigma_\alpha(n) - 2$ is defined for each n and such that $\sigma_\alpha(n)$ converges to $\alpha - 1$. Then $c_\alpha^{(m)} = \sum_{i \geq m} (a_m \dots a_i)e_{\sigma_\alpha(i)}$ for $m = 1, 2, \dots$ and G_α is generated by $G_{\alpha-1}$ and $\{c_\alpha^{(n)}\}_{n < \omega}$.

(iv) If $\rho_{\gamma+1}$ denotes the natural projection of P^* onto $\prod_{\lambda < \gamma} Re_\lambda$ and if $\gamma + 1 < \alpha < \beta$, then $\rho_{\gamma+1}(G_\alpha) = G_{\gamma+1}$.

(v) G_α is not a direct summand of $G_{\alpha+1}$ if α is a limit ordinal.

(vi) G_α is flat for $\alpha < \beta$.

If β is a limit ordinal we set $G_\beta = \bigcup_{\alpha < \beta} G_\alpha$ and if both $\beta - 1$ and $\beta - 2$ exist we set $G_\beta = G_{\beta-1} \oplus Re_{\beta-1}$. It is straightforward in either of the above two cases to show that (i)-(vi) hold for the collection $[G_\alpha]_{\alpha < \beta}$. Now suppose that $\beta - 1$ is a limit ordinal. Define $\sigma_\beta(n)$ and $c_\beta^{(m)}$ so that (iii) is satisfied and define G_β to be the submodule generated by $G_{\beta-1}$ and $\{c_\beta^{(n)}\}_{n < \omega}$. Suppose that $\gamma + 1 < \beta$ and consider $\rho_{\gamma+1}(G_\beta)$. To show that (iv) is satisfied, it clearly suffices to show that $\rho_{\gamma+1}(c_\beta^{(m)}) \in G_{\gamma+1}$. But this is a direct consequence of the fact that $c_\beta^{(m)} = \sum_{i \geq m} (a_m \dots a_i)e_{\sigma_\beta(i)}$ and that $\sigma_\beta(i) > \gamma + 1$ for all i larger than some integer i_0 . To see that (v) holds, let A_β be the set of ordinals $\{\sigma_\beta(1), \sigma_\beta(2), \dots\}$ and let I_β be the ordinals less than β that are not in A_β . Let $B = G_\beta \cap \prod_{I_\beta} Re_\lambda, A = G_\beta \cap \prod_{A_\beta} Re_\lambda$ and let S denote the finite sequences in $\prod_{I_\beta} Re_\lambda$. It is routine to show that $G_\beta = B \oplus A$ and that $G_{\beta-1} = B \oplus S$. We observe that (up to isomorphism) our A and S here are the same as the A and S , respectively, in Lemma 2.1. It follows that $G_{\beta-1}$ is not a direct summand of G_β . We also see that $G_{\beta-1}$ is flat since B is necessarily flat and since A is free. Thus the collection $[G_\alpha]_{\alpha \leq \beta}$ satisfies (i)-(vi) and hence we obtain $G = \bigcup_{\alpha < \Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha < \Omega}$ satisfies (i)-(vi). Note that G is torsionless since G is a submodule of P^* . G is flat

from (vi) since a direct limit of flat modules is flat. Property (v) implies that G is not a direct sum of countably generated modules. Finally, property (iv) guarantees that $\rho_{\gamma+1}$, when restricted to G , is a projection of G onto $G_{\gamma+1}$. Thus G is \aleph_1 -separable.

From the above proof, we obtain the following corollary.

