Pacific Journal of Mathematics

RANK k GRASSMANN PRODUCTS

Marion-Josephine Lim

Vol. 29, No. 2 June 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

RANK k GRASSMANN PRODUCTS

M. J. S. LIM

The general question concerning the structure of subspaces of a symmetry class of tensors in which every nonzero element has an irreducible representation as a sum of decomposable (or pure) elements of a given length is as yet largely unanswered. This problem relates to the problem of characterizing the linear transformations on such a symmetry class which map the set of tensors of "irreducible length" k into itself; i.e., preserves the *rank* k of the tensors. Another related problem is: "Is it possible to obtain algebraic relations involving the components of a tensor which imply it has rank ("Irreducible length") k, for any positive integer k"?

This paper is concerned mostly with the third question for the $\binom{n}{r}$-dimensional Grassmann Product Space $\wedge^r U$, where U is an n-dimensional vector space over a field F. It includes some discussion of the first question for F *algebraically closed* and $r = 2$.

A vector in $\wedge^r U$ is said to have *rank* k if it can be expressed as the sum of k, and *not less than* k, nonzero pure r-vectors in $\wedge^r U$. We denote the set of such vectors by $C_k^r(U)$. The nonzero pure products in $\wedge^r U$ have rank one.

The results obtained in this paper are as follows: (i) the rank of a vector in $\wedge^r U$ is unchanged if we extend U, (ii) in the Grassmann Algebra $\wedge^0 U + \wedge^1 U + \cdots + \wedge^r U + \cdots$, multiplication of a Grassmann product by a nonzero vector in U either annihilates it or preserves its rank, (iii) we can associate with each vector z in $C_k^r(U)$ a unique subspace $U(z)$ in U, (iv) if $z \in C_k^r(U)$ and dim $U(z)$ is rk, then z has rank k, (v)$x_1 \wedge y_1 + \cdots + x_s \wedge y_s \in C_s^2(U)$ if and only if $\{x_1, y_1, \cdots, x_s, y_s\}$ is independent. Finally, we discuss the rank two subspaces in $\wedge^2 U$ when dim $U = 4$. If F is *algebraically closed*, these subspaces are of dimension one. Otherwise, they can be different, as the examples show.

In this paper, $Q(k, t, n)$ will denote the totality of strictly increasing sequences of k integers chosen from $t, t+1, \cdots, n$; $S(k, t, n)$ the totality of sequences of k integers chosen from $t, t+1, \cdots, n$.

Let $x_1, \cdots, x_n$ be a basis of U. For $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$, we denote the product $x_{i_1} \wedge \cdots \wedge x_{i_r}$ by $\boldsymbol{x}_\omega$.

Let p be an r-linear alternating function from $\pi_{i=1}^r E \rightarrow F$, $E = \{1, \cdots, n\}$.

We will need the following known result.

THEOREM 1. (See [2], p. 289–312.) *Let*

$$z = \sum p(\omega)\boldsymbol{x}_\omega, \ (\omega \in Q(r, 1, n)) .$$

Then z is a pure vector if and only if

$$(1) \qquad \sum_{\mu=0}^{r} (-1)^\mu p(\alpha, j_\mu) p(j_0, \cdots, j_{\mu-1}, j_{\mu+1}, \cdots, j_r) = 0$$

for all $\alpha \in S(r-1, 1, n)$ *and all* $(j_0, \cdots, j_r) \in S(r+1, 1, n)$.

Furthermore, there are $(n-r)$ independent equations in the system of equations (1).

The following lemma will be useful.

LEMMA 2. *Let* $z = \sum p(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 1, n); z \in C_k^r(U))$. *Let* s, m *be integers,* $0 \leqq s \leqq r, 0 \leqq m \leqq n$, *and let*

$$z' = \sum p(1, \cdots, s, \alpha) x_1 \wedge \cdots \wedge x_s \wedge \boldsymbol{x}_\alpha, \qquad (\alpha \in Q(m-s, s+1, m)) .$$

Then $z' \in C_l^r(U)$, *for some* $l, 0 \leqq l \leqq k$.

