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PACIFIC JOURNAL OF MATHEMATICS
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ON BOUNDARY BEHAVIOR OF THE BERGMAN KERNEL FUNCTION AND RELATED DOMAIN FUNCTIONALS

BRUCE L. CHALMERS

Bergman and Hörmander have studied the boundary behavior of the Bergman kernel function for domains in $\underset{\sim}{C}^n$, $2 \leqq n < \infty$, with strictly pseudo-convex boundaries. It is the purpose of this paper to extend these results to a larger and more general class of domains with so called strictly (p, q)-pseudo-convex boundaries.

Let B be a bounded domain in the space $\underset{\sim}{C}^n$ of n complex variables, say $z_k = x_k + iy_k$, $k = 1, 2, \cdots, n$, where $2 \leqq n < \infty$. (The euclidean norm of a point $z = (z_1, z_2, \cdots, z_n)$ will be denoted by

$$|z| = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}$$

and we let $\bar{z} = (\bar{z}_1, \bar{z}_2, \cdots, \bar{z}_n)$ and $0 = (0, 0, \cdots, 0)$.) The space $H(B)$ of all square-integrable holomorphic functions on B forms a Hilbert space with the Bergman (reproducing) kernel function $K_B(z, \bar{t})$ with respect to the inner product $(f, g) = \int_B f\bar{g}\, d\omega$, where $f, g \in H(B)$ and $d\omega$ is the euclidean volume element of B. If $\{\varphi_n(z)\}_{n=1}^{\infty}$ is any complete orthonormal system for $H(B)$, then

$$K_B(z, \bar{t}) = \sum_{n=1}^{\infty} \varphi_n(z)\, \overline{\varphi_n(t)}\,, \tag{1}$$

which converges uniformly in the interior of $B \times B$. Then, if $z^* = z^*(z)$ is a biholomorphic map from B to $B^* \subset \underset{\sim}{C}^n$,

$$K_B(z, \bar{t}) = K_{B^*}(z^*(z), \overline{t^*(t)})\, \frac{\partial(z^*)}{\partial(z)}\, \overline{\frac{\partial(t^*)}{\partial(t)}}\,, \tag{2}$$

and we say $K_B(z, \bar{t})$ is a relative (pseudoconformal) invariant. Furthermore

$$K_B(z, \bar{z}) = \sup\{|f(z)|^2 : f \in H(B), \|f\| \leqq 1\} \tag{3}$$

is a decreasing domain functional; i.e.,

$$B \subset A \text{ implies } K_A(z, \bar{z}) \leqq K_B(z, \bar{z})\,. \tag{4}$$

In [1] and [10] respectively Bergman and Hörmander have investigated the behavior of $K(B; z) = K_B(z, \bar{z})$ near a point $Q \in \partial B$, the boundary of B, where B is strictly pseudoconvex at Q. In each case sufficient conditions were attained for $K(B; z)$ to become infinite of order $n + 1$ at $Q \in \partial B$ (i.e., $|z - Q|^{n+1} K(B; z)$ remains between two

positive finite bounds as z nears Q) in a certain "nontangential" approach A. Both procedures use as "basic" domains the complex n-dimensional hypersphere $[|z| < 1]$ whose kernel function is known and becomes infinite of order $n + 1$ at the boundary.

We here introduce an extension of the notion of strictly pseudo-convex as follows:

DEFINITION. Let B be a domain in $\underset{\sim}{C}^n$. Suppose that there exists an analytic change of coordinates, one-to-one in a neighborhood η of $Q \in \partial B$, so that, with respect to the new (normal) coordinates, $Q = 0$ and, in a neighborhood of 0, $B = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + o(r^2_{1(q-1)} + r^{2/p}_{qn})]$, where $p > 0$, $2 \leqq q \leqq n$, and $r^2_{uv} = \sum_{k=u}^{v} |z_k|^2$. Then B is said to be strictly (p, q)-pseudo-convex at Q. If also $B \subset \eta$, then B is said to be globally representable (with respect to normal coordinates) at Q.

Note that if B is strictly pseudo-convex at Q then B is strictly $(1, n)$-pseudo-convex at Q.

As examples of domains in $\underset{\sim}{C}^2$ which are strictly $(p, 2)$-pseudo-convex at the points of the circle $[z_1 = 0, |z_2| = 1]$ consider the Reinhardt circular domains given by $[|z_2|^2 < R(|z_1|^{2/p})]$, where $|z_1| < R_0$, R is decreasing and $R'(0) \neq 0$.

In §2 and §3 we prove the following extensions of theorems by Hörmander and Bergman respectively:

THEOREM A. *In $\underset{\sim}{C}^n$ let B be a bounded strictly (p, q)-pseudo-convex domain at Q. Suppose further that B is a domain of holomorphy. Then $K(B; z)$ becomes infinite of order $p(n - q + 1) + q$ in a "nontangential" (see §1) approach to Q.*

THEOREM B. *In $\underset{\sim}{C}^n$ let B be a bounded strictly (p, q)-pseudo-convex globally representable domain at Q. Suppose further that passing through Q is a "normal" (see §3) analytic hypersurface lying entirely outside B. Then $K(B; z)$ becomes infinite of order $p(n - q + 1) + q$ in a "nontangential" approach to Q.*

Since p is an arbitrary positive number, these theorems provide sufficient conditions for the kernel function to become infinite of any order $r > 2$. Setting $p = 1$ and $q = n$ in Theorem A yields Hörmander's result, while taking $p = 1$ and $q = n = 2$ in Theorem B gives Bergman's result.

The proofs of Theorems A and B will use as "basic" domains the nonhomogeneous (if $p \neq 1$) domains $R_p(n, s) = [r^{2/p}_{1(s-1)} + r^2_{sn} < 1]$ which we introduce in §1. There we will determine the kernel function of these domains and show that it becomes infinite of order $p(n - q + 1) + q$, $q = n - s + 2$, at the points on the boundary given by $r_{1(s-1)} = 0$.

Under the hypotheses of Theorem B, we will determine in §4 the boundary behavior of other domain functionals related to the kernel function K, such as the Bergman metric given by $ds^2 = T^{\nu\bar{\mu}}dz_\nu d\bar{z}_\mu$, where $T^{\nu\bar{\mu}} = \partial^2 \log K/\partial z_\nu \partial \bar{z}_\mu$, and the scalar (pseudoconformal) invariant $I = K/\det(T^{\nu\bar{\mu}})$.

1. Determination of the Bergman kernel function and its boundary behavior for the domains $R_p(n, s)$.

Notation. If $f(x) \in C^k$, then denote the k-th partial derivative of $f(x)$ with respect to x by $[f(x)]_{(x,k)}$.

