Pacific Journal of Mathematics

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Vol. 29, No. 2

June 1969

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Let $\{e_i, E_i\}$ be a total biorthogonal system in a linear topological space X. The multiplier algebra of X with respect to $\{e_i, E_i\}$ written M(X) is the set of all scalar sequences $(t^{(i)})$ such that for each $x \in X$ there is $y \in X$ with

$$E_i(y) = t^{(i)}E_i(x)$$
.

The form of M(X) is determined when $\{e_i, E_i\}$ is a norming complete biorthogonal system in a Banach space or a basis in a complete barreled space. It is shown that a sequence space is the multiplier algebra for a basis in a Banach space if and only if it is a γ -perfect *BK*-algebra.

A biorthogonal system is a double sequence $\{e_i, E_i\}$ with each e_i in a locally convex space X and each E_i a continuous linear functional on X (i.e., in X^{*}) which satisfies the relationship

$$E_i(e_j) = \delta_{ij}$$
 (Kronecker δ) $i, j = 1, 2, \cdots$.

The biorthogonal system is total if $\{E_i\}$ is total on X; that is, $E_i(x) = 0$ for each *i* implies x = 0. If $\{e_i, E_i\}$ is a total biorthogonal system then the space X can be identified with the space of all sequences $(E_i(x))$ by means of the natural correspondence x to $(E_i(x))$. Under this correspondence e_i becomes the *i*th coordinate vector, the sequence which has a one in the *i*th coordinate and 0's elsewhere and E_i becomes the *i*th coordinate functional, the functional whose value on the sequence $(x^{(1)}, x^{(2)}, \cdots)$ is $x^{(i)}$. This identification will be assumed whenever a total biorthogonal system is considered.

DEFINITION 1.1. Let $\{e_i, E_i\}$ be a total biorthogonal system in a locally convex space X. A scalar sequence

$$t = (t^{(1)}, t^{(2)}, \cdots)$$

is a multiplier of X with respect to $\{e_i, E_i\}$ if for each $x \in X$ there is $y \in X$ for which

$$E_i(y)=t^{\scriptscriptstyle(i)}E_i(x) \qquad i=1,\,2,\,\cdots\, ,$$

The set of all such t is written $M(X; e_i, E_i)$ or simply M(X) and called the multiplier algebra of X (with respect to $\{e_i, E_i\}$).

In other words M(X) is the set of all t such that

$$tx \in X$$
 whenever $x \in X$

where X is now considered a sequence space and multiplication of sequences is defined coordinatewise. It is now obvious that M(X) forms a linear algebra of operators from X into X; namely the operators which are diagonal with respect to $\{e_i, E_i\}$. Multiplication in this algebra is defined coordinatewise. The purpose of this paper is to study the properties of the space M(X) and the possible forms which it can assume with varying hypotheses on X or on $\{e_i, E_i\}$. Results of this type were obtained by Yamazaki in [11] and [12] for $\{e_i\}$ a basis in a Banach space. The concept of imultiplier space is implicitly treated in [4].

Throughout this paper it is immaterial whether the scalar field considered consists of the real or complex numbers.

2. Sequence spaces: notation and basic facts. A set of scalar sequences which is closed under coordinatewise addition and scalar multiplication is a sequence space; if it is closed under coordinatewise multiplication as well it will be called a sequence algebra. The *i*th coordinate vector is written e_i ; the *i*th coordinate functional, E_i . If each E_i is continuous on a locally convex sequence space (algebra) X and $e_i \in X$ for each *i* then X is called a *K*-space (algebra). If in addition X is an F-space (complete metric linear space) X will be called an FK-space or FK-algebra as the case may be. If X is a Banach space (algebra), X will be called a BK-space (algebra). Note that in an FK-algebra X the functions $x \to tx$ and $x \to xt$ are continuous in x for fixed t by the continuity of the coordinate functionals and the closed graph theorem. This is enough to conclude that a BK-algebra is a Banach algebra without identity. See p. 860 and 861 of [3].

The following are well known sequence spaces. For additional discussion see Chapter IV of [2], [5] or p. 289 of [10]:

 ω sometimes called s is the set of all scalar sequences. Endowed with the topology of coordinatewise convergence it is an *FK*-algebra.

