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EXTREMAL STRUCTURE OF STAR-SHAPED SETS

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F. E. TIDMORE

It is shown that the convex kernel of a compact star-shaped subset S of a finite-dimensional linear topological space L_n is determined by the $(n - 1)$ -extreme points of S . The cardinality of the set of k -extreme points is determined for compact star-shaped sets of dimension greater than two. Also given is the result that any compact star-shaped subset S of L_n contains a countable set of $(n - 1)$ -extreme points which determines the convex kernel of S . Another result is that a compact nonconvex star-shaped set S in a locally convex space L is determined by the convex kernel of S and the subset of points that are extreme in S relative to the convex kernel of S .

The convex kernel of a star-shaped set S will be denoted by ckS , the line segment $\{\alpha x + (1 - \alpha)y: \alpha \in [0, 1]\}$ will be denoted by xy , the ray $\{\beta y + (1 - \beta)x: \beta \geq 1\}$ will be denoted by xy^∞ and $L(x, y)$ will denote the line containing x and y , $x \neq y$. The convex hull of a set S will be denoted by $\text{conv } S$. The notation $\text{intv } S$ will denote the interior of S relative to the minimal flat that contains S . The set $\{x: f(x) = \alpha\}$, where f is a linear functional, will be denoted $[f: \alpha]$. Set-theoretic difference will be denoted by \setminus , and the closure of a set S will be denoted by $\text{cl } S$.

The concept of k -extreme point was introduced by Asplund [1].

DEFINITION 1. If S is a subset of a linear space L , a point $x \in S$ is a k -extreme point of S if no k -simplex Δ exists such that $x \in \text{intv } \Delta \subset S$.

For a subset S of a linear space L , S_x will denote the x -star of S determined by the point $x \in S$; that is, the set of points y such that $xy \subset S$. If S is a closed (compact) subset of a linear topological space L , then for any $x \in S$, S_x is a closed (compact) set. If $T \subset S$, let

$$S_T = \bigcap_{x \in T} S_x .$$

A point p belongs to the convex kernel of S if, and only if, $xp \subset S$ for all $x \in S$, which is true if, and only if, $p \in S_x$ for all $x \in S$. Thus $ckS = S_S$, which motivates the following definition.

DEFINITION 2. In a linear space L a subset T of a star-shaped

set S is said to star-generate the convex kernel of S if $ckS = S_T$. Such a subset T is said to be a star-generating set for ckS .

THEOREM 1. *Let S be a compact star-shaped subset of L_{k+1} . Then the set $S(k)$ of k -extreme points of S is a star-generating set for ckS .*

Proof. Without loss of generality, suppose that $0 \in ckS$. If $S = ckS$, then S is convex and $S_x = S$ for each $x \in S$ and the result follows since $\emptyset \neq S(1) \subset S(k)$. Let $p \in S \setminus ckS$. Then there exists a point $y \in S$ such that $py \not\subset S$. Since S is compact, y can be chosen such that $S \cap \text{intv } py^\infty = \emptyset$. Since $py \not\subset S$, there exists a point $z \in (\text{intv } py) \setminus S$. If $y \in S(k)$, then $p \notin S_y$ implies $p \notin S_{S(k)}$. If $y \notin S(k)$ there exists a k -simplex Δ such that $y \in \text{intv } \Delta \subset S$. Consider the convex cone $C = \{\beta y + (\lambda - \beta + 1)z : \beta, \lambda \geq 0\}$, which has vertex z and is contained in the subspace L' with basis $\{p, y\}$. Since $S \cap \text{intv } py^\infty = \emptyset$, Δ must intersect L' in some line other than $L(p, y)$; thus, $S \cap \text{intv } C \neq \emptyset$. There exists a linear functional f defined on L_{k+1} such that $f(q) = 1$ for every $q \in L(p, y)$; clearly $0 \notin L(p, y)$ since $py \not\subset S$ and $0 \in ckS$. The continuous linear functional f_1 , the restriction of f to L' , attains a maximum on the compact set $C \cap S$ at some point $w \in \text{intv } C$. Let $H = [f : f(w)]$. Since $H \cap C \cap S$ is a compact subset of the 1-dimensional set $H \cap L'$, there exists a minimal closed line segment in $\text{intv } C$ which contains $H \cap C \cap S$. Each endpoint of this segment, which may be degenerate, must be a point in $S(k)$. Let v be one of these endpoints. The points p, y, z and v are in L' . If $pv \subset S$, then the fact that $0 \in ckS$ implies that $z \in \text{conv } \{0, p, v\} \subset S$, a contradiction. Hence, $pv \not\subset S$ and $p \notin S_{S(k)}$. Therefore, $S \setminus ckS \subset S \setminus S_{S(k)}$, which gives the desired equality, since clearly $ckS \subset S_{S(k)}$.

