

# Pacific Journal of Mathematics

**EXTREMAL STRUCTURE OF STAR-SHAPED SETS**

FREDDIE EUGENE TIDMORE

## EXTREMAL STRUCTURE OF STAR-SHAPED SETS

F. E. TIDMORE

It is shown that the convex kernel of a compact star-shaped subset  $S$  of a finite-dimensional linear topological space  $L_n$  is determined by the  $(n - 1)$ -extreme points of  $S$ . The cardinality of the set of  $k$ -extreme points is determined for compact star-shaped sets of dimension greater than two. Also given is the result that any compact star-shaped subset  $S$  of  $L_n$  contains a countable set of  $(n - 1)$ -extreme points which determines the convex kernel of  $S$ . Another result is that a compact nonconvex star-shaped set  $S$  in a locally convex space  $L$  is determined by the convex kernel of  $S$  and the subset of points that are extreme in  $S$  relative to the convex kernel of  $S$ .

The convex kernel of a star-shaped set  $S$  will be denoted by  $ckS$ , the line segment  $\{\alpha x + (1 - \alpha)y: \alpha \in [0, 1]\}$  will be denoted by  $xy$ , the ray  $\{\beta y + (1 - \beta)x: \beta \geq 1\}$  will be denoted by  $xy^\infty$  and  $L(x, y)$  will denote the line containing  $x$  and  $y$ ,  $x \neq y$ . The convex hull of a set  $S$  will be denoted by  $\text{conv } S$ . The notation  $\text{intv } S$  will denote the interior of  $S$  relative to the minimal flat that contains  $S$ . The set  $\{x: f(x) = \alpha\}$ , where  $f$  is a linear functional, will be denoted  $[f: \alpha]$ . Set-theoretic difference will be denoted by  $\setminus$ , and the closure of a set  $S$  will be denoted by  $\text{cl } S$ .

The concept of  $k$ -extreme point was introduced by Asplund [1].

**DEFINITION 1.** If  $S$  is a subset of a linear space  $L$ , a point  $x \in S$  is a  $k$ -extreme point of  $S$  if no  $k$ -simplex  $\Delta$  exists such that  $x \in \text{intv } \Delta \subset S$ .

For a subset  $S$  of a linear space  $L$ ,  $S_x$  will denote the  $x$ -star of  $S$  determined by the point  $x \in S$ ; that is, the set of points  $y$  such that  $xy \subset S$ . If  $S$  is a closed (compact) subset of a linear topological space  $L$ , then for any  $x \in S$ ,  $S_x$  is a closed (compact) set. If  $T \subset S$ , let

$$S_T = \bigcap_{x \in T} S_x.$$

A point  $p$  belongs to the convex kernel of  $S$  if, and only if,  $xp \subset S$  for all  $x \in S$ , which is true if, and only if,  $p \in S_x$  for all  $x \in S$ . Thus  $ckS = S_S$ , which motivates the following definition.

**DEFINITION 2.** In a linear space  $L$  a subset  $T$  of a star-shaped

set  $S$  is said to star-generate the convex kernel of  $S$  if  $ckS = S_r$ . Such a subset  $T$  is said to be a star-generating set for  $ckS$ .

**THEOREM 1.** *Let  $S$  be a compact star-shaped subset of  $L_{k+1}$ . Then the set  $S(k)$  of  $k$ -extreme points of  $S$  is a star-generating set for  $ckS$ .*

*Proof.* Without loss of generality, suppose that  $0 \in ckS$ . If  $S = ckS$ , then  $S$  is convex and  $S_x = S$  for each  $x \in S$  and the result follows since  $\emptyset \neq S(1) \subset S(k)$ . Let  $p \in S \setminus ckS$ . Then there exists a point  $y \in S$  such that  $py \notin S$ . Since  $S$  is compact,  $y$  can be chosen such that  $S \cap \text{intv } py^\infty = \emptyset$ . Since  $py \notin S$ , there exists a point  $z \in (\text{intv } py) \setminus S$ . If  $y \in S(k)$ , then  $p \notin S_y$  implies  $p \notin S_{S(k)}$ . If  $y \notin S(k)$  there exists a  $k$ -simplex  $\Delta$  such that  $y \in \text{intv } \Delta \subset S$ . Consider the convex cone  $C = \{\beta y + (\lambda - \beta + 1)z : \beta, \lambda \geq 0\}$ , which has vertex  $z$  and is contained in the subspace  $L'$  with basis  $\{p, y\}$ . Since  $S \cap \text{intv } py^\infty = \emptyset$ ,  $\Delta$  must intersect  $L'$  in some line other than  $L(p, y)$ ; thus,  $S \cap \text{intv } C \neq \emptyset$ . There exists a linear functional  $f$  defined on  $L_{k+1}$  such that  $f(q) = 1$  for every  $q \in L(p, y)$ ; clearly  $0 \notin L(p, y)$  since  $py \notin S$  and  $0 \in ckS$ . The continuous linear functional  $f_1$ , the restriction of  $f$  to  $L'$ , attains a maximum on the compact set  $C \cap S$  at some point  $w \in \text{intv } C$ . Let  $H = [f : f(w)]$ . Since  $H \cap C \cap S$  is a compact subset of the 1-dimensional set  $H \cap L'$ , there exists a minimal closed line segment in  $\text{intv } C$  which contains  $H \cap C \cap S$ . Each endpoint of this segment, which may be degenerate, must be a point in  $S(k)$ . Let  $v$  be one of these endpoints. The points  $p, y, z$  and  $v$  are in  $L'$ . If  $pv \subset S$ , then the fact that  $0 \in ckS$  implies that  $z \in \text{conv } \{0, p, v\} \subset S$ , a contradiction. Hence,  $pv \not\subset S$  and  $p \notin S_{S(k)}$ . Therefore,  $S \setminus ckS \subset S \setminus S_{S(k)}$ , which gives the desired equality, since clearly  $ckS \subset S_{S(k)}$ .