 φ is the linear span of $\{e_i\}$ in ω , i.e., the space of all finitely nonzero sequences.

 l^1 is the set of all sequences t such that

$$||\,t\,|| = \sum_{i=1}^{\infty} |\,t^{(i)}\,| < \infty$$

which is a BK-space with this norm.

m is the set of all sequences t such that

$$||t|| = \sup_i |t^{\scriptscriptstyle (i)}| < \infty$$

which is a BK algebra with this norm.

bs is the set of all sequences t such that

$$||t|| = \sup_n |\sum_{i=1}^n t^{(i)}| < \infty$$

which is a BK space with this norm.

cs is the closed linear span of $\{e_i\}$ in bs; it consists of all sequences t such that $\sum_{i=1}^{\infty} t^{(i)}$ converges.

bv is the set of all sequences t such that

$$||t|| = \lim_{n \to \infty} |t_n| + \sum_{i=1}^{\infty} |t_i - t_{i+1}| < \infty$$

which is a *BK*-algebra with this norm. See §3 of [4] or p. 3 of [11]. Yamazaki denoted bv by w.

NOTATION 2.1. (a) Let $t, u \in \omega$ be such that $ut \in cs$; the sum $\sum_{i=1}^{\infty} u^{(i)} t^{(i)} \text{ is denoted by } (u, t).$ (b) For $A \subseteq \omega$ A^{α} , the α -dual of A is $\{t: ut \in l^{1}, u \in A\}$ A^{β} , the β -dual of A is $\{t: ut \in cs, u \in A\}$ A^{γ} , the γ -dual of A is $\{t: ut \in bs, u \in A\}$. (c) For A and $B \subseteq \omega$ $AB = \{uv: u \in A, v \in B\}.$ (d) For $t \in \omega$ and $A \subseteq \omega$ $t^{-1}A = \{u \in \omega: tu \in A\}.$ (e) For $A \subseteq \omega$ $A^{\varphi} = \{t \in \varphi: | (t, u) | \leq 1, u \in A\}.$ (f) For $A \subseteq \varphi$ $A^{\omega} = \{t \in \omega: | (t, u) | \leq 1, u \in A\}.$ (g) For X aK-space, X^{δ} is the space of all sequences $(f(e_{i}))$ as

(g) For X aK-space, X is the space of all sequences $(f(e_i))$ as f ranges over X^{*}. Note that for $t \in X^{\delta}$ with $t^{(i)} = f(e_i)$, $E_i(t) = t^{(i)} = f(e_i)$.

Gamma-perfect *BK*-spaces can be constructed by means of sequential norms. A sequential norm (s.n.) is a function *P* from ω into R^* which is an extended norm and in addition satisfies the condition

$$P(x) = \sup_{n} P\left(\sum_{i=1}^{n} \boldsymbol{x}^{(i)} \boldsymbol{e}_{i}\right) x \in \boldsymbol{\omega}$$
.

 \mathbf{If}

$$0 < \inf_n P(e_n) \leq \sup_n P(e_n) < \infty$$

P is a proper sequential norm (p.s.n.). For *P* a s.n, S_P is the set of all $x \in \omega$ for which $P(x) < \infty$ endowed with the topology determined by $P\varepsilon$. The closed linear span of $\{e_1, e_2, \cdots\}$ in S_P is denoted by S_P° . The

following proposition contains information about sequential norms which was derived in [7] and [8] and which we shall use in §6.

PROPOSITION 2.2. (a) If P is a s.n. S_P is a γ -perfect BK-space. If X is a γ -perfect BK-space there is an s.n. P such that $X = S_P$. (b) If P is a s.n. the function P' given by

$$P'(x) = \sup \left\{ \sup_{n} |\sum_{i=1}^{n} x^{(i)} y^{(i)}| : P(y) \leq 1 \right\}$$

is a s.n. and P'' = P. If P is a p.s.n. so is P'.

- (c) $(S_P^0)^{\delta} = S_{P'}$ and $(S_{P'}^0)^{\delta} = S_{P}$.
- (d) An s.n. P is a p.s.n. if and only if $l^1 \subseteq S_P \subseteq m$.
- 3. Preliminary results.

PROPOSITION 3.1. If $\{e_i, E_i\}$ is a total biorthogonal system in X

$$(3-1) \hspace{1.5cm} M(X) = \ \cap \ \{y^{-1}X \colon y \in S\}$$

where S is any absorbing subset of X.

