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JAMES A. HUCKABA

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**Let $w$ be a pseudo-valuation defined on a commutative ring $R$ and let $S$ be an overring of $R$. This paper investigates conditions needed to imply that $w$ can be extended to $S$. These conditions are given in terms of a particular sequence of ideals $\{A_i\}_{i=0}^{\infty}$ in $R$ which is called the best filtration for $w$. The main theorem states that if $w$ is a pseudo-valuation on $R$ with best filtration $\{A_i\}$ and each $A_i$ is a contracted ideal with respect to $S$, then $w$ can be extended to $S$. The converse of this result is then proved.**

By using our main theorem and some recent results by Gilmer [1], we show in several important cases that if $S$ is an overring of $R$ and $w$ is any pseudo-valuation on $R$ possessing a best filtration, then $w$ can be extended to $S$. In particular, if $R$ is a Prüfer domain with quotient field $K$ and if $S$ is an overring of $R$ such that $S \cap K = R$, then $w$ can be extended from $R$ to $S$.

We begin in §1 by defining and developing properties of a best filtration and determining classes of pseudo-valuations which have best filtrations. The main results and applications are then proved in §2.

1. Filtrations. All rings are commutative, associative, and have identity. If $S$ is an overring of $R$, we assume that $S$ and $R$ have the same identity. A *pseudo-valuation* on the ring $R$ is a mapping $w$ from $R$ into the extended real number system such that:

(i) $w(0) = \infty, w(1) = 0$,

(ii) $w(x - y) \geqq \min\{w(x), w(y)\}$,

(iii) $w(xy) \geqq w(x) + w(y)$, for each $x, y \in R$.

$w$ is called a *homogeneous pseudo-valuation* in case:

(iv) $w(x^2) = 2w(x)$ for each $x \in R$.

$w$ is called a *valuation* in case:

(v) $w(xy) = w(x) + w(y)$ for each $x, y \in R$.

Pseudo-valuations were first introduced by Rees [3]. Rees proved in [3] that (iv) is equivalent to the condition that $w(x^n) = nw(x)$ for each positive integer $n$ and for each $x \in R$. These functions arise quite naturally in ring theory. If $A$ is a proper ideal of $R$, define $v_A(x) = n$ if $x \in A^n, x \notin A^{n+1}$ and $v_A(x) = \infty$ if $x \in A^n$ for all $n$. Then $v_A$ is a pseudo-valuation. We say that $v_A$ is *associated with the ideal* $A$. A sequence of ideals $\{A_i\}_{i=0}^{\infty}$ of $R$ such that $A_0 = R, A_{i+1} \subset A_i$, and $A_iA_j \subset A_{i+j}$ for all $i$ and $j$ is called a *filtration* on $R$. Notice that the nonnegative integral powers of an ideal $A$ of $R$ forms a filtration, where $A^\circ$ is defined to be $R$. Also note that any filtration $\{A_i\}$ determines a pseudo-valuation in exactly the same manner that the powers of an ideal $A$ determines $v_A$. For an arbitrary pseudo-valuation $w$ on $R$ and a subset $T$ of $R$, define $w(T) = \inf\{w(t): t \in T\}$.

DEFINITION 1. If $w$ is a pseudo-valuation on $R$, define

$$\begin{aligned} A_0 &= R \\ A_i &= \{x \in R: w(x) > w(A_{i-1})\}\ , \\ &\quad \text{if } w(A_{i-1}) < \infty\ . \\ A_i &= A_{i-1}, \text{ if } w(A_{i-1}) = \infty\ . \end{aligned} \tag{1.1}$$

Each member of the sequence $\{A_i\}$ is an ideal of $R$. The sequence defined by (1.1) has the property that $A_0 \supset A_1 \supset A_2 \supset \cdots$. $A_{i+1}$ is not necessarily a proper subset of $A_i$, as will be shown in Examples 1 and 2. Also note that for a given pseudo-valuation $w$, $\{x \in R: w(x) = \infty\} \subset \cap A_i$. The following example shows that there exists pseudo-valuations such that the sequence defined by (1.1) is not a filtration.