It is not always sufficient to consider only the set of familiar extreme points  $S(1)$  as a star-generating set for  $ckS$ . For example, in  $E_3$  let  $S$  be the union of three closed faces of a 3-simplex. In some cases, proper subsets of  $S(k)$  exist which will star-generate  $ckS$ . However, characterizing such subsets may be very difficult, as indicated by the following example.

**EXAMPLE 1.** In the plane  $E_2$  let  $B_u$  be the upper closed unit half-disc,  $B_r$  the right closed unit half-disc. Let

$$\begin{aligned} T_1 &= \text{conv} [\{-2e_1\} \cup (B_r + (2e_1 + e_2))] , \\ T_2 &= \text{conv} [\{-2e_2\} \cup (B_u + (2e_2 - e_1))] , \\ S &= T_1 \cup T_2 \cup (-T_1) \cup (-T_2) . \end{aligned}$$

Then any star-generating subset of  $S(1)$  must contain four distinct

sequences of carefully chosen extreme points.

**THEOREM 2.** *If  $S$  is a compact star-shaped set in  $L_n$ , and  $\dim(S) \geq 3$ , then  $S(n-1)$  is an uncountable set.*

*Proof.* Without loss of generality, it can be assumed that  $0 \in \text{ck}S$ . Since  $\dim(S) \geq 3$  there exists some point  $x \in S, x \neq 0$ . If  $\beta x \in S(n-1)$  for every  $\beta \in (0, 1)$ , then  $S(n-1)$  is uncountable. Otherwise, consider some  $w = \beta x$  such that  $w \notin S(n-1)$ . Then there exists an  $(n-1)$ -simplex  $\Delta$  such that  $w \in \text{int} \Delta \subset S$ . Since  $n-1 \geq 2$  there exists a nondegenerate line segment  $zw \subset \Delta$  such that  $zw \cap 0x = \{w\}$ . There exists a linear functional  $f$  on  $L_n$  such that

$$f(w) = f(z) = 1.$$

There exists a point  $y \in [f:0]$  such that the set  $\{y, z, w\}$  is linearly independent. For each  $\lambda \in [0, 1]$  consider the subspace  $L(\lambda)$  of  $L_n$  with basis  $\{y, \lambda z + (1-\lambda)w\}$ . Let  $f_\lambda$  be the restriction of  $f$  to  $L(\lambda)$ . The set  $L(\lambda) \cap S$  is compact; hence,  $f_\lambda$  attains a maximum on  $L(\lambda) \cap S$  at some point  $u, f_\lambda(u) \geq 1$ . Since  $\dim(L(\lambda) \cap [f:f(u)]) = 1$  and

$$L(\lambda) \cap S \cap [f:f(u)]$$

is compact, there exists a minimal closed line segment in  $L(\lambda)$  which contains  $L(\lambda) \cap [f:f(u)] \cap S$ . This line segment must have at least one endpoint, which must belong to  $S(n-1)$ . For each pair of distinct real numbers  $\lambda, \mu$  in  $[0, 1]$ ,  $L(\lambda) \cap L(\mu) \subset [f:0]$ . There exists points  $p_\lambda \in L(\lambda) \cap S(n-1), p_\mu \in L(\mu) \cap S(n-1)$  such that  $f(p_\lambda) \geq 1, f(p_\mu) \geq 1$ , which implies that  $p_\lambda \neq p_\mu$ . Thus, the set  $S(n-1)$  is uncountable.

