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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS. II

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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS II

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In a previous paper one of the authors considered groups G with r. b. n (representation bound n) and $n < p^2$ for some prime p. Here we continue this study. We first offer a new proof of the fact that if $n = p - 1$ then G has a normal Sylow p-subgroup. Next we show that if $n = p^{3/2}$ then $p^2 \nmid |G/O_p(G)|$. Finally we consider $n = 2p - 1$ and with the help of the modular theory we obtain a fairly precise description of the structure of G. In particular we show that its composition factors are either p-solvable or isomorphic to $PSL(2, p)$, $PSL(2, p - 1)$ for p a Fermat prime or $PSL(2, p + 1)$ for p a Mersenne prime.

Now the irreducible characters of $PSL(2, p)$ have degrees (see [10] p. 128) $1, p, p \pm 1, (p \pm 1)/2$ for p odd and those of $PSL(2, 2^a)$ have degrees (see [10] p. 134) $1, 2^a, 2^a \pm 1$. Thus for $p > 2$ the linear groups of the preceding paragraph do in fact have r.b. $(2p - 1)$.

The notation here is standard. In addition, if χ is a character of G we let $\det \chi$ denote the linear character which is the determinant of the representation associated with χ. Also $n_p(G)$ denotes the number of Sylow p-subgroups of G.

LEMMA 1. *Let G be a group with* r.b.n. *and let $N \neq G$* be a *subgroup. Suppose $G = \bigcup_{i=0}^{t} Nx_iN$ is the (N, N)-double coset decomposition of G with $x_0 = 1$. Set $a_i = |Nx_iN|/|N| = [N: N \cap N^{x_i}]$. Then $n \geqq (a_1 + a_2 + \cdots + a_t)/t$.*

Proof. Let $\theta = (1_N)^G$ be the character of the permutation representation of G on the cosets of N. Then $\theta(1) = [G: N]$, $[\theta, 1_G] = 1$ and $\|\theta\|^2 = 1 + t$. Since $[\theta, 1_G] = 1$ we can write

$$\theta = 1_G + b_1\chi_1 + \cdots + b_s\chi_s$$

where the χ_i are distinct nonprincipal irreducible characters of G. Thus since G has r.b.n we have

$$\begin{aligned} 1 + nt &= 1 + n(\|\theta\|^2 - 1) = 1 + n(b_1^2 + \cdots + b_s^2) \\ &\geqq 1 + n(b_1 + \cdots + b_s) \geqq 1 + b_1\chi_1(1) + \cdots + b_s\chi_s(1) \\ &= \theta(1) = 1 + (a_1 + a_2 + \cdots + a_t) \end{aligned}$$

and the result follows.

LEMMA 2. *Let G be a group with* r.b.n.
(i) Let $N \neq G$ *be a subgroup. Then*

$$n \geqq \min \{[N: N \cap N^x] \mid x \in G - N\} .$$

(ii) *Let π be a set of primes and let H be a maximal* π-subgroup *of G. Then either $H \triangle G$ or $n \geqq \min \{[H: H \cap H^x] \mid x \in G - \boldsymbol{N}(H)\}$.*

Proof. (i) follows immediately from Lemma 1. Now let H be as in (ii) and suppose H is not normal in G. Set $N = \boldsymbol{N}(H) \neq G$. Since H is a maximal π-subgroup it follows that $H = \boldsymbol{O}_\pi(N)$. Thus if $x \in G$ then $H^x = \boldsymbol{O}_\pi(N^x)$ so $H \cap N^x = H \cap H^x$ and

$$[N: N \cap N^x] \geqq [H: H \cap N^x] = [H: H \cap H^x] .$$

Thus the result follows from (i).

Applying Lemma 2(ii) with $\pi = \{p\}$ and H a Sylow p-subgroup of G yields

THEOREM 3. *Let p be a prime and let G be a group with* r.b.$(p-1)$. *Then $n_p(G) = 1$.*

This result was originally proved in [7] (Theorem E) in a much more complicated way.

LEMMA 4. *Let G have* r.b.(p^2-1) *and let Q_1 and Q_2 be p-subgroups of G with $\langle Q_1, Q_2 \rangle$ not a p-group. Then $n_p(\boldsymbol{C}(Q_1) \cap \boldsymbol{C}(Q_2)) = 1$. If further the Sylow p-subgroups of G are abelian, then*

$$n_p(\boldsymbol{N}(Q_1) \cap \boldsymbol{N}(Q_2)) = 1$$

Proof. Set $W = \langle Q_1, Q_2 \rangle$. Since W is not a p-group we see that $n_p(W) > 1$. We assume now that $n_p(C) > 1$ where $C = \boldsymbol{C}(Q_1) \cap \boldsymbol{C}(Q_2) = \boldsymbol{C}(W)$ and derive a contradiction. Set $Z = W \cap C$ so that Z is central in W and C and let $\bar{W} = W/Z$, $\bar{C} = C/Z$. Since Z is central we have easily $n_p(\bar{W}) > 1$, $n_p(\bar{C}) > 1$ and $(WC)/Z = \bar{W} \times \bar{C}$. By Theorem 3 both $\bar{W}$ and $\bar{C}$ have irreducible characters of degree $\geqq p$ and hence $\bar{W} \times \bar{C}$ has an irreducible character of degree $\geqq p^2$. This is a contradiction since G has r.b.(p^2-1) and this property is inherited by subgroups and quotient groups. If the Sylow p-subgroups of G are abelian then any p-group normalizing Q_i centralizes it. Thus the second result follows from the first.