Proof. We prove first the case $k = 1$.
Let $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$. We set

$$p'(i_1, \cdots, i_r) = p(i_1, \cdots, i_r)$$

if $i_1 = 1, \cdots, i_s = s$, and $s + 1 \leqq i_{s+1} < \cdots < i_r \leqq m$. Otherwise, $p'(i_1, \cdots, i_r) = 0$. Then $z' = \sum p'(\omega)\boldsymbol{x}_\omega; (\omega \in Q(r, 1, n))$. It is easy to show that the system of equations (1) holds for the p' s; (there are 3 cases to check; viz., $i_t > m$ or $j_t > m$ for some t; not all of the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ or not all of the integers $1, \cdots, s$ are present in $j_0, \cdots, j_r$; and, thirdly, all the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ and in $j_0, \cdots, j_r$ with $i_t \leqq m$ $(t = 1, \cdots, r-1)$ and $j_l \leqq m$ $(l = 0, \cdots, r)$). Thus, by Theorem 1, $z' \in C_1^r(U)$ or is zero.
For $z = z_1 + \cdots + z_k \in C_k^r(U), z_i \in C_1^r(U)$ $(i = 1, \cdots, k)$, we apply the above result to each term z_i, noting that

$$z' = (z_1 \cdots + z_k)' = z_1' + \cdots + z_k' .$$

THEOREM 3. *Let* $U' \subseteq U$ *be a subspace.*
Then $C_k^r(U') \subseteq C_k^r(U)$.

Proof. Let $x_1, \cdots, x_s$ be a basis of U', and let $x_1, \cdots, x_n$ be an extension of this basis to a basis of U. Let

$$y_1 + \cdots + y_k \in C_k^r(U'), y_i \in C_1^r(U') .$$

Suppose $y_1 + \cdots + y_k = z_1 + \cdots + z_l \in C_l^r(U), z_i \in C_1^r(U)$. Clearly

$l \leqq k$.

To show $l \geqq k$, let

$$z_j = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega,\ \omega \in Q(r, 1, n), \qquad 1 \leqq j \leqq l .$$

Since $y_i \in C_1^r(U')$, $1 \leqq i \leqq k$, then

$$\sum_{j=1}^{l} p^{(j)}(\omega) = 0$$

whenever $\omega = (i_1, \cdots, i_r)$ and $\{i_1, \cdots, i_r\} \not\subseteq \{1, \cdots, s\}$. Hence

$$z'_j = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega,\ \omega \in Q(r, 1, s) ,$$

is in $C_1^r(U')$ by Lemma 2, and since $z'_1 + \cdots + z'_l = z_1 + \cdots + z_l = y_1, \cdots, y_k$, the $l \geqq k$.

Definition. For $z \in C_k^r(U)$, we define $R_r(z) = k$; i.e., $R_r\colon \wedge^r U \to J$ such that $R_r(z) = k$ if and only if $z \in C_k^r(U)$.

We will drop the index r when no confusion arises.

If $x \in U, z \in \wedge^r U$ such that $z = \sum p(\omega)\boldsymbol{x}_\omega, \omega \in Q(r, 1, n)$, where $x_1, \cdots, x_n$ is a basis of U, then we write $x \wedge z$ for the vector

$$\sum p(\omega)x \wedge \boldsymbol{x}_\omega,\ \omega \in Q(r, 1, n) .$$

If $z = x_1 \wedge \cdots \wedge x_r$ is a nonzero pure vector in $\wedge^r U$, then we shall denote the r-dimensional space $\langle x_1, \cdots, x_n \rangle$ by $U(z)$.

Theorem 4. *Let* $y = y_1 + \cdots + y_k \in C_k^r(U), y_i \in C_1^r(U), 1 \leqq i \leqq k$.

(i) *Suppose* $x \wedge (y_1 + \cdots + y_k) = 0, x \in U$. *Then* $x \in U(y_i)$, $i = 1, \cdots, k$.