Proof. Let R denote the set on the right of (3-1). If $t \in R$ and $x \in X$ there is a > 0 such that $ax \in S$. Thus $atx \in X$ so that $tx \in X$ which implies $t \in M(X)$. If $t \in M(X)$ then $tx \in X$ for every $x \in X$ so in particular for every $x \in S$.

A complete biorthogonal system is a total biorthogonal system $\{e_i, E_i\}$ on X such that sp $\{e_i\}$ $(= \varphi)$, the linear span of $\{e_i\}$ is dense in X.

PROPOSITION 3.2. Let $\{e_i, E_i\}$ be a complete biorthogonal system in X.

(a) For each $t \in M(X)$, the mapping $x \to tx$ is a closed linear operator from X into X.

(b) The set $M_c(X)$ of all $t \in M(X)$ for which $x \to tx$ is continuous is a closed sub-algebra of $\mathscr{L}(X)$ where $\mathscr{L}(X)$ has any topology containing the topology of simple convergence. Here $\mathscr{L}(X)$ denotes the space of continuous operators from X into X.

Proof. (a) Obvious. (b) Define $E_i \otimes e_j$ on $\mathscr{L}(X)$ by

$$E_i \otimes e_j(T) = E_i(Te_j)$$
 .

Then $E_i \otimes e_j$ is a continuous linear functional on $\mathscr{L}(X)$ given the topology of simple convergence. Therefore

$$S = \cap \{ [E_i \otimes e_j]^{-1}(0) : i \neq j \}$$

is closed in this topology and every topology containing it.

The following statement generalizes Theorem 1 of [11].

COROLLARY 3.3. If $\{e_i, E_i\}$ is a complete biorthogonal system in a Banach space X then there is a topology on M(X) which makes it a BK algebra.

Proof. In this case $M_{\epsilon}(X) = M(X)$ since the mapping $x \to tx$ is closed. Thus by 3.2 M(X) is a Banach algebra with the norm

$$||t|| = \sup\{||tx|| : ||x|| \le 1\}$$
.

Each E_i is continuous on M(X) since

$$E_i(T) = E_i \otimes e_i(T_t)$$

where $T_i(x) = tx$; and $E_i \otimes e_i$ is continuous on $\mathcal{L}(X)$.

PROPOSITION 3.4. If $\{e_i, E_i\}$ is a total biorthogonal system in a linear topological space X then

$$M_c(X; e_i, E_i) \subseteq M_c(X_0; e_i, E_i)$$

where X_0 is the closed linear span of $\{e_i\}$ in X.

Proof. If $t \in M_c(X)$ then $tx \in \varphi$ for $x \in \varphi$. Since φ is dense in X_0 and t is continuous $tx \in X_0$ for $x \in X_0$ so that $t \in M_c(X_0)$.

If $\{e_i, E_i\}$ is a complete biorthogonal system on X, X^* is isomorphic to X^{δ} under the correspondence of f in X^* to $(f(e_i))$ in X^{δ} and $\{e_i, E_i\}$ is a total biorthogonal system on X^{δ} .

PROPOSITION 3.5. If $\{e_i, E_i\}$ is a complete biorthogonal system in a locally convex space X then

$$M_c(X; e_i, E_i) \subseteq M(X^{\circ}; e_i, E_i)$$
.

Proof. If $t \in M_{\sigma}$ for $f \in X^*$ let $f_i(x) = f(tx), x \in X$. Then $f_i \in X^*$ and $f_i(e_i) = t^{(i)}f(e_i)$ for each i so that $ty \in X^{\delta}$ for $y \in X^{\delta}$.

4. Multiplier algebras of a norming biorthogonal system in a Banach space. A biorthogonal system $\{e_i, E_i\}$ in a normed space X is called *norming* if the topology of X is determined by a norm of the type

$$||x|| = \sup \{|f(x)| : f \in A\}$$

where A is a subset of the linear span of $\{E_i\}$ in X^* . An equivalent condition is that the above norm be given by

$$(4-1) || x || = \sup \{| (x, t) | : t \in A\}$$

where A is a subset of φ .