THEOREM 5. *Let p be a prime and let G be a group with* r.b.$p^{3/2}$. *Then $p^2 \nmid |G/\boldsymbol{O}_p(G)|$.*

Proof. If $p = 2$ then G has r.b.2 and the result follows from Theorem C of [7]. Thus we can assume that $p \geqq 3$ and clearly also that $O_p(G) = \langle 1 \rangle$. Since $p^2 - p - 1 \geqq [p^{3/2}]$ for $p \geqq 3$, Proposition 1.3 of [6] implies that a Sylow p-subgroup P of G is abelian. We assume that $|P| \geqq p^2$ and derive a contradiction. Set $n = p^{3/2}$.

Let $N = N(P)$ so that $N \neq G$. By Lemma 2(i) there exists $w \in G - N$ with $n \geqq [N: N \cap N^w] \geqq [P: P \cap P^w]$. Set $Q = P \cap P^w$ so since $p^2 > n$ and $w \notin N$ we see that $[P: Q] = p$ and hence $Q \neq \langle 1 \rangle$. Let $M = N(Q)$. Since $P \triangle N$, $P^w \triangle N^w$ we have $Q \triangle (N \cap N^w)$. Also $Q \triangle P$ and $P \nsubseteq N^w$. Hence $M \cap N \supseteqq \langle P, N \cap N^w \rangle$ so $[N: N \cap M] \leqq n/p = p^{1/2}$.

We now make the following crucial observation. If $[M: M \cap M^x] < p^2$ for some $x \notin M$ then Q and Q^x commute elementwise and $x \in MNM$. To see this suppose that Q and Q^x do not commute. Then since the Sylow p-subgroups of G are abelian, $\langle Q, Q^x \rangle$ is not a p-group. By Lemma 4, $n_p(M \cap M^x) = 1$ so if $U = O_p(M \cap M^x)$ then U is also a Sylow p-subgroup of $M \cap M^x$. Now $p^2 \nmid [M: M \cap M^x]$ and $Q \nsubseteq M^x$ clearly so QU is a Sylow p-subgroup of M. Since $N_M(QU) \supseteqq \langle Q, M \cap M^x \rangle$ we have $[M: N_M(QU)] < p$ and hence by Sylow's theorem $QU \triangle M$ and $n_p(M) = 1$. This is a contradiction since $Q = P \cap P^w$ and $P, P^w \subseteqq M$. Thus Q and Q^x commute and since $Q \neq Q^x$ and $[P: Q] = p$ it follows that $QQ^x = P^{y^{-1}}$ is a Sylow p-subgroup of G. Thus Q, Q^y and Q^{xy} are all contained in P. By Burnside's lemma these three groups are conjugate in N. Thus $Q^y = Q^h$, $Q^{xy} = Q^k$ for some $h, k \in N$. This yields $yh^{-1} \in M$, $xyk^{-1} \in M$ so

$$x = (xyk^{-1})kh^{-1}(yh^{-1})^{-1} \in MNM .$$

Since $Q \neq \langle 1 \rangle$ we have $M \neq G$. Let $G = \bigcup_{i=0}^{t} Mx_iM$ be the (M, M)-double coset decomposition of G with $x_0 = 1$. Set $a_i = |Mx_iM|/|M|$ and suppose that there are precisely r such $i \neq 0$ with $a_i < p^2$ and s with $a_i \geqq p^2$. Then by Lemma 1, $p^{3/2} = n \geqq (r + p^2s)/(r + s)$. Clearly $r \neq 0$ here so

$$p^{3/2} \geqq (r + p^2s)/(r + s) > p^2/(1 + r/s) .$$

If $s = 0$ then by the preceding paragraph Q commutes with all its conjugates. This implies that $\langle Q^x \mid x \in G \rangle$ is a nontrivial normal p-subgroup of G, a contradiction. Thus $s \geqq 1$. Also if $a_i < p^2$ then $Mx_iM \subseteqq MNM$ by the above. Since we have seen that $[N: N \cap M] \leqq p^{1/2}$ we have $r \leqq p^{1/2} - 1$ since the double coset M itself is not counted. Thus $r/s \leqq p^{1/2} - 1$ and

$$p^{3/2} > p^2/(1 + r/s) \geqq p^{3/2}$$

a contradiction and the result follows.

We now turn to the main result of this paper.

THEOREM 6. *Let p be a prime and let G be a nonabelian simple group with* r.b.$(2p-1)$. *Then $p > 2$ and we have one of the following.*

(i) *G is a p'-group.*

(ii) $G \cong PSL(2, p)$ for $p > 3$.

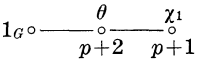
(iii) $G \cong PSL(2, p-1)$ *for p a Fermat prime,* $p > 3$.

(iv) $G \cong PSL(2, p+1)$ *for p a Mersenne prime.*

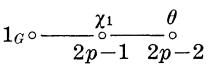
Proof. Since groups with r.b.3 are solvable (Corollary 6.5 of [8]) we have $p > 2$. By Theorem 5 since $p^{3/2} \geqq 2p-1$ for $p > 2$ we have $p^2 \nmid |G|$. If $p \nmid |G|$ then G satisfies (i) above. Thus we can assume now that G has a Sylow p-subgroup P of order p. Let $B_1(p)$ denote the principal p-block of G. We will use freely the structure of $B_1(p)$ and its associated tree as described in [1] and [2]. Since $B_1(p)$ contains a nonprincipal irreducible character χ and $\chi(1) < 2p$, Lemma 1 of [3] implies that P is self centralizing. Let $N = N(P)$ and let $e = |N/P|$ so that $e \mid (p-1)$. By Burnside's transfer theorem $e > 1$. We assume now that $G \ncong PSL(2, p)$ and $G \ncong PSL(2, p-1)$ for p a Fermat prime.