It is not always sufficient to consider only the set of familiar extreme points $S(1)$ as a star-generating set for ckS . For example, in E_3 let S be the union of three closed faces of a 3-simplex. In some cases, proper subsets of $S(k)$ exist which will star-generate ckS . However, characterizing such subsets may be very difficult, as indicated by the following example.

EXAMPLE 1. In the plane E_2 let B_u be the upper closed unit half-disc, B_r the right closed unit half-disc. Let

$$\begin{aligned} T_1 &= \text{conv} \{ \{-2e_1\} \cup (B_r + (2e_1 + e_2)) \}, \\ T_2 &= \text{conv} \{ \{-2e_2\} \cup (B_u + (2e_2 - e_1)) \}, \\ S &= T_1 \cup T_2 \cup (-T_1) \cup (-T_2). \end{aligned}$$

Then any star-generating subset of $S(1)$ must contain four distinct

sequences of carefully chosen extreme points.

THEOREM 2. *If S is a compact star-shaped set in L_n , and $\dim(S) \geq 3$, then $S(n - 1)$ is an uncountable set.*

Proof. Without loss of generality, it can be assumed that $0 \in \text{ck}S$. Since $\dim(S) \geq 3$ there exists some point $x \in S, x \neq 0$. If $\beta x \in S(n - 1)$ for every $\beta \in (0, 1)$, then $S(n - 1)$ is uncountable. Otherwise, consider some $w = \beta x$ such that $w \notin S(n - 1)$. Then there exists an $(n - 1)$ -simplex Δ such that $w \in \text{intv } \Delta \subset S$. Since $n - 1 \geq 2$ there exists a nondegenerate line segment $zw \subset \Delta$ such that $zw \cap 0x = \{w\}$. There exists a linear functional f on L_n such that

$$f(w) = f(z) = 1 .$$

There exists a point $y \in [f: 0]$ such that the set $\{y, z, w\}$ is linearly independent. For each $\lambda \in [0, 1]$ consider the subspace $L(\lambda)$ of L_n with basis $\{y, \lambda z + (1 - \lambda)w\}$. Let f_λ be the restriction of f to $L(\lambda)$. The set $L(\lambda) \cap S$ is compact; hence, f_λ attains a maximum on $L(\lambda) \cap S$ at some point $u, f_\lambda(u) \geq 1$. Since $\dim(L(\lambda) \cap [f: f(u)]) = 1$ and

$$L(\lambda) \cap S \cap [f: f(u)]$$

is compact, there exists a minimal closed line segment in $L(\lambda)$ which contains $L(\lambda) \cap [f: f(u)] \cap S$. This line segment must have at least one endpoint, which must belong to $S(n - 1)$. For each pair of distinct real numbers λ, μ in $[0, 1]$, $L(\lambda) \cap L(\mu) \subset [f: 0]$. There exists points $p_\lambda \in L(\lambda) \cap S(n - 1), p_\mu \in L(\mu) \cap S(n - 1)$ such that $f(p_\lambda) \geq 1, f(p_\mu) \geq 1$, which implies that $p_\lambda \neq p_\mu$. Thus, the set $S(n - 1)$ is uncountable.

THEOREM 3. *Let S be a closed subset of a linear topological space L and let T be a subset of S that star-generates $\text{ck}S$, which may be empty. If M is a dense subset of T , then M star-generates $\text{ck}S$.*

Proof. Since $M \subset T$ then clearly $S_T \subset S_M$. Suppose that M is a proper subset of T and $\text{ck}S$ is a proper subset of S_M . Then there exists a point $q \in S_M \setminus S_T$. But $S_T = S_M \cap S_{T \setminus M}$; thus $q \notin S_{T \setminus M}$. This implies that $q \notin S_x$ for some $x \in T \setminus M$. Since $q \in S_M, M \subset S_q$, which is closed. Hence, $x \in T \subset \text{cl } M \subset S_q$, which implies that $xq \subset S$ and that $q \in S_x$, a contradiction. Therefore, $\text{ck}S = S_M$.

THEOREM 4. *If S is a compact star-shaped subset of a normed linear space L , then any subset T of S which star-generates the convex kernel of S contains a countable subset M which also star-generates the convex kernel of S .*

Proof. The norm of L induces a metric on L . The compact set S can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space is separable, which implies that S is second countable [2]. Any nonempty subset T of S is a second countable topological space with the relative topology, which implies that T is separable. There exists a countable subset M of T such that $T \subset \text{cl } M$. Theorem 3 implies that M star-generates ckS and the theorem is proved.