EXAMPLE 1. Let $\mathfrak{A}$ be an ideal of a ring $R$ in which $\mathfrak{A}^i \supsetneqq \mathfrak{A}^{i+1}$ for all $i$. Define a sequence of ideals $\{B_i\}$ as follows: $B_0 = R$, $B_1 = \mathfrak{A}^2$, $B_2 = \mathfrak{A}^3$, $B_3 = B_4 = \mathfrak{A}^5$, and $B_i = \mathfrak{A}^7 (i \geqq 5)$. Then $\{B_i\}$ is a filtration in $R$ and determines some pseudo-valuation $w$, where $w(x) = n$ if $x \in \varepsilon B_n$, $x \notin B_{n+1}$ and $w(x) = \infty$ if $x \in \cap B_n$. Now use Definition 1 to define $A_i$ with respect to $w$. We obtain $A_i = B_i (i = 0, 1, 2, 3)$ and $A_i = B_{i+1} (i = 4, 5, \cdots)$. But $\{A_i\}$ is not a filtration, since $(A_2)^2 \not\subset A_4$.

DEFINITION 2. Let $w$ be a pseudo-valuation on $R$ and let $\{A_i\}$ be defined by (1.1). If $\{A_i\}$ is a filtration in $R$ such that $x \in \cap A_i$ if and only if $w(x) = \infty$, then $\{A_i\}$ is called a *best filtration* for $w$. Let $B(R)$ denote the class of all pseudo-valuations on $R$ which have a best filtration.

Example 1 then implies that there are pseudo-valuations which do not have best filtrations. It is clear from the definition that if $w$ has a best filtration, then it is unique. From now on we will talk about *the* best filtration for $w$.

EXAMPLE 2. Let $w$ be a pseudo-valuation on $R$ and let $\{A_i\}$ be the sequence defined by (1.1). It is possible for $\{A_i\}$ to be a filtration in $R$, yet not be the best filtration for $w$. Let $v$ be a real valued nondiscrete valuation on a field $K$ and consider $v$ as a pseudo-valuation on its valuation ring $R_v$. Since the value group of $v$ has no smallest positive element, $v(A_1) = 0$. Then $A_2 = \{x \in R: v(x) > v(A_1) = 0\} = A_1$.

By induction, we see that $A_i = A_1$ for each $i \geqq 1$. Hence the sequence $\{A_i\}$ is such that $A_0 \supsetneqq A_1 = A_2 = \cdots$. Therefore $v \notin B(R)$. However, it is clear that $\{A_i\}$ is a filtration.

We will be interested only when the sequence defined by (1.1) is a filtration. This always happens in one important case.

REMARK 1. If $v$ is a valuation on a ring $R$ and if $\{A_i\}$ is the sequence of ideals defined by (1.1), then $\{A_i\}$ is a filtration in $R$.

*Proof.* It is clear that $A_i \supset A_{i+1}$ for each $i$. Hence, to complete the proof we need to show that $A_iA_j \subset A_{i+j}$ for all nonnegative integers $i$ and $j$. We fix $j$ and use induction on $i$. Clearly $A_0A_j \subset A_{0+j}$. Assume that $A_{i-1}A_j \subset A_{i+j-1}$ for $i \geqq 1$. Let $x \in A_iA_j$, then $x = \sum_{k=1}^{n} a_kb_k$ where $a_k \in A_i$ and $b_k \in A_j$. We may assume without loss of generality that $v(a_1) + v(b_1) = \min_{k=1}^{n}(v(a_k) + v(b_k))$. Then $v(x) \geqq v(a_1) + v(b_1)$. Case 1: If $v(A_{i-1}) < \infty$, then $v(a_1) > v(A_{i-1})$, and thus $v(x) > v(A_{i-1}) + v(A_j) = v(A_{i-1}A_j) \geqq v(A_{i+j-1})$. By Definition 1, $x \in A_{i+j}$. Case 2: If $v(A_{i-1}) = \infty$, then $v(x) = \infty$, which implies that $x \in A_{i+j}$. Therefore $A_iA_j \subset A_{i+j}$.