Step 1. Let θ denote an exceptional character in $B_1(p)$. Then the tree of $B_1(p)$ must be one of the following.

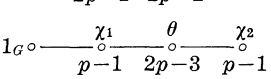
(1) $e = 2$



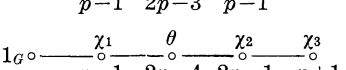
(2) $e = 2$



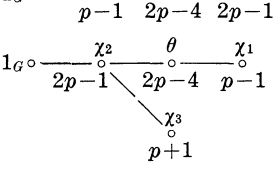
(3) $e = 3$



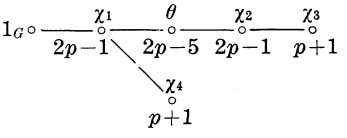
(4) $e = 4$



(5) $e = 4$



(6) $e = 5$



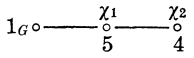
.

Here the degree of the character is written below the character designation.

Since G is nonabelian and simple it follows that every nonprincipal irreducible representation of G is faithful. Thus by Theorem 2 of [5] the degrees of the nonprincipal ordinary irreducible characters of G are all $\geqq p-1$ and by Theorem 1 of [4] the degrees of the nonprincipal

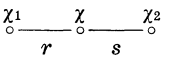
irreducible Brauer characters of G are $\geqq 2(p-1)/3$. Now G has r.b.$(2p-1)$ and the degrees of the ordinary irreducible characters of $B_1(p)$ satisfy $\chi(1) \equiv \pm 1(p)$ if χ is nonexceptional and $\theta(1) \equiv \pm e(p)$. This yields easily $\chi(1) = p-1, p+1, 2p-1$ or $\chi = 1_G$ and $\theta(1) = 2p-e, p+e$ or $\theta(1) = e$ if $e = p-1$.

Suppose first that $p = 3$. Since $e \mid (p-1)$ we have $e = 2$ and the tree is a line with three vertices. Now the center degree is maximal and the principal character must occur so we have clearly



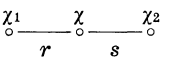
This is either tree (1) or tree (2) according to which of χ_1 of χ_2 we consider exceptional.

Now assume that $p \geqq 5$. Let $\chi \in B_1(p)$ with $\chi(1) = p-1$ and suppose that χ is modular reducible. We can denote this latter fact graphically by

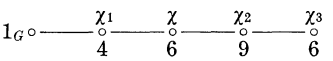


where r and s are degrees of modular constituents. Since $r + s \leqq p-1$ we cannot have both $r, s \geqq 2(p-1)/3$. This implies that one of the modular constituents is the principal character and the other has degree $p-2$.

Now let $\chi \in B_1(p)$ with $\chi(1) = p+1$ and suppose χ is modular reducible. Say we have



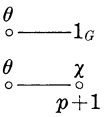
Here by the alternating nature of the tree $\chi_1, \chi_2 \neq 1_G$ so $r, s \geqq 2(p-1)/3$. Since $r + s \leqq p+1$ this yields $4(p-1)/3 \leqq p+1$ so $p = 5$ or 7. In fact either $p = 5, r = s = 3$ or $p = 7, r = s = 4$. We consider these in turn. Let $p = 5$. Since $e \mid (p-1)$ we have $e = 2$ or 4. Certainly $e \neq 2$ here since $B_1(p)$ contains $e+1$ ordinary irreducible characters including 1_G and hence $e = 4$. Thus the degrees of the nonprincipal ordinary irreducible characters of $B_1(p)$ are 4, 6 or 9 and the nonprincipal Brauer characters have degrees $\geqq 3$. The former fact implies that χ_1 and χ_2 are modular reducible so the tree having five vertices is a straight line. This yields easily



and we obtain tree (4). Now let $p = 7$ so that $e = 2, 3$ or 6. Clearly neither χ_1 nor χ_2 can be modular irreducible so the tree has at least five vertices and hence $e = 6$. Thus the degrees of the nonprincipal

ordinary irreducible characters of $B_1(p)$ are 6, 8 or 13 and the non-principal Brauer characters have degrees $\geqq 4$. This latter fact implies that neither χ_1 nor χ_2 can have degree 6 and hence $\chi_1(1) = \chi_2(1) = 13$. Now at least one of χ_1 or χ_2, say χ_1, is not adjacent to 1_G. Hence all characters adjacent to χ_1 have degree $p + 1$ and as we have seen these are either modular irreducible or have constituents of degree 4. This shows that all modular constituents of χ_1 have degree divisible by 4, a contradiction since $\chi_1(1) = 13$. Thus this case does not occur. It now clearly suffices to assume for the remainder of this step that all $\chi \in B_1(p)$ with $\chi(1) = p + 1$ are modular irreducible.