THEOREM 1.1. *For the kernel function of $R = R_p(n, s)$ we have*

$$K(R; z) = \frac{[[t^{n'}(1-t^p)^{-1}]_{(t,n-s')}]_{t^p,s')}}{\pi^n(1-r_{sn}^2)^{n'+1}}, \tag{1}$$

where $t = r_1^{2/p}(s-1)(1-r_{sn}^2)^{-1}$, $s' = s-1$, $p' = p-1$, and $n' = n + p's'$.

Proof. The monomials $z^m = {z_1}^{m_1}{z_2}^{m_2}\cdots{z_n}^{m_n}$, $m = (m_1, m_2, \cdots, m_n) \geqq 0$ form a complete set of orthogonal functions in $H(R)$. Thus by (0.1) $K(z) = K(R; z) = \sum_{m\geqq 0} c_m |z^m|^2$, where $c_m^{-1} = \int_R |z^m|^2 d\omega$. It is clear that R admits the group G_1 of unitary transformations of the first $s-1$ variables $z_1, \cdots, z_{s-1}$ and the group G_2 of unitary transformations of the last $n-s+1$ variables $z_s, \cdots, z_n$. The Jacobian of a unitary transformation has modulus one and so $K(z) = K(r_{1(s-1)}, 0, \cdots, 0, r_{sn}, 0, \cdots, 0) = \sum_{m_1=0}^{\infty}\sum_{m_s=0}^{\infty} c_{m_1\circ\cdots\circ m_s\circ\cdots\circ} r_{1(s-1)}^{2m_1} r_{sn}^{2m_s}$ and our problem has been reduced to only a double sum. After a straightforward calculation we find

$$c^{-1}_{m_1\circ\cdots\circ m_s\circ\cdots\circ} = \frac{\pi^n m_1!\, m_s!}{(s+m_1-1)![p(s+m_1-1)+1]\cdots[p(s+m_1-1)+m_s+n-s+1]}.$$

Now

$$\sum_{m_s=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+m_s-1)}{m_s!}(r_{sn}^2)^{m_s} = \frac{1}{(1-r_{sn}^2)^\alpha}, \quad \text{where}$$
$$\alpha = p(s+m_1-1)+n-s+2.$$

Thus,

$$K(z) = \frac{1}{\pi^n(1-r_{sn}^2)^{n'+1}} \sum_{m=0}^{\infty} (m+1)\cdots(m+s')[p(m+s')+1]\cdots \tag{2}$$
$$\cdots[p(m+s')+n-s'](t^p)^m.$$

Now if Q is a boundary point of $R_p(n, s)$ with $r_{1(s-1)} = 0$, then by a unitary transformation in G_2 followed by a translation, we can effect an analytic change of coordinates with respect to which $Q = 0$ and the domain can be written as

$$(3) \qquad R'_p(n, s) = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn}], \qquad q = n - s + 2,$$

with kernel function (obtained from formulas (2) and (0.2)):

$$(4) \quad K(R'_p(n, s); z) = \frac{1}{\pi^n (2x_1 - r^2_{1(q-1)})^{p(s-1)+q}} \sum_{m=0}^{\infty} a_m \left[\frac{r^2_{qn}}{(2x_1 - r^2_{1(q-1)})^p}\right]^m,$$

where the a_m are given in (2).

Notation. We shall write $\lim^A_{z\to 0}$, or sometimes simply $\lim^A$, to indicate a limit is being taken as $z \to 0$ in the set

$$(5) \qquad \left[0 < \beta < \frac{x_1}{|z|}\right] \text{ if } p < 2, \text{ and } \left[0 < \beta < \frac{x_1^p}{|z|}\right] \text{ if } p \geqq 2,$$

and say $z \to 0$ via an A-approach. (If for a strictly (p, q)-pseudoconvex domain the z-coordinates are normal, such an approach to 0 is nontangential in the usual sense.)

Thus from (4) we have

THEOREM 1.2. *$K(R'_p(n, s); z)$ becomes infinite of order $p(n-q+1)+q$ at 0 in a "nontangential" A-approach; in fact*

$$(6) \qquad \lim^A_{z\to 0} (2x_1)^{ps'+q} K(R'_p(n, s); z) = \pi^{-n} s'! [ps' + 1] \cdots [ps' + q'] = L.$$

2. **Poof of Theorem A.** The following lemma is due to Hörmander (see [10], p. 146):

LEMMA 2.1. *(Hörmander) Let B be a bounded domain of holomorphy. Let $Q \in \partial B$ and assume that for some neighborhood U of Q there is an analytic function u_0 in $B' = B \cap U$ such that $|u_0| \leqq 1$ in B', $|u_0(z)| \to 1$ when $z \to Q$, and $|u_0(z)|$ has an upper bound < 1 in $B' \cap {}^\sim U_0$ for some neighborhood U_0 of Q with compact closure contained in U. Then $K(B; z)/K(B'; z) \to 1$, $z \to Q$.*

LEMMA 2.2. *$R_p(n, s)$ is a domain of holomorphy.*

Proof. It follows easily from (1.1) that $K(R_p(n, s); z)$ becomes infinite at every boundary point of $R_p(n, s)$. But it is well known that a domain whose kernel function becomes infinite at every boundary point is a domain of holomorphy (see e.g., [7], p. 357).

LEMMA 2.3. *If B is strictly (p, q)-pseudo-convex at Q, then, with respect to local normal coordinates*

$$\overline{\lim}^{A} (2x_1)^{ps'+q} K(B; z) \leqq L . \tag{1}$$

Proof. B can be written locally in a neighborhood of $Q = 0$ as $B = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + o(r^2_{1(q-1)} + r^{2/p}_{qn})]$. For $\varepsilon > 0$ let

$$R_\varepsilon = [2x_1 > r^2_{1(q-1)} + r^{2/p}_{qn} + \varepsilon(r^2_{1(q-1)} + r^{2/p}_{qn})] .$$

Then $R^\delta_\varepsilon = R_\varepsilon \cap [|z| < \delta]$ is included in B if δ is sufficiently small. Since R_ε is obtained from $R'_p(n, s)$ by the change of coordinates $z_k \leftarrow (1 + \varepsilon)^{-1} z_k$, $k = 1, \cdots, q-1$, $z_l \leftarrow (1 + \varepsilon)^{-p} z_l$, $l = q, \cdots, n$, R_ε is also a domain of holomorphy. Thus

$$\begin{aligned} \overline{\lim}^{A} (2x_1)^{ps'+q} K(B; z) &\leqq \overline{\lim}^{A} (2x_1)^{ps'+q} K(R^\delta_\varepsilon; z) \\ &= \overline{\lim}^{A} (2x_1)^{ps'+q} K(R_\varepsilon; z) = (1+\varepsilon)^{ps'+q-2} L \end{aligned}$$

by (0.4), Lemma 2.1 (taking $u_0(z) = e^{-z_1}$ and $U = [|z| < \delta]$), and (0.2) together with Theorem 1.2, respectively.