If $\{e_i, E_i\}$ is a complete biorthogonal system which is norming on X and the norm of X is given by (4-1) it may be assumed that A consists of all sequences t in φ for which

$$(4-2) |(t, x)| \le ||x|| x \in X.$$

Denote by \hat{X} the space of all $x \in \omega$ for which

$$(4-3) || x || = \sup \{ | (x, t) | : t \in A \} < \infty$$

The function defined in (4-3) is a norm since $a_n^{-1}e^n \in A$ for $n = 1, 2, \cdots$ where a_n is the norm of E_n as a member of X^* . With this norm \hat{X} is a *BK*-space in which X is the closed linear span of $\{e_i\}$.

PROPOSITION 4.1. The space X^s consists of all $y \in \omega$ for which $||y||' = \sup \{|(y, x)|: x \in A^{\varphi}\} < \infty$.

The correspondence

$$(4-4) f to (f(e_i)) f \in X^*$$

is an isometry from X^* onto $(X^{\delta}, || ||')$. The correspondence

$$(4-5) g to (g(e_i)) g \in (X_0^{\delta})^*$$

is an isometry from $(X_0^{\delta})^*$ onto \hat{X} , where X_0^{δ} denotes the closed linear span of $\{e_i\}$ in X^{δ} .

Proof. The correspondence in (4-4) is clearly well defined and linear.

If $f \in X^*$ and $x \in A^{\circ}$ then $x \in X$ and $||x|| \leq 1$. Thus

$$egin{aligned} |(f(e_i), x)| &= |\sum\limits_i x^{(i)} f(e_i)| \ &= |f(x)| \leq ||f|| \end{aligned}$$

so that

$$|(f(e_i))||' \leq ||f||$$
.

If $y \in X^{\delta}$ define f on $\{\varphi, || ||\}$ by

$$f(x) = (y, x) \; .$$

Then f is a bounded linear functional on $\{\varphi, || ||\}$ for which

 $|f(x)| \leq ||y||' x \in \varphi, ||x|| \leq 1$

because

$$A^arphi = \{x \in arphi \colon || \ x \, || \leq 1\}$$
 .

Since φ is dense in X, f can be continuously extended to X with

 $||f|| \leq ||y||'$.

For this extended f

$$(f(e_i)) = (y, e_i) = y_i$$
 $i = 1, 2, \cdots$.

Therefore, the correspondence in (4-4) is an isometry.

That (4-5) is an isometry from $(X_0^s)^*$ onto \hat{X} will follow from an analogous argument if it is shown that

 $A^{\varphi\varphi} = A$.

When A has the form given by (4-2). That $A^{\varphi\varphi} \supseteq A$ is clear, if $z \in A^{\varphi\varphi}$ then

 $|(z, x)| \leq 1$ $x \in A^{\varphi}$

but A^{φ} is dense in the unit ball of X so that

 $|(z, x)| \leq 1$ $x \in X, ||x|| \leq 1$.

Thus if

f(x) = (z, x) $x \in X$

we have $||f|| \leq 1$ and

$$f(e_i) = z_i \qquad i = 1, 2, \cdots$$

so that $z \in A$. It is here that the assumption that $\{e_i, E_i\}$ is norming was used.

THEOREM 4.2. If $\{e_i, E_i\}$ is a norming complete biorthogonal system in the Banach space X then M(X) is of the form

$$(4-6) \qquad \qquad \bigcup_{n=1}^{\infty} n(AA^{\omega})^{\omega}$$

where A is a coordinatewise bounded subset of φ which contains a multiple of e_i for each *i*.

Proof. Let A be given by (4-2) and let Z denote the sequence space (4-6).

By 3.4, 3.5 and the fact that $M_{c}(Y) = M(Y)$ for Y a Banach space we have

$$M(X) \subseteq M(X^{\delta}) \subseteq M(X^{\delta}_{0}) \subseteq M(\hat{X}) \subseteq M(X)$$

so that M(X) and $M(\hat{X})$ are equal. It will be shown that $M(\hat{X}) = Z$. Suppose $t \in M(\hat{X})$, then there is k such that

 $||tx|| \leq k ||x|| \qquad x \in \hat{X}.$

If $s \in A$ and $x \in A^{\omega}$

$$|(sx, t)| = |(s, tx)| \leq k$$

so that

$$t \in k(AA^{\omega})^{\omega} \subseteq Z$$
 .