(ii) *Suppose* $x \in U, x \notin U(y_1) + \cdots + U(y_k)$. *Then* $x \wedge y \in C_k^{r+1}(U)$.

Proof. (i) Suppose on the contrary that $x \notin U(y_1)$. Then

$$x \wedge y_1 = x \wedge \left(-\sum_{i=2}^{k} y_i\right) \neq 0 .$$

Thus, we can choose a basis $x_1, \cdots, x_n$ of U such that

$$x = x_1, y_1 = x_2 \wedge \cdots \wedge x_{r+1} .$$

Then

$$\left(-\sum_{i=2}^{k} y_i\right) = x_2 \wedge \cdots \wedge x_{r+1} + \sum p(1, \alpha)x_1 \wedge \boldsymbol{x}_\alpha, (\alpha \in Q(r-1, 2, n)) .$$

Hence $(-\sum_{i=2}^{k} y_i) = y_1 + x \wedge v$, where $v = \sum p(1, \alpha)\boldsymbol{x}_\alpha \in \wedge^{r-1}U$. Taking

$s = 1$, $m = n$ in Lemma 2, it is easy to see that since $R(-\sum_{i=2}^{k}) = k - 1$, then $R(x \wedge v) \leqq k - 1$. But $x \wedge v = -(y_1 + \cdots + y_k)$ which implies $R(x \wedge v) = k$. We have a contradiction. Therefore $x \in U(y_1)$. Similarly, $x \in U(y_i)$, $i = 2, \cdots, k$.

(ii) Suppose that

$$x \wedge y = z_1 \cdots + z_l \in C_l^{r+1}(U), z_i \in C_1^{r+1}(U)\ , \qquad 1 \leqq i \leqq l\ .$$

Clearly $l \leqq k$.

To show $l \geqq k$, we choose a basis $x_1, \cdots, x_n$ of U such that $x = x_1$ and $x_2, \cdots, x_s$ is a basis of $U(y_1) + \cdots + U(y_k)$. Then

$$y = \sum p(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, n))\ .$$

Using (i) and the fact that $x \wedge (x \wedge y) = x_1 \wedge (z_1 + \cdots + z_l) = 0$, we can express each $z_j = x_1 \wedge (\sum p^{(j)}(\omega)\boldsymbol{x}_\omega; \omega \in Q(r, 2, n))$, $1 \leqq j \leqq l$.

Now $\sum_{j=1}^{l} p^{(j)}(\omega) = 0, (\omega = (i_1, \cdots, i_r))$, *unless*

$$\{i_1, \cdots, i_r\} \subseteq \{2, \cdots, s\}\ .$$

In the latter case, $\sum_{j=1}^{l} p^{(j)}(\omega) = p(\omega)$. Therefore, $z_1 + \cdots + z_l = z'_1 + \cdots + z'_l = x \wedge y$ where

$$z'_j = \sum p^{(j)}(\omega)x_1 \wedge \boldsymbol{x}_\omega, (\omega \in Q(r, 2, s))\ .$$

Hence $y = z''_1 + \cdots + z''_l$, where $z''_j = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, s))$, which implies $R(y) \leqq l$, i.e., $k \leqq l$.

THEOREM 5. *Let* $y_i \in C_1^r(U), z_i \in C_1^r(U)$, $(i = 1, \cdots, k)$ *such that* $y_1 + \cdots + y_k = z_1 + \cdots + z_k$.

Then $U(y_1) + \cdots + U(y_k) = U(z_1) + \cdots + U(z_k)$.

Proof. Suppose on the contrary that there exists a vector $x \in U(y_1)$ such that $x \notin U(z_1) + \cdots + U(z_k)$. Since $x \wedge (y_1 + \cdots + y_k) = x \wedge (z_1 + \cdots + z_k)$, then

$$R(x \wedge (y_1 + \cdots + y_k)) = R(x \wedge (z_1 + \cdots + z_k)) \leqq k - 1\ .$$

But, by Theorem 4 (ii), $R(x \wedge (z_1 + \cdots + z_k)) = k$, which is a contradiction.