COROLLARY. *Let S be a compact star-shaped subset of L_{k+1} . Then there exists a countable subset of $S(k)$ which star-generates ckS .*

Klee [3] introduced the concept of relative extreme point.

DEFINITION 3. If S and T are subsets of a linear space L , then $x \in S$ is said to be extreme in S relative to T if there do not exist points $y \in S, z \in T$ such that $x \in \text{intv } yz$.

If S is a star-shaped set, $\text{exk } S$ will denote the points of S which are extreme relative to ckS , and $E_s = (\text{exk } S) \setminus ckS$.

THEOREM 5. *Let S be a compact nonconvex star-shaped set in a locally convex space L . Then $C = S$, where*

$$C = \bigcup_{y \in E_s} \text{conv}(ckS \cup \{y\}) .$$

Proof. Since $E_s \subset S$, $\text{conv}(ckS \cup \{y\}) \subset S$ for each $y \in E_s$. Thus, $C \subset S$. Consider $z \in ckS \cup \text{exk } S$; since $E_s \neq \emptyset$, as shown below, $z \in C$. Let $K = ckS$. Suppose that $z \in S \setminus (ckS \cup \text{exk } S)$ and without loss of generality, suppose that $z = 0$. Since K is compact and convex, K^* and $-K^*$ are closed convex cones with vertex 0, where $K^* = \{\lambda x : x \in K, \lambda \geq 0\}$. Since $z \notin \text{exk } S$ there exist points $x \in K$ and $w \in S$ such that $0 \in \text{intv } xw$. Clearly $w \in -K^* \setminus \{0\}$, $S \cap (-K^* \setminus \{0\}) \neq \emptyset$ and $S \cap (-K^*)$ is compact. Let u be an arbitrary point in $-K^* \setminus \{0\}$; since L is locally convex and K^* is closed and convex, there exists a closed hyperplane $H = [f : f(u)]$ such that $u \in H$ and $H \cap K^* = \emptyset$, where f is a continuous linear functional. It can be assumed that $f(K^*) \leq 0$, which implies that $f(u) > 0$. The functional f then attains a maximum on $S \cap (-K^*)$ at some point $v \in S \cap (-K^*)$. Suppose that $v \notin \text{exk } S$. There exist points $p \in K, q \in S$ such that $v \in \text{intv } pq$. Since $v \in -K^*$, $v = -\lambda p', p' \in K, \lambda > 0$, and

$$v = \alpha p + (1 - \alpha)q , \quad 0 < \alpha < 1 .$$

Therefore, $v = -\lambda p' = \alpha p + (1 - \alpha)q$ and $q = \tau q'$, where $\tau < 0$ and $q' \in K$. Thus, $q \in S \cap (-K^*)$. But it can be easily shown that

$f(q) > f(v)$, which contradicts the fact that $f(v) \geq f(x)$ for each $x \in S \cap (-K^*)$. Hence, $v \in (\text{exk } S) \cap (-K^*)$ and $0 \in C$, which implies that $S \subset C$. This inclusion, along with the one given earlier, implies that $S = C$.

The following result shows that the set E_s is minimal in its use in Theorem 5.

THEOREM 6. *Let S be a compact nonconvex star-shaped set in a locally convex space L . If T is a proper subset of E_s then*

$$C(T) = \bigcup_{y \in T} \text{conv}(ckS \cup \{y\})$$

is a proper subset of S .

Proof. Consider any proper subset T of E_s ; there exists some point $x \in E_s \setminus T$. If $x \in C(T)$ there exists some $y \in T$ such that $x \in \text{conv}(ckS \cup \{y\})$. Hence, $x = \lambda z + (1 - \lambda)y$, where $\lambda \in [0, 1]$, $z \in ckS$. But $\lambda \in (0, 1)$ since $x \notin ckS \cup T$. This implies that $x \in \text{exk } S$, a contradiction. Thus, $x \notin C(T)$, which must be a proper subset of S .

BIBLIOGRAPHY

1. E. Asplund, *A k -extreme point is the limit of k -exposed points*, Israel J. Math. **1** (1963), 161-162.
2. D. W. Hall and G. L. Spencer II, *Elementary topology*, John Wiley and Sons, Inc., New York, 1955.
3. V. L. Klee, *Relative extreme points*, Proceedings of the International Symposium on Linear Spaces, Pergamon Press, New York, 1961.
4. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

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