If $z \in Z$ and $x \in \hat{X}$, $x/||x|| \in A^{\omega}$ so if n is such that $z \in n(AA^{\omega})^{\omega}$

 $|(t, zx)| = |(tx, z)| \le n ||x||$

for $t \in A$. Therefore $zx \in \hat{X}$.

Question. If $\{e_i, E_i\}$ is a complete biorthogonal system in the Banach space X and M(X) has the form (4-6) is $\{e_i, E_i\}$ norming?

5. Multiplier algebras of bases. A biorthogonal system $\{e_i, E_i\}$ in a linear topological space X is a (Schauder) basis for X if

(5-1)
$$x = \sum_{i=1}^{\infty} E_i(x)e_i \qquad x \in X$$
.

It is an unconditional basis if the convergence in (5-1) is unconditional. If X is a l.c.s. $\{e_i, E_i\}$ is an absolute basis if the convergence in (5-1) is absolute, i.e., if

$$\sum\limits_{i=1}^\infty |\, E_i(x)\, |\, p(e_i) < \, \infty$$

for each $x \in X$ and each continuous seminorm p on X. It is clear that an absolute basis is unconditional.

The proofs of 5.1, 5.2, and 5.3 are omitted since these statements are essentially known. See p. 205 of [10] and Propositions 4 and 5 of [1].

LEMMA 5.1. For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled l.c.s. X it is always true that $X^{\gamma} \subseteq X^{\delta}$.

LEMMA 5.2. For $\{e_i, E_i\}$ a basis in a locally convex space $X, X^{\beta} \subseteq X^{\beta}$.

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PROPOSITION 5.3. For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled locally convex space X the following are equivalent.

- (a) $\{e_i, E_i\}$ is a basis for X.
- (b) $X^{\delta} = X^{\beta}$.
- (c) $X^{\delta} = X^{\gamma}$.

THEOREM 5.4. If $\{e_i, E_i\}$ is a basis of a complete barreled space X then

$$M_c(X)=M(X)=(XX^{\,eta})^{eta}=(XX^{\,ar{\gamma}})^{\gamma}$$
 .

Proof. Suppose $t \in M(X)$, $x \in X$ and $y \in X^{\beta}$. Then $tx \in X$ so that $txy \in cs$ by 5.1 which implies $t \in (XX^{\beta})^{\beta}$.

Let P denote the family of all continuous seminorms on X. Let \hat{X} be the linear space of all $x \in \omega$ such that

$$p'(x) = \sup_n \, p\Bigl(\sum\limits_{i=1}^n x^{(i)} e_i \Bigr) < \infty$$
 , $p \in P$.

Since X is barreled, p' restricted to X is continuous and since φ is dense in $X, p'(x) \ge p(x)$ for $x \in X$. Thus X is the closed linear span of $\{e_i\}$ in the space \hat{X} with the topology determined by the seminorms $\{p': p \in P\}$.

For $t \in (XX^{\gamma})^{\gamma}$ define

$$p_t(x) = \sup_n p\left(\sum_{i=1}^n t^{(i)} x^{(i)} e_i\right)$$
 .

Since $t \in (XX^{\gamma})^{\gamma}$, $\{\sum_{i=1}^{n} t^{(i)} x^{(i)} e_i : n = 1, 2, \dots\}$ is a weakly bounded, thus a strongly bounded subset of X so that

$$P_t(x) < \infty$$
 $x \in X$

and p_t is a continuous seminorm on X. If $x \in \hat{X}$, $tx \in \hat{X}$ and

$$p'(tx) = p'_t(x)$$

so that $t \in M_{c}(\hat{X})$. By 3.4, $(XX^{\tau})^{\tau} \subseteq M_{c}(X)$. Thus

$$M_{\mathfrak{o}}(X) \subseteq M(X) \subseteq (XX^{\gamma})^{\beta} \subseteq (XX^{\gamma})^{\gamma} \subseteq M_{\mathfrak{o}}(X)$$

which establishes the result.

COROLLARY 5.5. If $\{e_i, E_i\}$ is an unconditional basis of a complete barreled space X then

$$M(X) = (XX^{\alpha})^{\alpha}$$
.