Let θ denote an exceptional character of $B_1(p)$ and we consider the possible branches leaving the vertex associated with θ. Suppose first that $\theta(1) \equiv +e(p)$. The above implies easily that we can only have

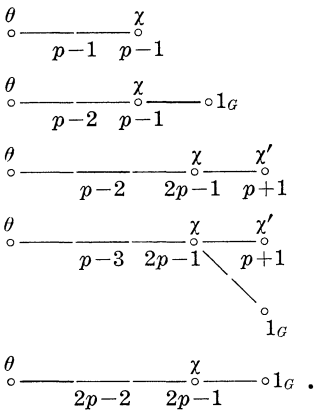


Now the first branch must occur precisely once and let the second branch occur a times. Since the tree has e edges we have

$$1 + a = e$$
$$1 + a(p + 1) = \theta(1) .$$

Now $e \geqq 2$ so $a \geqq 1$ and hence $\theta(1) > p$. Thus $\theta(1) = p + e$ and we obtain $a = 1, e = 2$ and this is tree (1).

Now let $\theta(1) \equiv -e(p)$ so that $\theta(1) = 2p - e$. Using the above information and the alternating nature of the tree we see easily that the only possible branches leaving the vertex associated with θ are



If the last branch occurs then since $2p - e = \theta(1) \geqq 2p - 2$ we have $e = 2$ and this is tree (2). Thus we can assume that only the first

four branches occur say with multiplicities a, b, c, d respectively. Since there are precisely e edges in the tree we have

$$a + 2b + 2c + 3d = e$$
$$a(p-1) + b(p-2) + c(p-2) + d(p-3) = \theta(1) = 2p - e .$$

Adding these two and dividing by p yields

$$a + b + c + d = 2 .$$

In addition the vertex of 1_G occurs precisely once so $b + d = 1$. Thus $a + c = 1$ and there are four possibilities which are easily seen to be trees (3), (4), (5) and (6).

Step 2. Let $N = N(P)$. We consider the restriction of the ordinary irreducible characters of G to N.

Now $N = PE$ is a Frobenius group of order pe with $E = \langle x \rangle$ cyclic of order e. N has precisely e linear characters, namely those of $N/P \cong E$, and the remaining irreducible characters have degree e. Let Δ denote any sum of irreducible characters of N of degree e. Clearly $\Delta_E = \Delta(1)/e \cdot \rho_E$ where ρ_E is the regular character of E. This yields easily

$$\Delta(x) = 0$$
$$\det \Delta(x) = \begin{cases} 1 & \text{if } e \text{ is odd} \\ (-1)^{\Delta(1)/e} & \text{if } e \text{ is even} . \end{cases} \tag{1}$$

If e is even let δ denote the linear character of N given by $\delta(x) = -1$.

Let ψ be an ordinary irreducible character of G with $\psi \notin B_1(p)$. Since P is self centralizing it follows that $B_1(p)$ is the unique p-block of positive defect and hence ψ belongs to a block of defect 0. Thus $p \mid \psi(1)$. Since G has r.b.$(2p-1)$ this yields $\psi(1) = p$ and clearly $\psi_P = \rho_P$. Thus $\psi_N = \lambda + \Delta$ where λ is linear and $\Delta(1) = p - 1$. Now G is simple so the linear character $\det \psi$ must be principal and hence $1 = \det \psi(x) = \lambda(x) \det \Delta(x)$. This yields by the above

$$\psi_N = \begin{cases} 1_N + \Delta & \text{if } e \text{ is odd} \\ 1_N + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is even} \\ \delta + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is odd} . \end{cases} \tag{2}$$

Now let $\chi \in B_1(p)$ and let $m(\chi)$ denote the number of linear characters counting multiplicities which occur in χ_N. Obviously $m(\chi)$ is the multiplicity of 1_P in χ_P. Suppose first that χ_i, χ_j are nonexceptional ordinary irreducible characters of $B_1(p)$ which are adjacent in the tree. Then $\chi_i + \chi_j = \Phi$ is a projective character and since $\Phi_P = \Phi(1)/p \cdot \rho_P$ we have easily

$$m(\chi_i) + m(\chi_j) = [\chi_i(1) + \chi_j(1)]/p .$$

Now suppose that χ_i is adjacent to the vertex of the exceptional characters $\{\theta_k\}$. Clearly $m(\theta_r) = m(\theta_s)$ for all r, s so since $\chi_i + \sum_{k=1}^{(p-1)/e} \theta_k = \Phi$ we have with $\theta = \theta_1$

$$m(\chi_i) + (p-1)/e \cdot m(\theta) = [\chi_i(1) + (p-1)/e \cdot \theta(1)]/p.$$

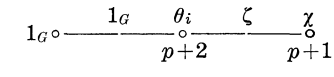
Using the above two equations, the fact that the tree of $B_1(p)$ is connected and the obvious fact that $m(1_G) = 1$ we obtain easily for irreducible χ

$$(3) \qquad m(\chi) = \begin{cases} k-1 & \text{if } \chi(1) = kp - 1 \\ k & \text{if } \chi(1) = kp \pm e \\ k+1 & \text{if } \chi(1) = kp + 1 . \end{cases}$$

There is of course additional information available, for example the fact that $\det \chi(x) = 1$ and the position of χ in the tree, which further limits the structure of χ_N.

Step 3. If tree (1) occurs in step 1, then p is a Mersenne prime and $G \cong PSL(2, p+1)$.

By assumption $e = 2$ and the tree of $B_1(p)$ has the form



where θ_i represents the $(p-1)/2$ exceptional characters with $\theta = \theta_1$. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of irreducible characters of G not in $B_1(p)$.