Conclusion of proof of Theorem A. Let $0 < \varepsilon < 1$. Then, analogously as in the proof of Lemma 2.3., $B^\delta = B \cap [|z| < \delta] \subset R_{-\varepsilon}$ if δ is sufficiently small. Now since B is a domain of holomorphy we can apply Lemma 2.1 with $u_0(z) = e^{-z_1}$ and $U = [|z| < \delta]$ to obtain

$$\begin{aligned} \underline{\lim} (2x_1)^{ps'+q} K(B; z) &= \underline{\lim} (2x_1)^{ps'+q} K(B^\delta; z) \geqq \underline{\lim} (2x_1)^{ps'+q} K(R_{-\varepsilon}; z) \\ &= (1 - \varepsilon)^{ps'+q-2} L , \end{aligned}$$

using also (0.4) and (0.2) together with Theorem 1.2. Combining this result with that of Lemma 2.3, we conclude that under the hypotheses of Theorem A,

$$\lim^{A} (2x_1)^{p(n-q+1)+q} K(B; z) = L . \tag{2}$$

3. Proof of Theorem B.

DEFINITION 3.1. Suppose that B is a strictly (p, q)-pseudo-convex globally representable domain at Q. Suppose h is an analytic hypersurface through Q which can be represented as $h = [z_1 = 0]$ with respect to normal coordinates for B and $h \cap \bar{B} = Q$. Then we say that through Q passes the "normal" analytic hypersurface h lying entirely outside B.

LEMMA 3.2. *Let B be a bounded domain whose closure intersects the hyperplane $[z_1 = 0]$ only at the origin. Then an arbitrary neigh-*

borhood of the origin includes the section $B \cap [z_1 = \gamma]$ if γ is sufficiently small.

Proof. Let $P = \sup_{z \in B} r_{2n}$. Since the set of points $[z_1 = 0, b \leqq r_{2n} \leqq P]$ is closed, for each $b > 0$ there exists a number $\tau(b)$, $0 < \tau(b) \leqq b$, such that $[|z_1| \leqq \tau(b), b \leqq r_{2n} \leqq P]$ contains no point of B. Now if U is any neighborhood of the origin, there is a $\nu > 0$ such that $[|z_1| < \nu, r_{2n} < \nu] \subset U$. Then if $z \in B$ and $b \leqq \nu$, we have that $|z_1| < \tau(b)$ implies $r_{2n} < b$, and so the section $B \cap [z_1 = \gamma] \subset U$ for $|\gamma| < \tau(b)$.

LEMMA 3.3. *Let B be a domain satisfying the hypotheses of Theorem B. Then there exists a transformation*

$$W: z_1 \leftarrow \frac{z_1}{1 - \alpha z_1}, z_k \leftarrow z_k \frac{1 + (\beta - \alpha) z_1}{1 - \alpha z_1}, k = 2, \cdots, n$$

such that for α and β positive and sufficiently large, the domain $R_{-\varepsilon}^{\alpha\beta} = W(R_{-\varepsilon})$ includes B in its interior, where $0 < \varepsilon < 1$.

Proof.

$$B = [2x_1 > r_{1(q-1)}^2 + r_{qn}^{2/p} + o(r_{1(q-1)}^2 + r_{qn}^{2/p})] \text{ and}$$

$R_{-\varepsilon} = [2x_1 > (1-\varepsilon)(r_{1(q-1)}^2 + r_{qn}^{2/p})]$. Thus $\exists$ a neighborhood η of 0 such that $B \cap \eta \subset R_{-\varepsilon} \cap \eta$. Hence for $|\gamma| < \delta$, $B \cap [z_1 = \gamma] \subset R_{-\varepsilon} \cap [z_1 = \gamma]$ if δ is sufficiently small, by Lemma 3.2.

Now let $R_{-\varepsilon}^{\alpha} = W^{\alpha}(R_{-\varepsilon})$, where $W^{\alpha}: z_1 \leftarrow z_1/(1 - \alpha z_1)$, $z_k \leftarrow z_k$, $k = 2, \cdots, n$. Then consider $R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0] = \sim c_{\alpha}$, where c_{α} is the circle

$$\left[\left| z_1 + \frac{1}{2\alpha - (1 - \varepsilon)} \right| \leqq \frac{1}{2\alpha - (1 - \varepsilon)} \right].$$

We can assume α is so large that $c_{\alpha} \cap \bar{B} \cap [r_{2n} = 0] = 0$. Then the closed set $M = \bar{B} \cap [r_{2n} = 0] \cap [|z_1| \geqq \delta]$ lies entirely in the interior of the section $R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0]$. Now the sections $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$, $\gamma \in M$, include hyperspheres with center $(\gamma, 0, \cdots, 0)$ and the radii of these hyperspheres have a positive lower bound r. If one applies to $R_{-\varepsilon}^{\alpha}$ the transformation $W_{\beta}: z_1 \leftarrow z_1, z_k \leftarrow z_k(1 + \beta z_1)$, $k = 2, \cdots, n$, to obtain the domain $R_{-\varepsilon}^{\alpha\beta} = W_{\beta}(R_{-\varepsilon}^{\alpha})$, then the section $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$ goes into a domain including a hypersphere with center $(\gamma, 0, \cdots, 0)$ and radius $\geqq r|1 + \beta\gamma|$. β can be chosen so large that $r|1 + \beta\gamma| > P$ for $|\gamma| \geqq \delta$, where $P = \sup_{z \in B} r_{2n}$. Thus $B \cap [z_1 = \gamma] \subset R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma]$ if $|\gamma| \geqq \delta$.

On the other hand consider $\gamma \in R_{-\varepsilon}^{\alpha} \cap [r_{2n} = 0] = \sim c_{\alpha}$. Thus if $\beta > 2\alpha - (1 - \varepsilon)$, then $|\gamma + 1/\beta| > 1/\beta$ and so $|1 + \beta\gamma| > 1$. Hence $R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma] \supset R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma]$. But for $|\gamma| < \delta$ we know $R_{-\varepsilon}^{\alpha} \cap [z_1 = \gamma] \supset B \cap [z_1 = \gamma]$. Thus $R_{-\varepsilon}^{\alpha\beta} \cap [z_1 = \gamma] \supset B \cap [z_1 = \gamma]$ for all γ.