**THEOREM 3.** *Let  $S$  be a closed subset of a linear topological space  $L$  and let  $T$  be a subset of  $S$  that star-generates  $\text{ck}S$ , which may be empty. If  $M$  is a dense subset of  $T$ , then  $M$  star-generates  $\text{ck}S$ .*

*Proof.* Since  $M \subset T$  then clearly  $S_T \subset S_M$ . Suppose that  $M$  is a proper subset of  $T$  and  $\text{ck}S$  is a proper subset of  $S_M$ . Then there exists a point  $q \in S_M \setminus S_T$ . But  $S_T = S_M \cap S_{T \setminus M}$ ; thus  $q \notin S_{T \setminus M}$ . This implies that  $q \notin S_x$  for some  $x \in T \setminus M$ . Since  $q \in S_M, M \subset S_q$ , which is closed. Hence,  $x \in T \subset \text{cl} M \subset S_q$ , which implies that  $xq \subset S$  and that  $q \in S_x$ , a contradiction. Therefore,  $\text{ck}S = S_M$ .

**THEOREM 4.** *If  $S$  is a compact star-shaped subset of a normed linear space  $L$ , then any subset  $T$  of  $S$  which star-generates the convex kernel of  $S$  contains a countable subset  $M$  which also star-generates the convex kernel of  $S$ .*

*Proof.* The norm of  $L$  induces a metric on  $L$ . The compact set  $S$  can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space is separable, which implies that  $S$  is second countable [2]. Any nonempty subset  $T$  of  $S$  is a second countable topological space with the relative topology, which implies that  $T$  is separable. There exists a countable subset  $M$  of  $T$  such that  $T \subset \text{cl } M$ . Theorem 3 implies that  $M$  star-generates  $ckS$  and the theorem is proved.

**COROLLARY.** *Let  $S$  be a compact star-shaped subset of  $L_{k+1}$ . Then there exists a countable subset of  $S(k)$  which star-generates  $ckS$ .*

Klee [3] introduced the concept of relative extreme point.

**DEFINITION 3.** If  $S$  and  $T$  are subsets of a linear space  $L$ , then  $x \in S$  is said to be extreme in  $S$  relative to  $T$  if there do not exist points  $y \in S, z \in T$  such that  $x \in \text{intv } yz$ .

If  $S$  is a star-shaped set,  $\text{exk } S$  will denote the points of  $S$  which are extreme relative to  $ckS$ , and  $E_S = (\text{exk } S) \setminus ckS$ .

**THEOREM 5.** *Let  $S$  be a compact nonconvex star-shaped set in a locally convex space  $L$ . Then  $C = S$ , where*

$$C = \bigcup_{y \in E_S} \text{conv}(ckS \cup \{y\}) .$$

*Proof.* Since  $E_S \subset S$ ,  $\text{conv}(ckS \cup \{y\}) \subset S$  for each  $y \in E_S$ . Thus,  $C \subset S$ . Consider  $z \in ckS \cup \text{exk } S$ ; since  $E_S \neq \emptyset$ , as shown below,  $z \in C$ . Let  $K = ckS$ . Suppose that  $z \in S \setminus (ckS \cup \text{exk } S)$  and without loss of generality, suppose that  $z = 0$ . Since  $K$  is compact and convex,  $K^*$  and  $-K^*$  are closed convex cones with vertex  $0$ , where  $K^* = \{\lambda x : x \in K, \lambda \geq 0\}$ . Since  $z \notin \text{exk } S$  there exist points  $x \in K$  and  $w \in S$  such that  $0 \in \text{intv } xw$ . Clearly  $w \in -K^* \setminus \{0\}$ ,  $S \cap (-K^* \setminus \{0\}) \neq \emptyset$  and  $S \cap (-K^*)$  is compact. Let  $u$  be an arbitrary point in  $-K^* \setminus \{0\}$ ; since  $L$  is locally convex and  $K^*$  is closed and convex, there exists a closed hyperplane  $H = [f : f(u)]$  such that  $u \in H$  and  $H \cap K^* = \emptyset$ , where  $f$  is a continuous linear functional. It can be assumed that  $f(K^*) \leq 0$ , which implies that  $f(u) > 0$ . The functional  $f$  then attains a maximum on  $S \cap (-K^*)$  at some point  $v \in S \cap (-K^*)$ . Suppose that  $v \notin \text{exk } S$ . There exist points  $p \in K, q \in S$  such that  $v \in \text{intv } pq$ . Since  $v \in -K^*$ ,  $v = -\lambda p', p' \in K, \lambda > 0$ , and

$$v = \alpha p + (1 - \alpha)q , \quad 0 < \alpha < 1 .$$

Therefore,  $v = -\lambda p' = \alpha p + (1 - \alpha)q$  and  $q = \tau q'$ , where  $\tau < 0$  and  $q' \in K$ . Thus,  $q \in S \cap (-K^*)$ . But it can be easily shown that

$f(q) > f(v)$ , which contradicts the fact that  $f(v) \geq f(x)$  for each  $x \in S \cap (-K^*)$ . Hence,  $v \in (\text{exk } S) \cap (-K^*)$  and  $0 \in C$ , which implies that  $S \subset C$ . This inclusion, along with the one given earlier, implies that  $S = C$ .