LEMMA 1. *Let $w \in B(R)$ and let $\{A_i\}$ be the best filtration for $w$. Then:*

(1) *$A_i = A_{i+1}$ if and only if $w(A_i) = \infty$.*

(2) *Let $x \in A_i$ and $x \notin A_{i+1}$. Then $y \in A_i$ and $y \notin A_{i+1}$ if and only if $w(x) = w(y)$. In fact, $w(x) = w(A_i)$.*

(3) *If $y \in A_i$ and $z \notin A_i$, then $w(y) > w(z)$.*

(4) *If $w(x) < \infty$, then there exists an integer $i$ such that $x \in A_i$ and $x \notin A_{i+1}$.*

*Proof.* (1) Suppose $A_i = A_{i+1}$. By induction we see that $A_i = A_{i+t}$ for each positive integer $t$. If $w(A_i) < \infty$, then there is an element $x \in A_i$ such that $w(x) < \infty$. But $x \in \cap A_i$ which implies that $w(x) = \infty$, a contradiction. Conversely, if $w(A_i) = \infty$, then $A_i = A_{i+1}$ by definition of the best filtration.

(2) First note that $x \in A_i$, $x \notin A_{i+1}$ implies that $w(x) = w(A_i)$. If $y \in A_i$, $y \notin A_{i+1}$, then clearly $w(x) = w(y)$. Conversely, assume $w(x) = w(y)$. If $i = 0$, then $w(y) \leqq w(A_1)$ and hence $y$ is in $A_0$, but not in $A_1$. If $i > 0$, then $w(A_{i-1}) \leqq w(A_i)$. If equality holds, then $A_i = A_{i+1}$, which implies that $x \in A_{i+1}$. Therefore $w(A_{i-1}) < w(A_i)$, which implies that $y \in A_i$. Also $y \notin A_{i+1}$, for if so, then $w(y) > w(A_i)$.

(3) and (4) are clear.

The converse of the above result is also true.

LEMMA 2. *Let $w$ be a pseudo-valuation on $R$ and let $\{B_i\}$ be a filtration in $R$ satisfying properties* (1)–(4). *Then $\{B_i\}$ is the best filtration for $w$.*

*Proof.* Clearly $x \in \cap B_i$ if and only if $w(x) = \infty$. Suppose that $w(B_{i-1}) < \infty$. By properties (2) and (3) $B_i = \{x \in R: w(x) \geqq w(B_i)\}$. Thus $B_i \subset \{x \in R: w(x) > w(B_{i-1})\}$. On the other hand, suppose that $w(x) > w(B_{i-1})$. If $w(x) = \infty$, then $x \in \cap B_j$ and hence $x \in B_i$. If $w(x) < \infty$, choose $k$ such that $x \in B_k$ and $x \notin B_{k+1}$. Suppose that $k \leqq i - 1$, then $B_k \supset B_{i-1}$, so $w(x) = w(B_k) \leqq w(B_{i-1})$, a contradiction. So we must have $k > i - 1$ and hence $x \in B_i$. Therefore $B_i = \{x \in R: w(x) > w(B_{i-1})\}$.

By (1), if $w(B_{i-1}) = \infty$, then $B_{i-1} = B_i$.

We assume from now on that all pseudo-valuations $w$ which are considered have the property that there exists at least one $x$ such that $0 < w(x) < \infty$.

LEMMA 3. (a) *If $w$ is a homogeneous pseudo-valuation on $R$ and if $\{A_i\}$ is the sequence of ideals defined by* (1.1), *then $w(A_i) < \infty$ for each $i$.*

(b) *If $w$ is a pseudo-valuation on a ring $R$ and if $\{A_i\}$ is the sequence of ideals defined by* (1.1) *such that each $A_i$ is finitely generated, then $w(A_{i-1}) < \infty$ implies that $w(A_i) > w(A_{i-1})$.*

*Proof.* (a) Suppose, to the contrary, that $i$ is the smallest positive integer such that $w(A_i) = \infty$. Since $w$ is nontrivial, $i \geqq 2$. Choose $x \in A_{i-1}$, $x \notin A_i$. Then $0 < w(x) < \infty$, and $w(x^2) > w(x) \geqq w(A_{i-1})$, so $x^2 \in A_i$. But, $w(A_i) \leqq w(x^2) = 2w(x) < \infty$, contradicting the assumption that $w(A_i) = \infty$.