Conclusion of proof of Theorem B. Since $R_{-\varepsilon}^{\alpha\beta} \supset B$, we have $\underline{\lim}^A (2x_1)^{ps'+q} K(B; z) \geqq \underline{\lim}^A (2x_1)^{ps'+q} K(R_{-\varepsilon}^{\alpha\beta}; z) = (1-\varepsilon)^{ps'+q-2} L$. This result together with Lemma 2.3 shows that (2.2) also holds for B under the hypotheses of Theorem B.

4. Boundary behavior of other functionals on the domains of Theorem B.

THEOREM 4.1. *Under the hypotheses of Theorem B and using normal coordinates, we have further, denoting* $\lim^A (2x_1)^\lambda F(B; z)$ *by* (λ, F) *and letting* $m = p(q-1)$ *and* $\alpha = m + s$,

(1) $(\alpha + 1, K_{z_1}) = -L\alpha$; $(\alpha + 1/2, K_{z_k}) = 0, 2 \leqq k \leqq q-1$;
$(\alpha + p/2, K_{z_k}) = 0, q \leqq k \leqq n$,

(2) $(2, ds^2) = \alpha |dz_1|^2$ *if* $0 < p < 2$;

$(2, ds^2) = \alpha |dz_1|^2 + c \sum_{k=s}^{n} |dz_k|^2$ *if* $p = 2$,

$(p, ds^2) = c \sum_{k=s}^{n} |dz_k|^2$ *if* $p > 2$, *where*

$$c = q(p+m+1)\cdots(p+\alpha-1)/[(m+1)\cdots(\alpha-1)],$$

(3) (o, I)

$$= \pi^{-n}(q-1)!\,\alpha^{-n-1}[(m+1)\cdots\alpha]^q[q(p+m+1)\cdots(p+\alpha-1]^{1-q}.$$

Proof (Sketch). One can show analogously as in Lemma 3.3 that B includes in its interior the domain $R_\varepsilon^{\alpha'\beta'} = W(R_\varepsilon)$ where

$$W: z_1 \leftarrow \frac{z_1}{1+\alpha' z_1}, \quad z_k \leftarrow \frac{z_k(1+\alpha' z_1)}{1+(\alpha'+\beta')z_1}, \quad k = 2, \cdots, n,$$

for sufficiently large α' and β'. Then if $F(B; z)$ is any decreasing functional of the domain B, *e.g.*, $K(B; z)$, we have

$$F(R_\varepsilon^{\alpha'\beta'}; z) \geqq F(B; z) \geqq F(R_{-\varepsilon}^{\alpha\beta}; z).$$

It is well known that the quantities ds^2 and I are rational functions of decreasing domain functionals F_i which can be determined from the kernel function and its derivatives. But from our results in §1, and using (0.2), we can calculate the kernel function and its derivatives for the domains $R_\varepsilon^{\alpha'\beta'}$ and $R_{-\varepsilon}^{\alpha\beta}$. Thus we can calculate the decreasing domain functionals F_i and show that there exist λ_i such that $\lim_{\varepsilon\to 0} (\lambda_i, F_i(R_\varepsilon^{\alpha'\beta'}; z)) = \lim_{\varepsilon\to 0} (\lambda_i, F_i(R_{-\varepsilon}^{\alpha\beta}; z)) = l_i$, whence $(\lambda_i, F_i(B; z)) = l_i$. From these results (2) and (3) follow immediately. Although the derivatives K_{z_k} are not themselves rational functions of monotone domain

functionals, they can be approximated above and below by such functions and the same method yields (1).

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UNIVERSITY OF CALIFORNIA, RIVERSIDE

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PEIRCE DECOMPOSITION IN SIMPLE LIE-ADMISSIBLE POWER-ASSOCIATIVE RINGS

WILLIAM E. COPPAGE

The main result is

THEOREM. **If A is a simple Lie-admissible power-associative ring with characteristic prime to six, and if A has an idempotent e relative to which A has a Peirce decomposition such that $A_{00}=0$, then either e is a unity element of A or $A \cong B$, where B is a three-dimensional algebra having a basis $\{e, x, y\}$ such that $e^2=e$, $ex=x$, $ye=y$, $xy=-yx=e$ and $xe=ey=x^2=y^2=0$.**

If A is a simple Lie-admissible power-associative ring then A belongs to a class of rings which includes associative rings, Lie rings, commutative power-associative rings, Jordan rings, anti-flexible rings, rings of type (γ, δ) and others. Lie rings do not have idempotent elements, and simple (γ, δ) rings with an idempotent $e \neq 1$ have been shown [2, 3, 4, 5, 6, 8] to be associative. Thus if A has an idempotent element $e \neq 1$ then A belongs to a class which includes rings of the associative, commutative power-associative, and antiflexible types. Assuming that A has an idempotent e satisfying,

$$(1) \qquad (e, e, x) = (e, x, e) = (x, e, e) = 0 ,$$

suffices to establish a Peirce decomposition,

$$A = A_{11} + A_{10} + A_{01} + A_{00} ,$$

where $A_{ij} = \{x \in A \mid ex = ix, xe = jx\}$ for $i, j = 0, 1$. This assumption eliminates the possibility that A is commutative, for then $A_{10} = A_{01} = 0$, so [2] $A = A_{11} \oplus A_{00}$ and simplicity implies that $A = A_{11}$, hence e is a unity element for A.

The class of rings under consideration does contain members which are not associative. Kosier [7] has given examples of simple Lie-admissible power-associative finite-dimensional algebras, the so-called anti-flexible algebras. These have the property that $A = A_{11} + A_{00}$ in every Peirce decomposition.

There are no rings with unity element, 1, which possess a Peirce decomposition with respect to an idempotent $e \neq 1$ in which $A_{00} = 0$. This is because $1 - e \in A_{00}$.

The algebra B of our theorem was introduced in [9]. It has the property that $B^{(-)}$ is a simple Lie algebra.

The associator, $(x, y, z) = (xy)z - x(yz)$, and the commutator, $[x, y] = xy - yx$, are functions which, defined on any ring, are linear

in each variable and related by the identity,

$$(2) \qquad [xy, z] + [yz, x] + [zx, y] = (x, y, z) + (y, z, x) + (z, x, y) .$$

A Lie-admissible ring satisfies,

$$(3) \qquad [xy - yx, z] + [yz - zy, x] + [zx - xz, y] = 0 ,$$

and a power-associative ring whose characteristic is prime to two satisfies

$$(4) \qquad [xy + yx, z] + [yz + zy, x] + [zx + xz, y] = 0 ,$$

hence in the ring A the function

$$\begin{aligned} H(x, y, z) &= (x, y, z) + (y, z, x) + (z, x, y) \\ &= [xy, z] + [yz, x] + [zx, y] \end{aligned}$$

is identically zero. Also, the fourth-power-associativity identities $(x^2, x, x) = 0$ and $(x, x, x^2) = 0$ may be linearized to yield functions $P(a, b, x, y) = \sum (ab, x, y)$ and $Q(a, b, x, y) = \sum (a, b, xy)$ which are identically zero. The $\sum$ here in both cases indicates a sum to be taken over the twenty-four permutations of a, b, x and y.