The following result shows that the set  $E_s$  is minimal in its use in Theorem 5.

**THEOREM 6.** *Let  $S$  be a compact nonconvex star-shaped set in a locally convex space  $L$ . If  $T$  is a proper subset of  $E_s$  then*

$$C(T) = \bigcup_{y \in T} \text{conv}(ckS \cup \{y\})$$

*is a proper subset of  $S$ .*

*Proof.* Consider any proper subset  $T$  of  $E_s$ ; there exists some point  $x \in E_s \setminus T$ . If  $x \in C(T)$  there exists some  $y \in T$  such that  $x \in \text{conv}(ckS \cup \{y\})$ . Hence,  $x = \lambda z + (1 - \lambda)y$ , where  $\lambda \in [0, 1]$ ,  $z \in ckS$ . But  $\lambda \in (0, 1)$  since  $x \notin ckS \cup T$ . This implies that  $x \notin \text{exk } S$ , a contradiction. Thus,  $x \notin C(T)$ , which must be a proper subset of  $S$ .

#### BIBLIOGRAPHY

1. E. Asplund, *A  $k$ -extreme point is the limit of  $k$ -exposed points*, Israel J. Math. **1** (1963), 161-162.
2. D. W. Hall and G. L. Spencer II, *Elementary topology*, John Wiley and Sons, Inc., New York, 1955.
3. V. L. Klee, *Relative extreme points*, Proceedings of the International Symposium on Linear Spaces, Pergamon Press, New York, 1961.
4. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

Received May 13, 1968. This work was done to partially fulfill the requirements for the degree of Doctor of Philosophy at Oklahoma State University under the direction of Professor E. K. McLachlan. During this time the author was an NSF Cooperative Graduate Fellow.

OKLAHOMA STATE UNIVERSITY  
TEXAS TECHNOLOGICAL COLLEGE



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN  
Stanford University  
Stanford, California

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. R. PHELPS  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON  
\* \* \*  
AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Bruce Langworthy Chalmers, <i>On boundary behavior of the Bergman kernel function and related domain functionals</i> .....	243
William Eugene Copping, <i>Peirce decomposition in simple Lie-admissible power-associative rings</i> .....	251
Edwin Duda, <i>Compactness of mappings</i> .....	259
Earl F. Ecklund Jr., <i>On prime divisors of the binomial coefficient</i> .....	267
Don E. Edmondson, <i>A modular topological lattice</i> .....	271
Phillip Alan Griffith, <i>A note on a theorem of Hill</i> .....	279
Marcel Herzog, <i>On finite groups with independent cyclic Sylow subgroups</i> .....	285
James A. Huckaba, <i>Extensions of pseudo-valuations</i> .....	295
S. A. Huq, <i>Semivarieties and subfunctors of the identity functor</i> .....	303
I. Martin (Irving) Isaacs and Donald Steven Passman, <i>Finite groups with small character degrees and large prime divisors. II</i> .....	311
Carl Kallina, <i>A Green's function approach to perturbations of periodic solutions</i> .....	325
Al (Allen Frederick) Kelley, Jr., <i>Analytic two-dimensional subcenter manifolds for systems with an integral</i> .....	335
Alistair H. Lachlan, <i>Initial segments of one-one degrees</i> .....	351
Marion-Josephine Lim, <i>Rank k Grassmann products</i> .....	367
Raymond J. McGivney and William Henry Ruckle, <i>Multiplier algebras of biorthogonal systems</i> .....	375
J. K. Oddson, <i>On the rate of decay of solutions of parabolic differential equations</i> .....	389
Helmut R. Salzmann, <i>Geometries on surfaces</i> .....	397
Annemarie Schlette, <i>Artinian, almost abelian groups and their groups of automorphisms</i> .....	403
Edgar Lee Stout, <i>Additional results on modules over polydisc algebras</i> .....	427
Lajos Tamássy, <i>A characteristic property of the sphere</i> .....	439
Mark Lawrence Teply, <i>Some aspects of Goldie's torsion theory</i> .....	447
Freddie Eugene Tidmore, <i>Extremal structure of star-shaped sets</i> .....	461
Leon Jarome Weill, <i>Unconditional and shrinking bases in locally convex spaces</i> .....	467