COROLLARY 2.3. *A ring R is left perfect if and only if each \aleph_1 -separable left R -module is projective.*

3. Some remarks on pure injective modules over artinian rings. An interesting consequence of our Lemma 2.1 is that the direct sum of \aleph_0 copies of a ring R (as a left R -module) is not a direct summand of the corresponding direct product of \aleph_0 copies of R if R is not left perfect. In this section we wish to consider in part the question of when the direct sum of infinitely many copies of R (as a left R -module) is a direct summand of the corresponding direct product of copies of R . More generally, we consider the problem of determining when projective modules are pure injective modules in the sense of Warfield [7]. For commutative Noetherian rings we obtain a complete answer to both of the above questions. A submodule A of a left R -module B is called a pure submodule provided, for any right module M , the natural homomorphism $M \otimes A \rightarrow M \otimes B$ is injective. A module Q is called pure injective, if for every module B and pure submodule A , each homomorphism of A into Q extends to a homomorphism of B into Q . Hence, if a pure injective module Q is a pure submodule of a module B , then Q is a direct summand of B . Our main theorems of this section follow the next lemma.

LEMMA 3.1. *If R is a left artinian ring, then any pure submodule of a left projective R -module is a direct summand.*

Proof. Suppose that A is a pure submodule of a left projective module P and suppose that M is an arbitrary right R -module. From the exact sequence

$$0 = \text{Tor}_1^R(M, P) \rightarrow \text{Tor}_1^R(M, P/A) \rightarrow M \otimes A \rightarrow M \otimes P,$$

we obtain that $\text{Tor}_1^R(M, P/A) = 0$ since the homomorphism $M \otimes A \rightarrow M \otimes P$ is injective. Hence P/A is a flat left R -module. By Theorem P [1], P/A is projective and thus A is a direct summand of P .

In what follows, $\sum A_i$ will denote the finitely nonzero vectors in the direct product $\prod A_i$.

THEOREM 3.2. *If R is a commutative artinian ring, then each projective R -module is pure injective. Moreover, if R is a commutative Noetherian ring and if each projective R -module is pure injective, then R is artinian.*

Proof. First suppose that R is a commutative artinian ring. It suffices to show that each free R -module is pure projective. By Proposition 9 [7], R is pure injective as a module over itself. Let $F = \sum_{\alpha} R$ be an arbitrary free R -module and let P denote the direct product $P = \prod_{\alpha} R$ containing F . It is elementary to see that F is a pure submodule of P and that P is pure injective since R is pure injective. By Theorem 3.4 [2], P is also a projective R -module. Hence, by Lemma 3.1, F is a direct summand of P and therefore is pure injective.

Now suppose that R is a commutative Noetherian ring for which each projective module is pure injective. Let S and A be as in Lemma 2.1. Note that $S = \sum_{\aleph_0} R$ and that $S \subseteq A \subseteq \prod_{\aleph_0} R$. Therefore S is pure in A and is therefore a direct summand of A . Hence Lemma 2.1 yields that R is a perfect ring. Since R is also Noetherian, we have that R is artinian.

COROLLARY 3.3. *If R is a commutative artinian ring, then the direct sum $\sum_{\alpha} R$ is a direct summand of the direct product $\prod_{\alpha} R$ for each cardinal number α . Moreover, if R is a commutative Noetherian ring and if $\sum_{\aleph_0} R$ is a direct summand of $\prod_{\aleph_0} R$, then R is artinian.*

We conclude our consideration of pure injective modules with an answer to the converse problem answered in Theorem 3.2, that is, we classify those rings for which every pure injective R -module is projective. Our solution here needs no initial assumptions on the ring.

THEOREM 3.4. *A ring R has the property that each pure injective left R -module is projective if and only if R is semi-simple and artinian.*

Proof. The sufficiency is clear. Hence suppose that R has the property that each pure injective left R -module is projective. Since each injective left module is pure injective, it follows that each injective left R -module is also projective. By Theorem 5.3 [3] of Faith and Walker, we have that R is quasi-Frobenius. Since each left R -module can be embedded as a pure submodule of a pure injective left R -module by Corollary 6 [7], we have that any left R -module is isomorphic to a pure submodule of a projective module. Since a quasi-Frobenius ring is left artinian, it follows by Lemma 3.1 that each

left R -module is projective. It is well-known that such a ring is a semi-simple artinian ring.

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Received April 29, 1968. This work was supported in part by NASA Grant NGR-44-005-037.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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