Pacific Journal of Mathematics

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Vol. 29, No. 2 June 1969

PACIFIC JOURNAL OF MATHEMATICS
Vol. 29, No. 2, 1969

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It is shown that the convex kernel of a compact star-shaped subset S of a finite-dimensional linear topological space L_n is determined by the $(n-1)$-extreme points of S. The cardinality of the set of k-extreme points is determined for compact star-shaped sets of dimension greater than two. Also given is the result that any compact star-shaped subset S of L_n contains a countable set of $(n-1)$-extreme points which determines the convex kernel of S. Another result is that a compact nonconvex star-shaped set S in a locally convex space L is determined by the convex kernel of S and the subset of points that are extreme in S relative to the convex kernel of S.

The convex kernel of a star-shaped set S will be denoted by ckS, the line segment $\{\alpha x + (1-\alpha)y : \alpha \in [0, 1]\}$ will be denoted by xy, the ray $\{\beta y + (1-\beta)x : \beta \geqq 1\}$ will be denoted by xy^∞ and $L(x, y)$ will denote the line containing x and $y, x \neq y$. The convex hull of a set S will be denoted by conv S. The notation intv S will denote the interior of S relative to the minimal flat that contains S. The set $\{x: f(x) = \alpha\}$, where f is a linear functional, will be denoted $[f: \alpha]$. Set-theoretic difference will be denoted by $\backslash$, and the closure of a set S will be denoted by cl S.

The concept of k-extreme point was introduced by Asplund [1].

DEFINITION 1. If S is a subset of a linear space L, a point $x \in S$ is a k-extreme point of S if no k-simplex Δ exists such that $x \in \text{intv } \Delta \subset S$.

For a subset S of a linear space L, S_x will denote the x-star of S determined by the point $x \in S$; that is, the set of points y such that $xy \subset S$. If S is a closed (compact) subset of a linear topological space L, then for any $x \in S, S_x$ is a closed (compact) set. If $T \subset S$, let

$$S_T = \bigcap_{x \in T} S_x .$$

A point p belongs to the convex kernel of S if, and only if, $xp \subset S$ for all $x \in S$, which is true if, and only if, $p \in S_x$ for all $x \in S$. Thus $ckS = S_S$, which motivates the following definition.

DEFINITION 2. In a linear space L a subset T of a star-shaped set S is said to star-generate the convex kernel of S if $ckS = S_T$. Such a subset T is said to be a star-generating set for ckS.

THEOREM 1. *Let S be a compact star-shaped subset of L_{k+1}. Then the set $S(k)$ of k-extreme points of S is a star-generating set for ckS.*

Proof. Without loss of generality, suppose that $0 \in ckS$. If $S = ckS$, then S is convex and $S_x = S$ for each $x \in S$ and the result follows since $\emptyset \neq S(1) \subset S(k)$. Let $p \in S \backslash ckS$. Then there exists a point $y \in S$ such that $py \not\subset S$. Since S is compact, y can be chosen such that $S \cap \text{intv}\ py^\infty = \emptyset$. Since $py \not\subset S$, there exists a point $z \in (\text{intv}\ py) \backslash S$. If $y \in S(k)$, then $p \notin S_y$ implies $p \notin S_{S(k)}$. If $y \notin S(k)$ there exists a k-simplex Δ such that $y \in \text{intv}\ \Delta \subset S$. Consider the convex cone $C = \{\beta y + (\lambda - \beta + 1)\ z \colon \beta, \lambda \geqq 0\}$, which has vertex z and is contained in the subspace L' with basis $\{p, y\}$. Since $S \cap \text{intv}\ py^\infty = \emptyset$, Δ must intersect L' in some line other than $L(p, y)$; thus, $S \cap \text{intv}\ C \neq \emptyset$. There exists a linear functional f defined on L_{k+1} such that $f(q) = 1$ for every $q \in L(p, y)$; clearly $0 \notin L(p, y)$ since $py \not\subset S$ and $0 \in ckS$. The continuous linear functional f_1, the restriction of f to L', attains a maximum on the compact set $C \cap S$ at some point $w \in \text{intv}\ C$. Let $H = [f \colon f(w)]$. Since $H \cap C \cap S$ is a compact subset of the 1-dimensional set $H \cap L'$, there exists a minimal closed line segment in intv C which contains $H \cap C \cap S$. Each endpoint of this segment, which may be degenerate, must be a point in $S(k)$. Let v be one of these endpoints. The points p, y, z and v are in L'. If $pv \subset S$, then the fact that $0 \in ckS$ implies that $z \in \text{conv}\ \{0, p, v\} \subset S$, a contradiction. Hence, $pv \not\subset S$ and $p \notin S_{S(k)}$. Therefore, $S \backslash ckS \subset S \backslash S_{S(k)}$, which gives the desired equality, since clearly $ckS \subset S_{S(k)}$.