(b) Let $a_1, \cdots, a_n$ be a basis for $A_i$. Choose $a_k$ such that $w(a_k) = \min\{w(a_1), \cdots, w(a_n)\}$. Then $w(A_i) = w(a_k)$. Since $a_k \in A_i$, $w(a_k) > w(A_{i-1})$ and therefore $w(A_i) > w(A_{i-1})$.

The following theorem shows that there are many pseudo-valuations with best filtrations.

THEOREM 1. (1) *Any pseudo-valuation associated with an ideal is in $B(R)$. More generally, any pseudo-valuation determined by a filtration $\{B_i\}$, where $B_i = B_{i+1}$ implies that $B_i = B_{i+k}$ for each positive integer $k$, is in $B(R)$.*

(2) *If the sequence $\{A_i\}$ of ideals defined by* (1.1) *is a filtration and if $\lim_{i\to\infty} w(A_i) = \infty$, then $w \in B(R)$. Both of these conditions are satisfied if $w$ is a valuation on $R$ and $R$ is noetherian.*

(3) *A pseudo-valuation $w$ on $R$ such that the range of $w$ is equal to the set of all multiples of some positive real number $t > 0$ is in $B(R)$. This includes all integrally valued homogeneous pseudo-valuations $w$ such that there is an $x \in R$ for which $w(x) = 1$.*

(4) *All integrally valued pseudo-valuations and pseudo-valuations on a noetherian ring such that* (1.1) *forms a filtrations are in $B(R)$.*

*Proof.* (1) Clear.

(2) Let $w$ and $\{A_i\}$ satisfy the hypothesis of the first statement of (2). Clearly $w(x) = \infty$ implies that $x \in \cap A_i$. Let $x \in \cap A_i$, then $w(x) \geqq w(A_{i-1})$ for each $i$. Since $\lim_{i\to\infty} w(A_i) = \infty$, $w(x) = \infty$.

We will now prove the second statement of (2). Let $v$ be a valuation on a noetherian ring $R$. By Remark 1, the sequence of ideals $\{A_i\}$ defined by (1.1) is a filtration in $R$. We need to show that $\lim_{i\to\infty} v(A_i) = \infty$. Consider a basis $\{y_1, \cdots, y_r\}$ for the ideal $A_1$. Let $v(y_1) = \min\{v(y_1), \cdots, v(y_r)\}$. Then $y_1$ is an element of $R$ with the property that $v(y_1) = \varepsilon$ is a minimal positive element in the range of $v$. Assume that $\lim_{i\to\infty} v(A_i) = t < \infty$. By Lemma 3 (b), $v(A_i) > v(A_{i-1})$ for each $i$. Thus we can choose a sequence $\{x_j\} \in R$ so that $v(x_j) = a_j$ where $(t-\varepsilon) < a_1 < a_2 < \cdots$, and each $a_j < t$. Let $B$ be the ideal generated by $\{x_j\}$. Since $R$ is noetherian there exists a positive integer $n$ so that $\{x_1, \cdots, x_n\}$ is a basis of $B$. Let $p > n$, then $x_p \in B$ and so $x_p = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in R$. Then $v(x_p) \geqq \min\{v(\alpha_1 x_1), \cdots, v(\alpha_n x_n)\}$. Let $v(\alpha_j x_j)$ be this minimum. Case 1: If $v(\alpha_j) \neq 0$, then $v(x_p) \geqq v(\alpha_j) + a_j \geqq \varepsilon + a_j \geqq t$, which is a contradiction. Case 2: If $v(\alpha_j) = 0$, then $v(x_p) = a_p > a_j = v(x_j) = v(\alpha_j x_j)$. By properties of a valuation, $v(\alpha_j x_j) = v(\alpha_k x_k)$ for some $k \leqq n$, $k \neq j$. Since $v(x_j) \neq v(x_k)$, $v(\alpha_k) \neq 0$. Hence, $v(x_p) \geqq v(\alpha_k) + a_k \geqq t$, a contradiction. This proves that $\lim_{i\to\infty} v(A_i) = \infty$.

(3) Define $B_0 = R$ and inductively, $B_i = \{x \in R: w(x) \geqq i \cdot t\}$. The sequence $\{B_i\}$ satisfies the hypothesis of Lemma 2 and is a best filtration for $w$.