DEFINITION. Let

$$z = z_1 + \cdots + z_k \in C_k^r(U), z_i \in C_1^r(U), \qquad i = 1, \cdots, k\ .$$

Then we define $U(z)$ to be the subspace $U(z_1) + \cdots + U(z_k)$.

THEOREM 6. *Let* $z_i \in C_1^r(U)$, $i = 1, \cdots, k$, *and let*

$$\dim [U(z_1) + \cdots + U(z_k)] = rk .$$

Then $R(z_1 + \cdots + z_k) = k$.

Proof. Suppose the Theorem is false. Let k be the smallest integer for which it fails. Clearly $k \geqq 2$. Let

$$z_1 + \cdots + z_k = y_1 + \cdots + y_l \in C_l^r(U), y_i \in C_1^r(U) .$$

Let $x \in U(z_1)$. Then $x \notin U(z_2) + \cdots + U(z_k)$. By the choice of

$$k, z_2 + \cdots + z_k \in C_{k-1}^r(U) .$$

Hence, by Theorem 4 (ii),

$$x \wedge (z_2 + \cdots + z_k) = x \wedge (z_1 + \cdots + z_k) = x \wedge (y_1 + \cdots + y_l) ,$$

and $l \geqq k - 1$. But we assumed $l < k$. Therefore $l = k - 1$.

By Theorem 5,

$$U(x \wedge z_2) + \cdots + U(x \wedge z_k) = U(x \wedge y_1) + \cdots + U(x \wedge y_{k-1}) .$$

Hence $\langle x \rangle + U(z_2) + \cdots + U(z_k) = \langle x \rangle + U(y_1) + \cdots + U(y_{k-1})$.

Now let $x' \in U(z_1)$, independent of x. Then again

$$\langle x' \rangle + U(z_2) + \cdots + U(z_k) = \langle x' \rangle + U(y_1) + \cdots + U(y_{k-1}) .$$

Taking intersections, we obtain

$$U(z_2) + \cdots + U(z_k) = U(y_1) + \cdots + U(y_{k-1}) .$$

By a similar argument,

$$\begin{aligned} V_i &= U(z_1) + \cdots + U(z_{i-1}) + U(z_{i+1}) + \cdots + U(z_k) \\ &= U(y_1) + \cdots + U(y_{k-1}) . \end{aligned}$$

Hence $U(y_1) + \cdots + U(y_{k-1}) = \bigcap_{i=1}^k V_i = \{0\}$, which is impossible. The result follows.

THEOREM 7. $\sum_{i=1}^s x_i \wedge y_1 \in C_s^2(U)$ *if and only if* $(\{x_1, y_1, \cdots x_s, y_s\}$ *is independent.*

Proof. If $\{x_1, y_1, \cdots, x_s, y_s\}$ is dependent, it is easy to show that $R(\sum_{i=1}^s x_i \wedge y_i) \leqq s - 1$. It follows that the condition is necessary.

The converse follows easily from Theorem 6.

COROLLARY 8. *Let* $f = \sum_{i=1}^s x_i \wedge y_i$, *and* $\dim \langle x_1, y_1, \cdots, x_s, y_s \rangle$ $\langle 2k, k \leqq s$. *Then* $R(f) \leqq k - 1$.

We shall now direct our attention to the rank 2 subspaces of $\wedge^2 U$.

DEFINITION. A *rank* 2 *subspace* H in $\wedge^2 U$ is a subspace whose nonzero members are in $C_2^2(U)$.

In this paper, we shall restrict our considerations to the case dim $U = 4$. It is clear from Theorem 7 that $C_2^2(U)$ is empty when dim $U < 4$.