It is not always sufficient to consider only the set of familiar extreme points $S(1)$ as a star-generating set for ckS. For example, in E_3 let S be the union of three closed faces of a 3-simplex. In some cases, proper subsets of $S(k)$ exist which will star-generate ckS. However, characterizing such subsets may be very difficult, as indicated by the following example.

EXAMPLE 1. In the plane E_2 let B_u be the upper closed unit half-disc, B_r the right closed unit half-disc. Let

$$T_1 = \text{conv}\,[\{-2e_1\} \cup (B_r + (2e_1 + e_2))]\ ,$$
$$T_2 = \text{conv}\,[\{-2e_2\} \cup (B_u + (2e_2 - e_1))]\ ,$$
$$S = T_1 \cup T_2 \cup (-T_1) \cup (-T_2)\ .$$

Then any star-generating subset of $S(1)$ must contain four distinct sequences of carefully chosen extreme points.

THEOREM 2. *If S is a compact star-shaped set in L_n, and* dim $(S) \geqq 3$, *then $S(n-1)$ is an uncountable set.*

Proof. Without loss of generality, it can be assumed that $0 \in ckS$. Since $\dim(S) \geqq 3$ there exists some point $x \in S, x \neq 0$. If $\beta x \in S(n-1)$ for every $\beta \in (0, 1)$, then $S(n-1)$ is uncountable. Otherwise, consider some $w = \beta x$ such that $w \notin S(n-1)$. Then there exists an $(n-1)$-simplex Δ such that $w \in \text{intv}\, \Delta \subset S$. Since $n-1 \geqq 2$ there exists a nondegenerate line segment $zw \subset \Delta$ such that $zw \cap 0x = \{w\}$. There exists a linear functional f on L_n such that

$$f(w) = f(z) = 1 .$$

There exists a point $y \in [f: 0]$ such that the set $\{y, z, w\}$ is linearly independent. For each $\lambda \in [0, 1]$ consider the subspace $L(\lambda)$ of L_n with basis $\{y, \lambda z + (1-\lambda)w\}$. Let f_λ be the restriction of f to $L(\lambda)$. The set $L(\lambda) \cap S$ is compact; hence; f_λ attains a maximum on $L(\lambda) \cap S$ at some point $u, f_\lambda(u) \geqq 1$. Since $\dim(L(\lambda) \cap [f: f(u)]) = 1$ and

$$L(\lambda) \cap S \cap [f: f(u)]$$

is compact, there exists a minimal closed line segment in $L(\lambda)$ which contains $L(\lambda) \cap [f: f(u)] \cap S$. This line segment must have at least one endpoint, which must belong to $S(n-1)$. For each pair of distinct real numbers λ, μ in $[0, 1], L(\lambda) \cap L(\mu) \subset [f; 0]$. There exists points $p_\lambda \in L(\lambda) \cap S(n-1), p_\mu \in L(\mu) \cap S(n-1)$ such that $f(p_\lambda) \geqq 1, f(p_\mu) \geqq 1$, which implies that $p_\lambda \neq p_\mu$. Thus, the set $S(n-1)$ is uncountable.

THEOREM 3. *Let S be a closed subset of a linear topological space L and let T be a subset of S that star-generates ckS, which may be empty. If M is a dense subset of T, then M star-generates ckS.*

Proof. Since $M \subset T$ then clearly $S_T \subset S_M$. Suppose that M is a proper subset of T and ckS is a proper subset of S_M. Then there exists a point $q \in S_M \backslash S_T$. But $S_T = S_M \cap S_{T \backslash M}$; thus $q \notin S_{T\backslash M}$. This implies that $q \notin S_x$ for some $x \in T \backslash M$. Since $q \in S_M, M \subset S_q$, which is closed. Hence, $x \in T \subset \text{cl}\, M \subset S_q$, which implies that $xq \subset S$ and that $q \in S_x$, a contradiction. Therefore, $ckS = S_M$.

THEOREM 4. *If S is a compact star-shaped subset of a normed linear space L, then any subset T of S which star-generates the convex kernel of S contains a countable subset M which also star-generates the convex kernel of S.*

Proof. The norm of L induces a metric on L. The compact set S can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space is separable, which implies that S is second countable [2]. Any nonempty subset T of S is a second countable topological space with the relative topology, which implies that T is separable. There exists a countable subset M of T such that $T \subset \operatorname{cl} M$. Theorem 3 implies that M star-generates ckS and the theorem is proved.