(4) The first part is clear. For the second part use the same technique as in (2).

## 2. The main results.

The following notation will be used in this section. Let $S$ be an overring of $R$. If $A$ is an ideal of $R$, then the extension of $A$ to $S$, $A \cdot S$, will be denoted by $A^e$. If $B$ is an ideal of $S$, then the contraction of $B$ to $R$, $B \cap R$, will be denoted by $B^c$.

THEOREM 2. *Suppose that $S$ is an overring of $R$, $w_0 \in B(R)$, and $\{A_i\}$ is the best filtration for $w_0$. If each $A_i$ is a contracted ideal with respect to $S$, then $w_0$ can be extended to $S$.*

*Proof.* Define $B_i = A_i^e$ for each $i$. Then $\{B_i\}$ is a filtration on $S$.

Define a mapping $w$ on $S$ as follows: $w(x) = w_0(A_i)$ if $x \in B_i$, $x \notin B_{i+1}$ and $w(x) = \infty$ if $x \in \cap B_i$. We will show that $w$ is a pseudo-valuation on $S$ which extends $w_0$ to $S$. Property (i) of the definition of pseudo-valuation is obviously satisfied. Suppose that $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Without loss of generality, assume that $i \leqq j$. Then $x - y \in B_i$ and hence, $w(x-y) \geqq w_0(A_i) = \min\{w_0(A_i), w_0(A_j)\} = \min\{w(x), w(y)\}$. Similarly if either $x \in B_i$ for all $i$ or $y \in B_j$ for all $j$, then $w(x-y) \geqq \min\{w(x), w(y)\}$. This proves property (ii).

Finally, we wish to show $w(xy) \geqq w(x) + w(y)$. Again let $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Then $xy \in B_iB_j \subset B_{i+j}$, so that

$$w(xy) \geqq w_0(A_{i+j}) \ .$$

If $w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j})$, then $w(xy) \geqq w(x) + w(y)$. On the other hand, if $w_0(A_i) + w_0(A_j) > w_0(A_{i+j})$, there are two cases to consider. Case 1: Suppose there is a positive integer $t$ such that

$$w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j+t}) \ ,$$

but $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t-1})$. Then $A_iA_j \subset A_{i+j+t}$, and hence $B_iB_j \subset B_{i+j+t}$. Since $xy \in B_iB_j \subset B_{i+j+t}$, $w(xy) \geqq w(x) + w(y)$. Case 2: Suppose that $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t})$ for all $t$. Then $w_0(A_iA_j) > w_0(A_k)$ for all $k$, which implies that $A_iA_j \subset A_k$ for all $k$. Hence,

$$xy \in A_i^eA_j^e \subset (\textstyle\bigcap_{k=1}^{\infty} A_k)^e \subset \bigcap_{k=1}^{\infty} (A_k^e) = \bigcap_{k=1}^{\infty} B_k \ .$$

Therefore $w(xy) = \infty$ and $w(xy) \geqq w(x) + w(y)$. When either $x \in \cap B_k$ or $y \in \cap B_k$, clearly $w(xy) = w(x) + w(y) = \infty$. This proves property (iii), showing that $w$ is a pseudo-valuation on $S$.

It is easy to see that $w$ extends $w_0$. Take $z \in R$. If $z \in A_i$, $z \notin A_{i+1}$ then by Lemma 1, $w_0(z) = w_0(A_i)$. Clearly $z \in B_i$. Suppose $z \in B_{i+1}$, since $z$ is also in $R$, $z \in A_{i+1}{}^{ec} = A_{i+1}$ a contradiction. Therefore $z \notin B_{i+1}$, and hence $w(z) = w_0(A_i)$. If $z \in \cap A_i$, then $z \in \cap B_i$ which implies that $w_0(z) = w(z) = \infty$.