LEMMA 9. *Let $f \in C_2^2(U)$ and let $\{y_1, \cdots, y_4\}$ be any basis of $U(f)$. Then f has a representation $f = y_1 \wedge u + v \wedge w$, where $\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle$.*

Proof. Since $f \in \wedge^2\langle y_1, \cdots, y_4\rangle$, then

$$\begin{aligned} f &= \sum p(\omega)\boldsymbol{y}_\omega, \ (\omega \in Q(2, 1, 4)),\ p(\omega) \in F\ , \\ &= y_1 \wedge (\textstyle\sum_{j=2}^4 p(1, j)y_j) + \sum p(\alpha)\boldsymbol{y}_\alpha\ ; \qquad (\alpha \in Q(2, 2, 4))\ , \end{aligned}$$

which is of the form $y_1 \wedge u + v \wedge w$. It follows from Theorem 7 and its corollary, and the fact that $R(f) = 2$ that

$$\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle\ .$$

THEOREM 10. *Let* dim $U = 4$ *and let* H *be a rank* 2 *subspace in* $\wedge^2 U$. *Then* dim $H = 1$, *provided* F *is algebraically closed.*

Proof. Let f be a nonzero member of H. Then f has a representation $f = x_1 \wedge x_2 + x_3 \wedge x_4$ in $C_2^2(U)$. By Theorem 7,

$$U = U(f) = \langle x_1, \cdots, x_4\rangle\ .$$

If f' is any other nonzero member of H, then $U(f') = \langle x_1, \cdots, x_4\rangle$. By Lemma 9, $f' = x_1 \wedge u + v \wedge w$, $\langle u, v, w\rangle = \langle x_2, x_3, x_4\rangle$. Hence dim $\langle v, w\rangle \cap \langle x_3, x_4\rangle \leqq 1$. Without loss of generality, we shall assume $x_3 \in \langle v, w\rangle \cap \langle x_3, x_4\rangle$. Hence

$$f' = x_1 \wedge u + x_3 \wedge w',\ \langle u, w'\rangle \subset \langle x_2, x_3, x_4\rangle\ .$$

Let $u = \sum_{i=2}^4 b_i x_i$; $w' = \sum_{i=2,4} d_i x_i$; $b_i, d_i \in F$. Then for

$$\begin{aligned} \lambda \in F, z = \lambda f + f' = x_1 &\wedge (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4) \\ &+ x_3 \wedge (\lambda x_4 + d_2 x_2 + d_4 x_4)\ . \end{aligned}$$

The condition that the vectors

$$x_1, (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4), x_3, (\lambda x_4 + d_2 x_2 + d_4 x_4)$$

be independent; i.e., $R(z) = 2$, is equivalent to the condition that the determinant

$$\Gamma(\lambda, f_1, f_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda + b_2 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ 0 & d_2 & 0 & \lambda + d_4 \end{vmatrix} \quad \textit{be}$$

nonzero. Now

$$\Gamma(\lambda, f_1, f_2) = \lambda^2 + \lambda(d_4 + b_2) + (b_2 d_4 - d_2 b_4) = g(\lambda) .$$

Since u, w' are independent, $(b_2 d_4 - d_2 b_4) \neq 0$. Hence $g(\lambda)$ is a nontrivial polynomial in λ, and hence, for some nonzero λ in F, $g(\lambda) = 0$; i.e., $\Gamma(\lambda, f_1, f_2) = 0$. For such a λ, $R(z) \leqq 1$. It follows that dim $H = 1$.

The above theorem is false when F is *nonalgebraically closed*. For example, the vectors

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (x_3 + x_4) + (x_3 - x_2) \wedge x_4$$

in $C_2^2(U)$, where $U = \langle x_1, \cdots, x_4 \rangle$, dim $U = 4$, $F \equiv$ Reals, generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

It is interesting to note that if F (nonalgebraically closed) has an *irreducible quadratic* polynomial $h(\lambda)$, and dim $U = 4$, then we can construct 2 independent vectors f_1, f_2 in $C_2^2(U)$, which will generate a 2-dimensional rank 2 subspace in $\wedge^2 U$, and such that $\Gamma(\lambda, f_1, f_2) = h(\lambda)$ (see Theorem 10). The construction is as follows:

Let dim $U = 4$, $U = \langle x_1, \cdots, x_4 \rangle$. Let $h(\lambda) = \lambda^2 + a_1 \lambda + a_0$ be irreducible in F. The companion matrix of $h(\lambda)$ is

$$B = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}; \qquad \lambda I - B = \begin{bmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{bmatrix}.$$

Now

$$\det(\lambda I - B) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & a_0 & 0 & \lambda + a_1 \end{vmatrix} = h(\lambda) \neq 0 .$$

Taking this determinant to be $\Gamma(\lambda, f_1, f_2)$ corresponding to $z = \lambda f_1 + f_2$, where $f_1, f_2 \in C_2^2(U)$, $\lambda \in F$, we have

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$
$$f_2 = x_1 \wedge (-x_4) + x_3 \wedge (a_0 x_2 + a_1 x_4) .$$

The construction is complete. Thus, for example, if $F \equiv$ Rationals and $h(\lambda) = \lambda^2 - 2$, then

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (-x_4) + (-2)x_3 \wedge x_2 \ ,$$

and f_1, f_2 generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

For the work in this paper, I am greatly indebted to Dr. R. Westwick for his generous assistance.

REFERENCES

1. N. Bourbaki, *Elements De Mathematique, Algebre,* Chap. III, Algebre Multilineare, 1948.
2. W. V. D. Hodge, and D. Pedoe, *Methods of algebraic geometry,* Vol. I, Cambridge, 1947.
3. R. Westwick, *Linear transformations on Grassmann spaces,* Pacific J. Math. **14** (1964).

Received May 10, 1968.

UNIVERSITY OF BRITISH COLUMBIA

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is $8.00; single issues, $3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: $4.00 per volume; single issues $1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics
Vol. 29, No. 2 June, 1969

Bruce Langworthy Chalmers, *On boundary behavior of the Bergman kernel function and related domain functionals* . . . 243
William Eugene Coppage, *Peirce decomposition in simple Lie-admissible power-associative rings* . . . 251
Edwin Duda, *Compactness of mappings* . . . 259
Earl F. Ecklund Jr., *On prime divisors of the binomial coefficient* . . . 267
Don E. Edmondson, *A modular topological lattice* . . . 271
Phillip Alan Griffith, *A note on a theorem of Hill* . . . 279
Marcel Herzog, *On finite groups with independent cyclic Sylow subgroups* . . . 285
James A. Huckaba, *Extensions of pseudo-valuations* . . . 295
S. A. Huq, *Semivarieties and subfunctors of the identity functor* . . . 303
I. Martin (Irving) Isaacs and Donald Steven Passman, *Finite groups with small character degrees and large prime divisors. II* . . . 311
Carl Kallina, *A Green's function approach to perturbations of periodic solutions* . . . 325
Al (Allen Frederick) Kelley, Jr., *Analytic two-dimensional subcenter manifolds for systems with an integral* . . . 335
Alistair H. Lachlan, *Initial segments of one-one degrees* . . . 351
Marion-Josephine Lim, *Rank k Grassmann products* . . . 367
Raymond J. McGivney and William Henry Ruckle, *Multiplier algebras of biorthogonal systems* . . . 375
J. K. Oddson, *On the rate of decay of solutions of parabolic differential equations* . . . 389
Helmut R. Salzmann, *Geometries on surfaces* . . . 397
Annemarie Schlette, *Artinian, almost abelian groups and their groups of automorphisms* . . . 403
Edgar Lee Stout, *Additional results on modules over polydisc algebras* . . . 427
Lajos Tamássy, *A characteristic property of the sphere* . . . 439
Mark Lawrence Teply, *Some aspects of Goldie's torsion theory* . . . 447
Freddie Eugene Tidmore, *Extremal structure of star-shaped sets* . . . 461
Leon Jarome Weill, *Unconditional and shrinking bases in locally convex spaces* . . . 467