Let α denote the unique nonprincipal linear character of N. By (1) and (2) we have

$$(4) \qquad \begin{aligned} \psi_{jN} &= 1_N + \Delta'_j , \quad \psi_j(x) = 1 \quad \text{for } p \equiv 1 \ (4) \\ \psi_{jN} &= \alpha + \Delta'_j , \quad \psi_j(x) = -1 \quad \text{for } p \equiv 3 \ (4) . \end{aligned}$$

By (3), $\theta_{iN} = \lambda + \Delta_i$ and since the θ_i are all algebraically conjugate λ is the same for all i. Thus $\det \theta_i(x) = 1$ and equation (1) yield

$$(5) \qquad \begin{aligned} \theta_{iN} &= \alpha + \Delta_i , \quad \theta_i(x) = -1 \quad \text{for } p \equiv 1 \ (4) \\ \theta_{iN} &= 1_N + \Delta_i , \quad \theta_i(x) = 1 \quad \text{for } p \equiv 3 \ (4) . \end{aligned}$$

Now $\chi_N = a1_N + b\alpha + \Delta$ with $a + b = 2$ by (3) and since x is a p'-element $a - b = \chi(x) = \zeta(x) = \theta(x) - 1$. Thus (5) yields

$$(6) \qquad \begin{aligned} \chi_N &= 2\alpha + \Delta , \quad \chi(x) = -2 \quad \text{for } p \equiv 1 \ (4) \\ \chi_N &= 1_N + \alpha + \Delta , \quad \chi(x) = 0 \quad \text{for } p = 3 \ (4) . \end{aligned}$$

Equations (4), (5) and (6) and Frobenius reciprocity now yield

$$(7)\qquad \begin{aligned} (1_N)^G &= 1_G + \sum_1^k \psi_j\,, & \alpha^G &= 2\chi + \sum_1^{(p-1)/2} \theta_i \quad \text{for } p \equiv 1\ (4)\\ (1_N)^G &= 1_G + \chi + \sum_1^{(p-1)/2} \theta_i\,, & \alpha^G &= \chi + \sum_1^k \psi_j \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Thus since $(1_N)^G(1) = \alpha^G(1) = [G: N]$ and $|N| = 2p$ we obtain easily

$$(8)\qquad \begin{aligned} |G| &= p(p^2 + 5p + 2)\,, \quad k = (p+5)/2 \quad \text{for } p \equiv 1\ (4)\\ |G| &= p(p+1)(p+2)\,, \quad k = (p+1)/2 \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Now using $|C(x)| = \sum \eta(x)\overline{\eta(x)}$, where η runs over all ordinary irreducible characters of G, along with equations (4), (5), (6) and (8) we have

$$(9)\qquad \begin{aligned} |C(x)| &= p + 7 \quad \text{for } p \equiv 1\ (4)\\ |C(x)| &= p + 1 \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Since $(p+7) \nmid p(p^2 + 5p + 2)$ the case $p \equiv 1\ (4)$ is eliminated. Thus $p \equiv 3\ (4)$.

Set $S = C(x)$ so that $|S| = p + 1$ and $[G: S] = p(p+2)$. We consider $(1_S)^G$. Since this character is rational and the θ_i are algebraically conjugate we have

$$(1_S)^G = 1_G + a\sum_1^{(p-1)/2} \theta_i + b\chi + \sum_1^k c_j\psi_j\,.$$

Set $c = \sum_1^k c_j$. By considering degrees we have

$$p(p+2) = 1 + a(p+2)(p-1)/2 + b(p+1) + cp$$

and evaluating at x yields

$$0 < (1_S)^G(x) = 1 + a(p-1)/2 - c$$

by (4), (5), (6) and the fact that $x \in S$. Certainly $a \leqq 2$. Also $b \leqq \chi(1)/2 = (p+1)/2$ by Frobenius reciprocity and the fact that $\chi(x) = 0$. Thus $a = 0$ yields a contradiction. If $a = 1$ then $b \equiv 0\ (p)$ so $b = 0$ and $c = (p+3)/2 > 1 + a(p-1)/2$ again a contradiction. Thus $a = 2$ and we have easily

$$(10)\qquad (1_S)^G = 1_G + 2\sum_1^{(p-1)/2} \theta_i + \chi$$

so $(1_S)^G(x) = p$ by (5) and (6). By definition of induced character and the fact that $S = C(x)$ this implies that S contains precisely p distinct conjugates of x. Since $|S| = p + 1$ this shows that S is an elementary abelian 2-group and therefore that S is a Sylow 2-subgroup of G and p is a Mersenne prime. By Burnside's lemma the nonidentity elements of S are all conjugate in $N(S)$ so $N(S) > S$.

Set $H = N(S) > S$. Then $(1_H)^G$ is a national constituent of $(1_S)^G$ and $(1_H)^G(1) \leqq p(p+2)/3$. Thus by (10) we have easily

$$(1_H)^G = 1_G + \chi \,. \tag{11}$$

Therefore G is a doubly transitive permutation group on the set Ω where $H = G_\infty$ for some point $\infty \in \Omega$. By (10) χ_S contains 1_S with multiplicity one so (11) implies that S has two orbits on Ω. Hence since $|S| = |\Omega| - 1$, S is in fact a regular normal subgroup of G_∞. Now $|H| = p(p+1)$ so if $\tilde{P}$ is a Sylow p-subgroup of H, then since $\tilde{P}$ is self centralizing and $|\tilde{P}| = |S| - 1$ we see that G is sharply 3-transitive.