Proof. Since $\{e_i, E_i\}$ is an unconditional basis $X^{\alpha} = X^{\delta}$. If $t \in (XX^{\gamma})^{\gamma}$

and $u \in XX^{\gamma}$ let u = xy with $x \in X, y \in X^{\gamma} = X^{\delta} = X^{\alpha}$. Let $v^{(i)} =$ sgn $t^{(i)}u^{(i)}$ then $vy \in X^{\alpha}$ so that $vu \in XX^{\gamma}$. Hence, $vut \in bs$ so that $ut \in l^1$. Therefore, $(XX^{\gamma})^{\gamma} \subseteq (XX^{\alpha})^{\alpha}$ from which the conclusion follows.

THEOREM 5.6. Let $\{e_i, E_i\}$ be an absolute basis of a sequentially complete locally convex space X. If P is any family of continuous seminorms which determines the topology of X then

$$M(X) = (AA^{\alpha})^{\alpha}$$

where

$$A = \{(p(e_i)): p \in P\}$$
.

Proof. The hypotheses of this theorem imply that $X = A^{\alpha}$. Now $t \in M(X)$ if and only if $tx \in X$ whenever $x \in X$ which happens if and only if $txy \in l^1$ whenever $x \in A^{\alpha} = X$ and $y \in A$. This will hold if and only if $t \in (AA^{\alpha})^{\alpha}$.

EXAMPLES. $M(\omega) = M(\varphi) = \omega; M(cs) = M(bv_0) = bv; M(c_0) = M(l^1) = m.$

Let X be the space of all real sequences x for which

$$p_k(x) = \sum |x_n| k^n < \infty$$
 $k = 1, 2, \cdots$.

Then X with the seminorms p_1, p_2, \cdots is a nuclear *F*-space which is equivalent to the space of all infinitely differentiable real functions of period 2π . See §5 of [6]. If $A = (k^n)$: $k = 1, 2, \cdots$ } it is clear that $X = A^{\alpha}$. If $x \in A^{\alpha}$, $x = xe \in AA^{\alpha}$ and if $x \in AA^{\alpha}$, $x = y(k^n)$ for some k and $y \in A^{\alpha}$. But

$$\sum_{n=1}^{\infty} |t_n| k^n h^n = \sum_{n=1}^{\infty} |t_n| (kh)^n$$

so $x \in A^{\alpha}$. Thus $M(X) = A^{\alpha \alpha} = X^{\delta}$.

The following theorem is a version of Theorem 3 and Corollary 2 of [4]. For the definition of B_r -complete see p. 162 of [9].

THEOREM 5.7. A complete biorthogonal system $\{e_i, E_i\}$ in a space X which is barreled and B_r complete is a basis for X if and only if $M(X) \supseteq bv$. It is an unconditional basis for X if and only if $M(X) \supseteq m$.

6. Gamma-perfect BK-algebras. A proper sequential norm P which satisfies the inequality

$$(6-1) P(xy) \leq P(x)P(y)$$

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will be called an algebraic p.s.n. (a.p.s.n.).

THEOREM 6.1. The following statements are equivalent for a sequence space M:

- (a) M is a multiplier algebra for a basis in a Banach space;
- (b) M is a γ -perfect BK-algebra containing e;
- (c) $M = S_P$ for P an a.p.s.n. with $P(e) < \infty$.

Proof. (a) \Rightarrow (b). If *M* is a multiplier algebra for a basis in a Banach space *X* it is a *BK*-algebra containing *e* by 3.3 and γ -perfect since it is the γ -dual of XX^{γ} by 5.4.

(b) \Rightarrow (c). Suppose M is a γ -perfect BK-algebra containing e. By 2.2 (a) there is a sequential norm Q such that $M = S_q$. It is routine to verify that P given by

$$P(x) = \sup \{Q(xy) \colon Q(y) \le 1\}$$

is a s.n. equivalent to Q (i.e., $S_P = S_Q$) which satisfies (6-1). It remains to show

$$(6-2) 0 < \inf_n P(e_n) \leq \sup_n P(e_n) < \infty.$$

Since $P(e_n) = P(e_n e_n) \leq P(e_n)^2$, the left inequality of (6-2) is valid. Since $e \in S_P$, $bs \supseteq S_P^{\gamma}$ so that $bv \subseteq S_P$. That S_P is γ -perfect follows form 2.2 (a). The identity map from bv into S_P is continuous and $\{e_n: n = 1, 2, \dots\}$ is bounded in S_P . Hence the right inequality in (6-2) is true. Therefore, P is an a.p.s.n.