COROLLARY. *Let S be a compact star-shaped subset of L_{k+1}. Then there exists a countable subset of $S(k)$ which star-generates ckS.*

Klee [3] introduced the concept of relative extreme point.

DEFINITION 3. If S and T are subsets of a linear space L, then $x \in S$ is said to be extreme in S relative to T if there do not exist points $y \in S, z \in T$ such that $x \in \operatorname{intv} yz$.

If S is a star-shaped set, exk S will denote the points of S which are extreme relative to ckS, and $E_S = (\operatorname{exk} S)\backslash ckS$.

THEOREM 5. *Let S be a compact nonconvex star-shaped set in a locally convex space L. Then $C = S$, where*

$$C = \bigcup_{y \in E_S} \operatorname{conv} (ckS \cup \{y\}) .$$

Proof. Since $E_S \subset S$, $\operatorname{conv} (ckS \cup \{y\}) \subset S$ for each $y \in E_S$. Thus, $C \subset S$. Consider $z \in ckS \cup \operatorname{exk} S$; since $E_S \neq \emptyset$, as shown below, $z \in C$. Let $K = ckS$. Suppose that $z \in S\backslash(ckS \cup \operatorname{exk} S)$ and without loss of generality, suppose that $z = 0$. Since K is compact and convex, K^* and $-K^*$ are closed convex cones with vertex 0, where $K^* = \{\lambda x: x \in K, \lambda \geqq 0\}$. Since $z \notin \operatorname{exk} S$ there exist points $x \in K$ and $w \in S$ such that $0 \in \operatorname{intv} xw$. Clearly $w \in -K^*\backslash\{0\}$, $S \cap (-K^*\backslash\{0\}) \neq \emptyset$ and $S \cap (-K^*)$ is compact. Let u be an arbitrary point in $-K^*\backslash\{0\}$; since L is locally convex and K^* is closed and convex, there exists a closed hyperplane $H = [f: f(u)]$ such that $u \in H$ and $H \cap K^* = \emptyset$, where f is a continuous linear functional. It can be assumed that $f(K^*) \leqq 0$, which implies that $f(u) > 0$. The functional f then attains a maximum on $S \cap (-K^*)$ at some point $v \in S \cap (-K^*)$. Suppose that $v \notin \operatorname{exk} S$. There exist points $p \in K, q \in S$ such that $v \in \operatorname{intv} pq$. Since $v \in -K^*, v = -\lambda p', p' \in K, \lambda > 0$, and

$$v = \alpha p + (1 - \alpha)q , \qquad 0 < \alpha < 1 .$$

Therefore, $v = -\lambda p' = \alpha p + (1 - \alpha)q$ and $q = \tau q'$, where $\tau < 0$ and $q' \in K$. Thus, $q \in S \cap (-K^*)$. But it can be easily shown that

$f(q) > f(v)$, which contradicts the fact that $f(v) \geqq f(x)$ for each $x \in S \cap (-K^*)$. Hence, $v \in (\text{exk } S) \cap (-K^*)$ and $0 \in C$, which implies that $S \subset C$. This inclusion, along with the one given earlier, implies that $S = C$.

The following result shows that the set E_S is minimal in its use in Theorem 5.

THEOREM 6. *Let S be a compact nonconvex star-shaped set in a locally convex space L. If T is a proper subset of E_S then*

$$C(T) = \bigcup_{y \in T} \text{conv}\,(ckS \cup \{y\})$$

is a proper subset of S.

Proof. Consider any proper subset T of E_S; there exists some point $x \in E_S \backslash T$. If $x \in C(T)$ there exists some $y \in T$ such that $x \in \text{conv}\,(ckS \cup \{y\})$. Hence, $x = \lambda z + (1 - \lambda)y$, where $\lambda \in [0, 1]$, $z \in ckS$. But $\lambda \in (0, 1)$ since $x \notin ckS \cup T$. This implies that $x \notin \text{exk } S$, a contradiction. Thus, $x \notin C(T)$, which must be a proper subset of S.

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Received May 13, 1968. This work was done to partially fulfill the requirements for the degree of Doctor of Philosophy at Oklahoma State University under the direction of Professor E. K. McLachlan. During this time the author was an NSF Cooperative Graduate Fellow.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 29, No. 2 June, 1969