With the structure of H as given above we can clearly identify Ω with $GF(p+1) \cup \{\infty\}$ in such a way that S is the set of translations $\left\{\begin{pmatrix} z \\ z+r \end{pmatrix} \middle| r \in GF(p+1)\right\}$ and $\tilde{P}$ is the set $\left\{\begin{pmatrix} z \\ sz \end{pmatrix} \middle| s \in GF(p+1),\ s \neq 0\right\}$. Let $\tilde{x} \in G$ with $\tilde{x} = (0\ \infty)(1) \cdots$. Then $\tilde{x}$ has order 2 and normalizes $\tilde{P} = G_{\infty 0}$ so $\tilde{x}$ acts in a dihedral manner on $\tilde{P}$. If $\tilde{x} = \begin{pmatrix} z \\ f(z) \end{pmatrix}$ then for all $s \in GF(p+1)$, $s \neq 0$

$$\begin{pmatrix} z \\ sz \end{pmatrix}\begin{pmatrix} z \\ f(z) \end{pmatrix} = \begin{pmatrix} z \\ f(z) \end{pmatrix}\begin{pmatrix} z \\ s^{-1}z \end{pmatrix}$$

so $f(sz) = s^{-1}f(z)$. Setting $z = 1$ yields $f(s) = s^{-1}$. Thus $\tilde{x} = \begin{pmatrix} z \\ 1/z \end{pmatrix}$ and since $G = \langle H, \tilde{x} \rangle$ we have clearly $G \supsetneq PSL(2, p+1)$. By (8) we have in fact $G \cong PSL(2, p+1)$ and this step follows.

Step 4. Completion of the proof.

We now consider the remaining trees in turn. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of ordinary irreducible characters of G not in $B_1(p)$.

Suppose first that we have tree (2). If $p = 3$ this is the same as tree (1) so we assume that $p > 3$. From

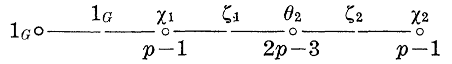
$$1_G \circ \overset{1_G}{\text{———}} \underset{2p-1}{\circ} \overset{\chi}{\text{———}} \overset{\zeta}{\text{———}} \underset{2p-2}{\circ} \overset{\theta_i}{}$$

and (3) and $\det \chi(x) = 1$ we have $\chi_N = 1_N + \Delta$. Let α be the unique nonprincipal linear character of N so that we have by (3) $\theta_{iN} = a1_N + b\alpha + \Delta_i$ with $a + b = 2$. Since x is a p'-element $a - b = \theta_i(x) = \zeta(x) = \chi(x) - 1 = 0$ so $\theta_{iN} = 1_N + \alpha + \Delta_i$. Now by (2) all the ψ_j occur in either $(1_N)^G$ or α^G depending on the parity of $(p-1)/2$. Since $(1_N)^G(1) = \alpha^G(1)$, the above and Frobenius reciprocity imply that the ψ_j occur in α^G and hence

$$(1_N)^G = 1_G + \chi + \sum_1^{(p-1)/2} \theta_i \,.$$

Since $|N| = 2p$ this yields $|G| = 2p(p^2 + 1)$. Now $\theta_i(1) \mid |G|$ so $(p - 1) \mid (p^2 + 1)$ and this is easily seen to be a contradiction for $p > 3$.

Now consider tree (3)



By (3), $\chi_{iN} = \Delta_i$ and $\chi_i(x) = 0$ for $i = 1, 2$. This implies that $\zeta_1(x) = -1$, $\zeta_2(x) = 0$ so $\theta_i(x) = -1$. Now by (3), $\theta_{iN} = a1_N + b\alpha + c\alpha^2 + \Delta'_i$ where α is a nonprincipal linear character and $a + b + c = 2$. Since $\theta_i(x) = -1$ we have easily $\theta_{iN} = \alpha + \alpha^2 + \Delta'_i$. Applying Frobenius reciprocity to the above and (2) we have

$$(1_N)^G = 1_G + \sum_1^k \psi_j\ , \qquad \alpha^G = \sum_1^{(p-1)/3} \theta_i$$

and this yields easily

$$|G| = p(p-1)(2p-3)\ , \qquad k = (2p-5)/3\ .$$

Using $|C(x)| = \sum \bar{\eta}(x)\eta(x)$ along with the above and (2) we obtain $|C(x)| = p - 1$. Now clearly x is a real element so $|C^*(x)| = 2(p-1)$ where $C^*(x) = \{g \in G \mid x^g = x \text{ or } x^{-1}\}$. Since $2(p-1)$ does not divide $|G|$ as given above, it follows that this tree does not occur.

Suppose tree (4) or (5) occurred. Since $\chi_1(1) = p - 1$ and $\det \chi_1(x) = 1$, (1) and (3) imply that $(p-1)/4$ is even. Hence by (2), $\psi_{jN} = 1_N + \Delta_j$. Now there are four linear characters of N and at most two occur in θ_N so choose $\alpha \neq 1_N$ such that α does not occur in θ_N. Thus α can occur only in χ_{2N} or χ_{3N} with multiplicity at most two. Hence

$$[G\colon N] = \alpha^G(1) \leqq 2\chi_2(1) + 2\chi_3(1) = 6p\ .$$

Now choose β so that β occurs in θ_N. Then

$$[G\colon N] = \beta^G(1) \geqq \sum_1^{(p-1)/4} \theta_i(1) = (2p-4)(p-1)/4\ .$$

Since $(p-1)/4$ is even and $(p-1)/4 \neq 2$ we have $(p-1)/4 \geqq 4$, $p \geqq 17$ and

$$6p \geqq \alpha^G(1) = \beta^G(1) \geqq 4(2p-4)\ ,$$

a contradiction.