A subring $R$ of a ring $S$ is said to have *property C* with respect to $S$ in case each ideal of $R$ is a contraction of an ideal in $S$. In [1], Gilmer shows that in several cases, if $S$ is an overring of $R$ which is integrally dependent on $R$, then $R$ has property $C$ with respect to $S$. Using Gilmer's theory we obtain several applications of Theorem 2, which are listed in the corollaries below. A *Prüfer domain* is a domain $R$ with identity in which each finitely generated ideal is invertible, or equivalently, in which $R_P$ is a valuation ring for each prime ideal $P$ in $R$. An ideal $A$ of a ring $R$ is called a *valuation ideal* in case there exists a valuation ring $R_v$ containing $R$ and an ideal $B$ of $R_v$ such that $B \cap R = A$.

COROLLARY 1. *Suppose that $R$ is a Prüfer domain with quotient field $K$ and that $R$ is a subdomain of $R_1$. If $R_1 \cap K = R$, then every $w \in B(R)$ can be extended to $R_1$.*

*Proof.* By [1; p. 563, Corollary 2], $R$ has property $C$ with respect to $R_1$. Then each ideal in a best filtration for $w$ is a contracted ideal with respect to $R_1$. By Theorem 2, $w$ can be extended to $R_1$.

COROLLARY 2. *Let $R$ be a domain, let $w \in B(R)$, suppose that $R_1$ is integral over $R$, and let $\{A_i\}$ be the best filtration for $w$. If each $A_i$ is an intersection of valuation ideals of $R$, then $w$ can be extended to $R_1$.*

*Proof.* Apply [1; p. 564, Th. 2] and Theorem 2.

It is known that if $R$ is an integrally closed domain, $A$ is a complete ideal in $R$ if and only if $A$ is the intersection of valuation ideals. Now let $R$ be an integrally closed domain with quotient field $K$, $L$ a finite algebraic extension of $K$, and $R'$ the integral closure of $R$ in $L$. By [1; p. 569, Th. 6] and Theorem 2, we have:

COROLLARY 3. *If $R'$ has an integral basis over $R$ and if $w \in B(R)$, then $w$ can be extended to $R'$.*

THEOREM 3. *Suppose that $R$ is a subring of the ring $S$ and suppose that $w_0$ is a pseudo-valuation on $R$ which has an extension to a pseudo-valuation $w$ on $S$. If $\alpha$ belongs to the set of extended reals, then the ideals $A_\alpha = \{x \in R: w_0(x) > \alpha\}$ and $B_\alpha = \{x \in R: w_0(x) \geqq \alpha\}$ are contractions of ideals of $S$.*

*Proof.* $A_\alpha$ is the contraction of $A'_\alpha = \{x \in S: w(x) > \alpha\}$ and $B_\alpha$ is the contraction of $B'_\alpha = \{x \in S: w(x) \geqq \alpha\}$.

The converse of Theorem 2 is also true.

THEOREM 4. *Let $S$ be an overring of $R$, let $w_0 \in B(R)$, and let $\{A_i\}$ be the best filtration for $w_0$. If $w_0$ can be extended to $S$, then each $A_i$ is a contracted ideal with respect to $S$.*

*Proof.* Apply Theorem 3.

COROLLARY 4. *Suppose $w_0 \in B(R)$ can be extended to some $w$ on $R_v$, where $R_v$ is a valuation ring. Then each ideal in the best filtration of $w_0$ is a valuation ideal.*

REMARK 2. Let $R$ be a domain with quotient field $K$ and $w_0$ a nonnegative pseudo-valuation on $R$. (Nonnegative pseudo-valuations were the most important types of pseudo-valuations studied in [2] and [3]). $w_0$ can always be extended to a nonnegative pseudo-valuation $w$ on a subring $R'$, where $R \subset R' \subset K$, in the following way. Let $M$ be the set of $y \in R$ such that $w_0(y) < \infty$ and $w_0(xy) = w_0(x) + w_0(y)$ for all $x \in R$. Then $M$ is a multiplicative subset of $R$ not containing zero. Hence we can form the quotient ring of $R$ with respect to $M$, $R_M$. A function $w'$ can be defined on $R_M$ by $w'(x/y) = w_0(x) - w_0(y)$. $w'$ is not necessarily nonnegative. However, if $R' = \{z \in R_M: w'(z) \geqq 0\}$ and $w$ is the restriction of $w'$ to $R'$, then $R'$ is a ring and $w$ is an extension of $w_0$ to $R'$. $R'$ is called the *natural domain* of $w_0$. This type of extension was discussed and used in [2].

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