We will use $\cdot$ as well as juxtaposition in denoting products, with juxtaposition taking precedence. Thus $a \cdot bc = a(bc)$.

LEMMA. *Let A be a ring whose characteristic is prime to six and in which the functions H, P and Q vanish identically. Suppose A contains an idempotent e relative to which A has a Peirce decomposition. If a_{mn} denotes the component of an element a in the module A_{mn} then*

$$(5) \qquad x_{ii}y_{jj} = 0 ,$$

$$(6) \qquad x_{ii}y_{ii} + y_{ii}x_{ii} \in A_{ii} ,$$

$$(7) \qquad x_{ii}^2 \in A_{ii} ,$$

$$(8) \qquad x_{ii}y_{ij} = (x_{ii}y_{ij})_{ij} + (x_{ii}y_{ij})_{jj} \in A_{ij} + A_{jj} ,$$

$$(9) \qquad y_{ij}x_{ii} = (x_{ii}y_{ij})_{jj} \in A_{jj} ,$$

$$(10) \qquad y_{ji}x_{ii} = (y_{ji}x_{ii})_{ji} + (y_{ji}x_{ii})_{jj} \in A_{ji} + A_{jj} ,$$

$$(11) \qquad x_{ii}y_{ji} = (y_{ji}x_{ii})_{jj} \in A_{jj} ,$$

$$(12) \qquad x_{ij}y_{ij} = y_{ij}x_{ij} \in A_{ii} + A_{jj} ,$$

$$(13) \qquad x_{ij}y_{ji} \in A_{ii} + A_{jj} ,$$

$$(14) \qquad x_{ii}y_{ii} \in A_{ii} + A_{jj} ,$$

(15) $$[A_{ij}^2, A_{ii} + A_{ji} + A_{jj}] = 0 ,$$

(16) $$[A_{ij}A_{ji}, A_{jj}] = 0 ,$$

(17) $$[A_{ji}A_{jj}, A_{ij}] = 0 ,$$

(18) $$[A_{jj}A_{ij}, A_{ji}] = 0 ,$$

(19) $$a_{ii}\cdot x_{ij}y_{ij} = x_{ij}y_{ij}\cdot a_{ii} = (1/2)((a_{ii}x_{ij}\cdot y_{ij})_{ii} + (a_{ii}y_{ij}\cdot x_{ij})_{ii}) ,$$

(20) $$a_{ii}\cdot x_{ji}y_{ji} = x_{ij}y_{ij}\cdot a_{ii} = (1/2)((x_{ji}a_{ii}\cdot y_{ji})_{ii} + (y_{ji}a_{ii}\cdot x_{ji})_{ii}) ,$$

(21) $$(a_{ii}(x_{ij}y_{ji} + y_{ji}x_{ij}))_{ii} = (a_{ii}x_{ij}\cdot y_{ji})_{ii} + (y_{ji}a_{ii}\cdot x_{ij})_{ii} ,$$

and

(22) $$((x_{ij}y_{ji} + y_{ji}x_{ij})a_{ii})_{ii} = (y_{ji}\cdot a_{ii}x_{ij})_{ii} + (x_{ij}\cdot y_{ji}a_{ii})_{ii} .$$

Proof of the lemma. Identities (5), (6) and (7) are derived in [1] using only the fact that the functions H, P and Q vanish identically. All of the identities are obtained by relatively straightforward substitution of elements into H, P and Q. Due to the excessive length of many of the computations involved we leave the proofs to the reader.

Since our theorem hypothesizes that $A_{00} = 0$ the multiplicative properties stated in (8) through (14) of the preceding lemma can be more compactly exhibited in our case by the module multiplication table:

(23)

	A_{11}	A_{10}	A_{01}
A_{11}	A_{11}	A_{10}	0
A_{10}	0	A_{11}	A_{11}
A_{01}	A_{01}	A_{11}	A_{11}

We will henceforth make free use of the properties shown in this table. Note also that (12) can be written

(24) $$[A_{10}, A_{10}] = [A_{01}, A_{01}] = 0 ,$$

and (15) can be written

(25) $$[A_{10}^2, A_{11} + A_{01}] = [A_{01}^2, A_{11} + A_{10}] = 0 .$$

From (16) we have

(26) $$[A_{01}A_{10}, A_{11}] = 0 ,$$

and (19) through (22) specialize to

(27) $$a_{11}\cdot x_{10}y_{10} = x_{10}y_{10}\cdot a_{11} = (1/2)(a_{11}x_{10}\cdot y_{10} + a_{11}y_{10}\cdot x_{10}) ,$$

(28) $$a_{11}\cdot x_{01}y_{01} = x_{01}y_{01}\cdot a_{11} = (1/2)(x_{01}a_{11}\cdot y_{01} + y_{01}a_{11}\cdot x_{01}) ,$$

$$a_{11}(x_{10}y_{01} + y_{01}x_{10}) = a_{11}x_{10}\cdot y_{01} + y_{01}a_{11}\cdot x_{10} \tag{29}$$

and

$$(x_{10}y_{01} + y_{01}x_{10})a_{11} = y_{01}\cdot a_{11}x_{10} + x_{10}\cdot y_{01}a_{11}\ , \tag{30}$$

respectively.

We assume throughout that e is not a unity element for A. We will show

$$(A, A_{11}, A_{11}) = (A_{11}, A, A_{11}) = (A_{11}, A_{11}, A) = 0\ . \tag{31}$$

The submodule A_{11} is a subring of A, and for $i \neq j$, two of the associators in $H(x_{11}, y_{11}, a_{ij}) = (x_{11}, y_{11}, a_{ij}) + (y_{11}, a_{ij}, x_{11}) + (a_{ij}, x_{11}, y_{11}) = 0$, are equal to zero, hence all three are equal to zero. Thus it suffices to show that A_{11} is associative.