Finally consider tree (6). By (1), (2) and (3) we have easily $\chi_{1N} = 1_N + \Delta$, $\chi_{2N} = 1_N \Delta'$, and $\psi_{jN} = 1_N + \Delta_j$. Now since $e = 5$ and $m(\theta) = 2$ we can choose a linear character α of N with $\alpha \neq 1_N$ and such that α does not occur in θ_N. Hence by the above and the fact that $m(\chi_3) = m(\chi_4) = 2$ we have $\alpha^G = a\chi_3 + b\chi_4$ with $a, b \leqq 2$. Thus since

$$[G : N] = \alpha^G(1) = a\chi_3(1) + b\chi_4(1) = (a + b)(p + 1)$$

and $[G:N] \equiv 1\ (p)$ we have $[G:N] = p+1$. Now choose β so that β occurs in θ_N. Then

$$p+1 = [G:N] = \beta^G(1) \geqq \sum_{1}^{(p-1)/5} \theta_i(1) = (2p-5)(p-1)/5\ ,$$

a contradiction since $5 \mid (p-1)$ implies that $p \geqq 11$. This therefore completes the proof of the theorem.

Finally we consider the remaining groups with r.b.$(2p-1)$.

THEOREM 7. *Let p be a prime and let G be a group with* r.b.$(2p-1)$. *Then we have one of the following.*

(i) *G has a normal abelian Sylow p-subgroup.*
(ii) *G is solvable and has p-length* 1.
(iii) *$G/\boldsymbol{Z}(G) \cong PSL(2,p)$ or $PGL(2,p)$ for $p > 3$.*
(iv) *$G/\boldsymbol{Z}(G) \cong PSL(2,p-1)$ for p a Fermat prime, $p > 3$.*
(v) *$G/\boldsymbol{Z}(G) \cong PSL(2,p+1)$ for p a Mersenne prime.*
(vi) *$G/\boldsymbol{Z}(G) \cong$* Sym (4) for $p = 2$.

Proof. If $p = 2$ then G has r.b.3. Thus by Corollary 6.5 of [8], G satisfies (ii) or (vi) above. Now let $p > 2$. Since $2p-1 \leqq p^{3/2}$, Theorem 5 implies that $p^2 \nmid |G/\boldsymbol{O}_p(G)|$. With this additional fact it is easy to see that the proof of the main theorem of [6] applies also to groups with r.b.$(2p-1)$ with $p > 2$ yielding the same conclusion. (The $p > 2$ assumption is used crucially in the last paragraph of the proof of Proposition 3.1 of [6].) Thus either G satisfies (i) or (ii) above or $G = P_1 \times G_1$ where P_1 is an abelian p-group and $p^2 \nmid |G_1|$. Clearly G_1 has r.b.$(2p-1)$ and if G_1 satisfies any of the above then so does G. Therefore it suffices to assume that $G = G_1$ or equivalently that $p^2 \nmid |G|$. We assume now that G does not satisfy (i). This of course implies that $p \,||\, |G|$.

Let $K = \boldsymbol{O}_{p'}(G)$ and let H/K be a minimal normal subgroup of G/K. Then $p \,||\, |H/K|$ and since $p^2 \nmid |G/K|$ this implies that H/K is the unique minimal normal subgroup. Now H/K is a product of isomorphic simple groups and $p^2 \nmid |H/K|$ so H/K is simple. If $|H/K| = p$ then G is p-solvable of p-length 1. Thus since G does not have a normal Sylow p-subgroup, Proposition 2.3 of [6] implies that G is solvable and G satisfies (ii). Hence it suffices to assume that $\bar{H} = H/K$ is a nonabelian simple group. It is convenient to first consider the possibility $p \geqq 5$.

Since $\bar{H}$ is the unique minimal normal subgroup of $\bar{G} = G/K$ we have $\boldsymbol{C}_{\bar{G}}(\bar{H}) = \langle 1 \rangle$ and thus $\bar{G} \subseteq \text{Aut}\, \bar{H}$. Suppose $\bar{T}$ is a subgroup of $\bar{H}$ with $1 < [\bar{H}:\bar{T}] < 2p$. Since $\bar{H}$ is simple and $p \,||\, |\bar{H}|$ we cannot have $[\bar{H}:\bar{T}] < p$. Thus $p \leqq [\bar{H}:\bar{T}] < 2p$ and $\bar{T}$ is maximal in $\bar{H}$ and hence self normalizing. If $\bar{T}$ were abelian it would follow easily that

$\bar{T}$ is a T.I. set and then $\bar{H}$ is a simple Frobenius group, a contradiction. Thus $\bar{T}$ is nonabelian.