(c) \Rightarrow (a). If P is an a.p.s.n. with $P(e) < \infty$, S_P is a BK-algebra with identity so $M(S_P) = S_P$. But since $S_P = (S_P^0)^{\delta}$ and $S_{P'} = (S_P^0)^{\delta}$, $M(S_P^0) = M$ and $\{e_i, E_i\}$ is a basis for S_P^0 (2.2).

The following theorem gives a means of constructing γ -perfect *BK*-algebras with identity which are distinct from bv and m. Let N denote the sequence of positive integers and N_k a subsequence of the form

(6-3)
$$N_k = \{k(1) < k(2) < \cdots \}$$
 .

THEOREM 6.2. (a) Let N_1, N_2, \dots, N_r be a partition of N with each N_k given by (6-3). For each k let P_k be an a.p.s.n. for which $P_k(e) < \infty$. Define P by

$$P(x) = \max \{P_k(x^{k(1)}, x^{k(2)}, \cdots): k = 1, 2, \cdots r\}$$
.

Then P is an a.p.s.n. and S_P is a γ-perfect BK-algebra containing e.
(b) Let N₁, N₂, ... be an infinite partition of N with each N_k

given by (6-3). For each k let P_k be an a.p.s.n. for which

(6-4)

$$ext{sup}_{k}P_{k}(e)<\infty$$
 .

Define P by

$$P(x) = \sup_{k} \{ P_k(x^{k(1)}, x^{k(2)}, \cdots) \}$$
.

Then P is an a.p.s.n. and S_P is a γ -perfect BK-algebra containing e.

Proof. The proof of (a) is omitted since it is similar to but less difficult than that of (b).

(b) It is straightforward to verify that P is a norm. That P is a.s.n. follows from the equalities:

$$egin{aligned} \sup_n P\Bigl(\sum\limits_{i=1}^n x^{(i)} e_i\Bigr) \ &= \sup_n \sup_k \left\{P_k\Bigl(\sum\limits_{k(i) \leq n} x^{k(i)} e_i\Bigr)
ight\} \ &= \sup_k \left\{\sup_n P_k\Bigl(\sum\limits_{k(i) \leq n} x^{k(i)} e_i\Bigr)
ight\} \ &= \sup_k \left\{P_k(x^{k(1)}, x^{k(2)}, \cdots)
ight\} \end{aligned}$$

since each P_k is an s.n. That S_P is an algebra follows since

$$egin{aligned} P(xy) &= \sup_k \left\{ P_k(x^{k(i)}y^{k(i)})
ight\} \ &\leq \sup_k \left\{ P_k(x^{k(i)}) P_k(y^{k(i)})
ight\} \ &\leq P(x) P(y) \; . \end{aligned}$$

Therefore, S_P is a γ -perfect *BK*-algebra; it contains *e* because of (6-4).

EXAMPLE. Let M be the set of all sequences x such that

$$P(x) = \sup_k \left\{ \sum_{i=1}^\infty |x^{k(i)} - x^{k(i+1)}| + \lim_{k(i)} |x^{k(i)}|
ight\} < \infty$$

for N_1, N_2, \cdots a partition of the integers with each N_k given by (6-3). Then M is a γ -perfect BK-algebra containing e but is neither m nor bv. The sequence y with

$$y^{\scriptscriptstyle (i)} = egin{cases} 1 & ext{for} \;\; i = k(1) \;\; ext{for} \;\; ext{each} \;\; k \ 0 \;\; ext{otherwise} \end{cases}$$

is in M but not in bv. The sequence z with

$$z^{\scriptscriptstyle (i)} = egin{cases} 1 ext{ for } i = k(2j-1) ext{ for each } j < k ext{ and all } k \ 0 ext{ otherwise} \end{cases}$$

is in m but not M.

Question. Are there γ -perfect BK-algebras other than those in

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the smallest class of BK-algebras which contain bv., and m and are closed under the operations described in Theorem 6.2.?

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Received August 5, 1968.

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Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

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