We assert that the submodule $I = (A_{11}, A_{11}, A_{11}) + (A_{11}, A_{11}, A_{11})A_{11}$ is an ideal of A. We will use the fact that the function $T(a, x, y, b) = (ax, y, b) - (a, xy, b) + (a, x, yb) - a(x, y, b) - (a, x, y)b$ is identically zero in any nonassociative ring. Thus $0 = T(a_{mn}, x_{11}, y_{11}, b_{ij})$, with $m + n = i + j = 2$ implies that $A_{11}(A_{11}, A_{11}, A_{11}) \subseteqq I$, and with $m + n = 2$, $i + j = 1$, implies that $(A_{11}, A_{11}, A_{11})A_{ij} = 0$, using the fact that $(A_{11}, A_{11}, A_{ij}) = 0$. If $m + n = 1$ and $i + j = 2$ then we get $A_{mn}(A_{11}, A_{11}, A_{11}) = 0$. Thus $(A_{11}, A_{11}, A_{11})A + A(A_{11}, A_{11}, A_{11})$ is in I. Furthermore,

$$(A_{11}, A_{11}, A_{11})A_{11}\cdot A \subseteqq ((A_{11}, A_{11}, A_{11}), A_{11}, A) + (A_{11}, A_{11}, A_{11})A \subseteqq I\ ,$$

so $IA \subseteqq I$. Finally,

$$\begin{aligned} A\cdot(A_{11}, A_{11}, A_{11})A_{11} &\subseteqq (A, (A_{11}, A_{11}, A_{11}), A_{11}) \\ &\quad + A(A_{11}, A_{11}, A_{11})\cdot A_{11} \subseteqq I + IA_{11} \subseteqq I\ , \end{aligned}$$

and it follows that $AI \subseteqq I$. Hence I is an ideal of A. If $A = I$ then e is a unity element for A, which contradicts our assumption. Therefore $I = 0$ and in particular $(A_{11}, A_{11}, A_{11}) = 0$, which proves (31).

We assert next that $A_{10}^2 = A_{01}^2 = 0$. First we prove that $J = A_{10}^2 + A_{10}^2A_{10}$ is an ideal of A. We have

$$A_{10}A_{10}^2 \subseteqq A_{10}A_{11} = 0\ , \qquad A_{01}A_{10}^2 = A_{10}^2A_{01} \subseteqq A_{11}A_{01} = 0$$

by using (25), and $A_{11}A_{10}^2 = A_{10}^2A_{11} \subseteqq A_{10}^2$ by (27). Thus $A_{10}^2A + AA_{10}^2 \subseteqq J$. Moreover, $(A_{10}^2A_{10})A_{11} \subseteqq A_{10}A_{11} = 0$, and, by using (31), $A_{11}(A_{10}^2A_{10}) = (A_{11}A_{10}^2)A_{10} \subseteqq A_{10}^2A_{10} \subseteqq J$. Letting $a_{11} = u_{10}v_{10}$ in (29) we have

$$u_{10}v_{10}(x_{10}y_{01} + y_{01}x_{10}) = (u_{10}v_{10}\cdot x_{10})y_{01} + (y_{01}\cdot u_{10}v_{10})x_{10}\ .$$

But $y_{01}\cdot u_{10}v_{10} = u_{10}v_{10}\cdot y_{01} = 0$ by using (25), so

$$(u_{10}v_{10}\cdot x_{10})y_{01} = u_{10}v_{10}(x_{10}y_{01} + y_{01}x_{10}) \in A_{10}^2A_{11} \subseteqq J$$

and therefore $A_{10}^2A_{10}\cdot A_{01} \subseteqq J$. Finally, $H(u_{10}v_{10}, x_{10}, y_{01}) = 0$ implies

$$y_{01}(u_{10}v_{10}\cdot x_{10}) = (u_{10}v_{10}\cdot x_{10})y_{01} \in A_{10}^2A_{10}\cdot A_{01} \subseteqq J\ .$$

Thus J is an ideal of A.

Since A is simple either $J = 0$ or $J = A$. If $J = A$ then $A_{10}^2 = A_{11}$, $A_{11}A_{10} = A_{10}$ and $A_{01} = 0$. Thus we may write

$$e = \sum_{i=1}^{t} x_{10}^{(i)}y_{10}^{(i)}\ .$$

But

$$\begin{aligned} 0 = H(x_{10}, x_{10}, y_{10}) &= [x_{10}^2, y_{10}] + [x_{10}y_{10}, x_{10}] + [y_{10}x_{10}, x_{10}] \\ &= x_{10}^2y_{10} + 2x_{10}y_{10}\cdot x_{10} \end{aligned}$$

and

$$\begin{aligned} 0 &= (1/4)P(x_{10}, x_{10}, y_{10}, y_{10}) \\ &= (x_{10}^2, y_{10}, y_{10}) + 2(x_{10}y_{10}, x_{10}, y_{10}) + 2(x_{10}y_{10}, y_{10}, x_{10}) + (y_{10}^2, x_{10}, x_{10}) \\ &= x_{10}^2y_{10}\cdot y_{10} - x_{10}^2y_{10}^2 + 2(x_{10}y_{10}\cdot x_{10})y_{10} - 2x_{10}y_{10}\cdot x_{10}y_{10} + 2(x_{10}y_{10}\cdot y_{10})x_{10} \\ &\quad - 2x_{10}y_{10}\cdot y_{10}x_{10} + y_{10}^2x_{10}\cdot x_{10} - y_{10}^2x_{10}^2\ , \end{aligned}$$

so using the fact that A_{10}^2 is in the center of the associative subring A_{11}, we obtain $x_{10}^2y_{10}^2 = -2(x_{10}y_{10})^2$. It follows that

$$(x_{10}y_{10})^4 = (1/4)(x_{10}^2y_{10}^2)^2 = (1/4)x_{10}^4y_{10}^4 = 0$$

since $x_{10}^3 = x_{10}\cdot x_{10}^2 = 0$. But then we have

$$e = e^{3t+1} = \left(\sum_{i=1}^{t} x_{10}^{(i)}y_{10}^{(i)}\right)^{3t+1} = 0\ ,$$

since every term in the multinomial expansion must contain, for some j, a factor $(x_{10}^{(i)}y_{10}^{(i)})^4 = 0$. From this contradiction we conclude that $J = 0$, hence $A_{10}^2 = 0$. Then also $A_{01}^2 = (A_{01}^{\#})^2 = 0$, where A is a ring which is anti-isomorphic to A.

We may now replace (23) with the table,

(32)

	A_{11}	A_{10}	A_{01}
A_{11}	A_{11}	A_{10}	0
A_{10}	0	0	A_{11}
A_{01}	A_{01}	A_{11}	0

.

We will continue to make free use of these multiplicative properties in the sequel. Of special interest are the identities,

(33) $$y_{01}\cdot z_{01}x_{10} = -z_{01}\cdot x_{10}y_{01}$$

and

$$y_{10}z_{01}\cdot x_{10} = -z_{01}x_{10}\cdot y_{10}\ , \tag{34}$$

obtained by using the function H and (32).

We show next that the subring A_{11} is itself a simple ring.