Let ψ be an irreducible character of K and let χ be an irreducible constituent of ψ^H. If $e = [\chi_K, \psi]_K$ then $\chi(1) = et\psi(1)$ where $t = [H: T]$ and T is the inertial group of ψ in H. Suppose $T < H$ and set $\bar{T} = T/K$. Since $\chi(1) < 2p$ we have $t < 2p$ and thus by the remarks of the preceding paragraph $t \geqq p$ and $\bar{T}$ is nonabelian. Thus we have $2p > \chi(1) = et\psi(1) \geqq ep\psi(1)$ so $e = \psi(1) = 1$. Now there exists an irreducible character η of T with $\eta^H = \chi$ and $\eta_K = e\psi = \psi$. Since $\bar{T}$ is nonabelian we can choose a nonlinear irreducible character β of T containing K in its kernel. Thus since η is linear, $\eta_0 = \eta\beta$ is also an irreducible character of T. Let χ_0 be an irreducible constituent of η_0^H. Then $\beta(1)\psi = \eta_{0K}$ occurs in χ_{0K} and therefore $[\chi_{0K}, \psi] \geqq \beta(1) > 1$. The above reasoning applied to χ_0 now yields a contradiction. Thus $H = T$ and H fixes all irreducible characters of K. By Brauer's lemma, H fixes all conjugacy classes of K. Let P be a Sylow p-subgroup of H. Then P fixes each class of K and since K is a p'-group, P centralizes K. Thus if $C = \boldsymbol{C}_H(K)$ then $KC > K$ and since H/K is simple we have $H = KC$. Now $C/(C \cap K) \cong \bar{H}$ and $\boldsymbol{Z}(C) \supseteq C \cap K$ so $\boldsymbol{Z}(C) = C \cap K$.

Let D denote the last term in the derived series of C. Then clearly $D = D'$, $D/\boldsymbol{Z}(D) \cong \bar{H}$ and $Z = \boldsymbol{Z}(D) = D \cap K$. Thus Z is a homomorphic image of the Schur multiplier of $\bar{H}$. By Theorem 6, $\bar{H} \cong PSL(2, p)$, $PSL(2, p-1)$ for p a Fermat prime or $PSL(2, p+1)$ for p a Mersenne prime. We have by assumption $p \geqq 5$. Also for $p = 5$, $PSL(2, p) \cong PSL(2, p-1)$ and we will view this group as $PSL(2, p)$. By [10] (Satz IX, p. 119) either $Z = \langle 1 \rangle$ or $\bar{H} \cong PSL(2, p)$, $D \cong SL(2, p)$ and $|Z| = 2$.

We show now that K is central. Suppose first that $Z = \langle 1 \rangle$ so that $H \cong D \times K$. Let χ be a fixed irreducible character of D with $\chi(1) = p$ and let λ be an irreducible character of K. Then $\chi\lambda$ is an irreducible character of H so $2p > \chi(1)\lambda(1) = p\lambda(1)$ and $\lambda(1) = 1$. Thus K is abelian and central in H. If K is not central in G, then some linear character λ of K is not fixed by G. This implies easily that if θ is a constituent of $(\chi\lambda)^G$ then $\theta(1) \geqq 2\chi(1) = 2p$, a contradiction. Thus K is central in G in this case. Now let $Z \neq \langle 1 \rangle$ so that $|Z| = 2$ and $D \cong SL(2, p)$. We have an epimorphism $D \times K \to DK = H$ where the kernel is the third subgroup W of order 2 in the group generated by the copies of Z in D and K. Let λ be an irreducible character of K. Since Z is central in K and $|Z| = 2$ it is easy to see from the character table of $SL(2, p)$ ([10], p. 128) that there exists an irreducible character χ of D with $\chi(1) \geqq p$ and with W in the kernel of $\chi\lambda$, an irreducible character of $D \times K$. Thus $\chi\lambda$ is a character of H. The preceding argument now shows first that K is abelian and then that K is central. We have therefore shown that $G/\boldsymbol{Z}(G) \cong \bar{G}$ and it remains

to identify $\bar{G}$.

Now $\bar{G} \supsetneq \text{Aut}\, \bar{H}$ and $\bar{H}$ is a 2-dimensional projective group so the possibilities for $\bar{G}$ are given by Satz 1 of [9]. Suppose first that $\bar{H} \cong PSL(2, p)$. Then either $\bar{G} \cong PSL(2, p)$ or $\bar{G} \cong PGL(2, p)$ and we have (iii). Note the fact that $PGL(2, p)$ has $r.b.(2p - 1)$ can be seen from the character table on page 136 of [10]. We consider the remaining two cases. Thus $\bar{H} \cong PSL(2, s)$ with $2^n = s = p \pm 1$ and $\bar{G}/\bar{H}$ is isomorphic to a subgroup of the Galois group of $GF(2^n)/GF(2)$, a cyclic group of order n. Suppose $\bar{G} > \bar{H}$ and let $t \in \bar{G}$ correspond to a nontrivial field automorphism $x \to x^j$. Then in the notation of page 134 of [10], but replacing upper case by lower case letters, we have $t^{-1}at = a^j \neq a$. Since $s > 4$ by our assumption for $p = 5$ it follows easily that $a^j \neq a^{-1}$ so a^j is not conjugate to a in $\bar{H}$. From the character table of $\bar{H}$ we now see easily that t moves some irreducible character of $\bar{H}$ of degree $s + 1$ and thus $\bar{G}$ has an irreducible character of degree at least $2(s + 1) \geqq 2p$, a contradiction. Hence $\bar{G} = \bar{H}$ and G satisfies (iv) or (v). This completes the proof of the theorem for $p \geqq 5$.

Finally let $p = 3$. Since $\bar{H}$ is a nonabelian simple group with $r.b.(2p - 1)$, $\bar{H} \cong PSL(2, 4) \cong PSL(2, 5)$ by Theorem 6. Certainly G is not 5-solvable and G has $r.b.(2 \cdot 5 - 1)$. Thus by the prime 5 case already proved, $G/\boldsymbol{Z}(G) \cong PSL(2, 5)$ or $PGL(2, 5)$. Since the latter group has an irreducible character of degree $6 > 2p - 1$ we have $G/\boldsymbol{Z}(G) \cong PSL(2, 5) \cong PSL(2, p + 1)$ and G satisfies (v). Thus the result follows.

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