Let B_{11} be any nonzero ideal of A_{11} and consider the submodule,

$$L = B_{11} + B_{11}A_{10} + A_{01}B_{11} + A_{01}\cdot B_{11}A_{10} + A_{01}B_{11}\cdot A_{10} + (A_{01}\cdot B_{11}A_{10})A_{11} + A_{11}(A_{01}B_{11}\cdot A_{10})\ .$$

We will show that L is an ideal of A.

Evidently, $AB_{11} + B_{11}A \subseteqq L$. Also $B_{11}A_{10}\cdot A_{11} \subseteqq A_{10}A_{11} = 0$; and by (31), $A_{11}\cdot B_{11}A_{10} = A_{11}B_{11}\cdot A_{10} \subseteqq B_{11}A_{10} \subseteqq L$. By (24),

$$A_{10}\cdot B_{11}A_{10} = B_{11}A_{10}\cdot A_{10} \subseteqq A_{10}^2 = 0\ .$$

Noting that $B_{11}A_{10}\cdot A_{01} \subseteqq A_{01}B_{11}\cdot A_{10} + B_{11} \subseteqq L$ by (29), and $A_{01}\cdot B_{11}A_{10} \subseteqq L$ by the definition of L, we see that $A\cdot B_{11}A_{10} + B_{11}A_{10}\cdot A \subseteqq L$. Moreover, $A\cdot A_{01}B_{11} + A_{01}B_{11}\cdot A \subseteqq L$ from the left-right symmetry of our identities. Similarly, verification that the fourth and sixth terms in in the definition of L yield elements of L when multiplied on the left or right by an element of A implies the same result for the fifth and seventh terms.

By (26), $[A_{01}\cdot B_{11}A_{10}, A_{11}] \subseteqq [A_{01}A_{10}, A_{11}] = 0$. Since $(A_{01}\cdot B_{11}A_{10})A_{11} \subseteqq L$ by definition of L, it follows that $A_{11}(A_{01}\cdot B_{11}A_{10}) \subseteqq L$ also. Clearly, $A_{10}(A_{01}\cdot B_{11}A_{10}) \subseteqq A_{10}A_{11} = 0$, and by (34), $(A_{01}\cdot B_{11}A_{10})A_{10} \subseteqq A_{10}A_{01}\cdot B_{11}A_{10} \subseteqq L$. Also $(A_{01}\cdot B_{11}A_{10})A_{01} \subseteqq A_{11}A_{01} = 0$, and by (30) and (33),

$$A_{01}(A_{01}\cdot B_{11}A_{10}) \subseteqq A_{01}(B_{11} + A_{10}\cdot A_{01}B_{11}) \subseteqq L + A_{01}(A_{10}\cdot A_{01}B_{11}) \subseteqq L + A_{01}B_{11}\cdot A_{01}A_{10} \subseteqq L\ .$$

Thus $A(A_{01}\cdot B_{11}A_{10}) + (A_{01}\cdot B_{11}A_{10})A \subseteqq L$.

Since $[A_{01}\cdot B_{11}A_{10}, A_{11}] \subseteqq [A_{01}A_{10}, A_{11}] = 0$ it suffices to show that $(A_{01}\cdot B_{11}A_{10})A_{11}\cdot A$ and $A\cdot A_{11}(A_{01}\cdot B_{11}A_{10})$ are in L. By (31),

$$(A_{01}\cdot B_{11}A_{10})A_{11}\cdot A = (A_{01}\cdot B_{11}A_{10})\cdot A_{11}A \subseteqq (A_{01}\cdot B_{11}A_{10})A \subseteqq L$$

and $A\cdot A_{11}(A_{01}\cdot B_{11}A_{10}) = AA_{11}\cdot(A_{01}\cdot B_{11}A_{10}) \subseteqq A(A_{01}\cdot B_{11}A_{10}) \subseteqq L$. This completes the verification that L is an ideal of A.

Since A is simple and $0 \neq B_{11} \subseteqq L$ we must have $L = A$, hence $B_{11}A_{10} = A_{10}$ and $A_{01}B_{11} = A_{01}$.

If $b_{11} \in B_{11}$ then $b_{11}(a_{11}x_{10})\cdot y_{01} + y_{01}b_{11}\cdot a_{11}x_{10} \in B_{11}$ and

$$(b_{11}a_{11})x_{10}\cdot y_{01} + y_{01}(b_{11}a_{11})\cdot x_{10} \in B_{11}$$

by (29). Taking the difference of these two elements and using (31) gives $(y_{01}b_{11})a_{11}\cdot x_{10} - y_{01}b_{11}\cdot a_{11}x_{10} \in B_{11}$. Since $A_{01}B_{11} = A_{01}$ it follows that $(A_{01}, A_{11}, A_{10}) \subseteqq B_{11}$.

If the intersection of all proper ideals of A_{11} is the zero ideal,

then $(A_{01}, A_{11}, A_{10}) = 0$. Hence, by (31) and (33),

$$z_{01}(a_{11}\cdot x_{10}y_{01}) = z_{01}a_{11}\cdot x_{10}y_{01} = -y_{01}(z_{01}a_{11}\cdot x_{10})$$
$$= -y_{01}(z_{01}\cdot a_{11}x_{10}) = z_{01}(a_{11}x_{10}\cdot y_{01})\ ;$$

i.e., $z_{01}(a_{11}, x_{10}, y_{01}) = 0$.

Since the set N_{11} of elements of A_{11} which annihilate A_{01} is an ideal of A_{11}, and since $0 = A_{01}N_{11} \neq A_{01}$, it follows that $N_{11} = 0$, hence $(A_{11}, A_{10}, A_{01}) = 0$. Thus $A_{10}A_{01} = (B_{11}A_{10})A_{01} = B_{11}(A_{10}A_{01}) \subseteqq B_{11}$ and, by using (29), $A_{01}A_{10} = (A_{01}B_{11})A_{10} \subseteqq B_{11}A_{10}\cdot A_{01} + B_{11} \subseteqq B_{11}$. This implies that the ideal L is given by $L = B_{11} + B_{11}A_{10} + A_{01}B_{11}$. Since A is simple, $B_{11} = A_{11}$ and A_{11} is simple.

The other possibility is that A_{11} contains a unique minimal ideal, M_{11}. If $(A_{01}, A_{11}, A_{10}) = 0$ we may proceed as above. Thus assume that there exists a nonzero element b_{11} of the form (y_{01}, a_{11}, x_{10}). Since $(A_{01}, A_{11}, A_{10}) \subseteqq B_{11}$ for every nonzero ideal B_{11} of A_{11}, we see that $b_{11} \in M_{11}$. Moreover b_{11} is in the center of A_{11} by (26). Since M_{11} is minimal, $M_{11} = b_{11}A_{11}$. If $b_{11}c_{11} = 0$ then, since $A_{01}M_{11} = A_{01}$, $A_{01}c_{11} = A_{01}A_{11}b_{11}c_{11} = 0$. Thus $c_{11} \in N_{11} = 0$; i.e., no nonzero element of A_{11} annihilates b_{11}. Hence $b_{11}^2 \neq 0$ and $M_{11} = b_{11}^2A_{11}$. Then there exists $b_{11} \in A_{11}$ such that $b_{11} = b_{11}^2d_{11}$, or $b_{11}(e - b_{11}d_{11}) = 0$. It follows that $e = b_{11}d_{11} \in M_{11}$ hence $M_{11} = A_{11}$ is simple in this case also.

By (31), (33), and (26), $z_{01}(x_{10}y_{01}\cdot a_{11}) = (z_{01}\cdot x_{10}y_{01})a_{11} = -(y_{01}\cdot z_{01}x_{10})a_{11} = -y_{01}(z_{01}x_{10}\cdot a_{11}) = -y_{01}(a_{11}\cdot z_{01}x_{10}) = -y_{01}a_{11}\cdot z_{01}x_{10} = z_{01}(x_{10}\cdot y_{01}a_{11})$; i.e., $z_{01}(x_{10}, y_{01}, a_{11}) = 0$, or $(A_{10}, A_{01}, A_{11}) \subseteqq N_{11} = 0$. Then (30) reduces to $y_{01}x_{10}\cdot a_{11} = y_{01}\cdot a_{11}x_{10}$, which, in view of (26), implies that $A_{01}A_{10}$ is an ideal of A_{11}. If $A_{01}A_{10} = 0$ then (34) implies that $A_{10}A_{01}$ annihilates A_{10}, hence $A_{10}A_{01} = 0$. But then we easily see from (32) that both A_{10} and A_{01} are ideals of A, hence $A_{10} = A_{01} = 0$, which implies that e is an unity element for $A = A_{11}$. From this contradiction we conclude that $A_{01}A_{10} = A_{11}$, hence by (26), A_{11} is commutative and therefore a field.

Let $A_{11} = \Phi e$. To prove that A_{01} is one-dimensional over Φ, choose $0 \neq z_{01} \in A_{01}$ such that $z_{01}A_{10} = A_{11} = \Phi e$. Suppose $z_{01}x_{10} = e$. Then for every $y_{01} \in A_{01}$ we have, by (33), $y_{01} = -z_{01}\cdot x_{10}y_{01} = \alpha z_{01}$ for $\alpha \in \Phi$. Also $A_{10} = A_{01}^{\sharp}$ is one-dimensional over Φ.

We now have $A_{11} = \Phi e$, $A_{10} = \Phi x$ and $A_{01} = \Phi y$. Since (34) gives $(xy + yx)x = 0$ and $xy + yx \in \Phi e$, we must have $xy + yx = 0$. Without loss of generality we may take $xy = -yx = e$, which completes the proof of the theorem.

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an extension to rings of the theorem originally proved only for finite-dimensional algebras.

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WRIGHT STATE UNIVERSITY
DAYTON, OHIO

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COMPACTNESS OF MAPPINGS

EDWIN DUDA

A class of spaces that admit to a special kind of mapping onto the nonnegative real numbers is considered and it is shown that this particular type of space is invariant under compact monotone mappings. It is also shown that if such a space admits to a one to one (monotone) mapping onto a "nice" subset of the plane then the mapping is a homeomorphism (compact monotone mapping). If the one to one mapping (monotone) is not a homeomorphism (compact monotone mapping) then its range necessarily separates E^2.

In [3] the author showed that if a locally connected generalized continuum X with the property that the complement of each compact set has only one nonconditionally compact component admits a one-to-one mapping f onto E^2, then f is a homeomorphism. This result was then used in [4] to show that reflexive compact mappings of E^2 onto E^2 are compact mappings. This condition that the domain be all of E^2 and the range be all of E^2 are naturally unsatisfactory conditions. In this paper we undertake to reduce these conditions by considering spaces which admit to certain special kinds of mappings onto the nonnegative real numbers. It is shown that these spaces include numerous subsets of E^2 and also include spaces which are not subsets of E^2. In general a completely satisfactory reduction on the hypotheses used for the main result in [4] is obtained.

2. Notation and definition. Throughout X and Y will be *metric spaces and mapping will mean continuous function.* A locally connected generalized continuum is a connected separable metric space which is locally compact and locally connected. A mapping is monotone f each point inverse is a continuum and a mapping is compact if the inverse image of each compact set is also compact. A continuum X is unicoherent if whenever $X = A \cup B$, where A and B are continua, then so is $A \cap B$. The nonnegative real numbers will be represented by $[0, \infty)$ and $[a, b]$ will be the real numbers interval from a to b or a simple arc joining a to b depending on the context. The cyclic element theory used is that of reference [10]. A topological ray is a homeomorphic image of a ray in the real line. A set B in a metric space is conditionally compact if its closure is a compact set. The Euclidean plane will be represented by E^2.

3. Preliminary results.

THEOREM 1. *Let X be a noncompact metric space which has the representation $X = \bigcup_{i=1}^{\infty} A_i$, where $A_i \subset \operatorname{int} A_{i+1}$, $i = 1, 2, 3, \cdots$ and each A_i is a locally connected unicoherent continuum which does not separate X. Then there is a mapping g of X onto $[0, \infty)$ and a monotone increasing sequence of positive real numbers a_i, $(a_i) \to \infty$, such that $g^{-1}[0, a_i]$ is a locally connected unicoherent continuum for $i = 1, 2, 3, \cdots$.*

Proof. For convenience of notation assume $\operatorname{int} A_1 \neq \phi$. Let $B_i = A_i \cap Cl(X - A_i)$ for each positive integer i. Each B_i is a continuum, [5], the B_i are pairwise disjoint and the union of any subcollection of B_i is a closed subset. By a modified version of Urysohn's Lemma there is a mapping f_1 of A_1 onto $[0, 1]$ with $f_1(B_1) = 1$, $f_1^{-1}f_1(B_1) = B_1$. Similarly there is a mapping f_2 of $B_1 \cup A_2 - A_1$ onto $[1, 2]$ with

$$f_2(B_1) = 1, f_2^{-1}f_2(B_1) = B_1, f_2(B_2) = 2, f_2^{-1}f_2(B_2) = B_2 .$$

In general there exists a mapping f_n of $B_{n-1} \cup A_n - A_{n-1}$ onto $[n-1, n]$ with $f_n(B_{n-1}) = n - 1$, $f_n^{-1}f_n(B_{n-1}) = B_{n-1}$, $f_n(B_n) = n$, $f_n^{-1}f_n(B_n) = B_n$. Let g be the mapping of X onto $[0, \infty)$ which is defined in the following manner. For $x \in A_j$, set $g(x) = f_j(x)$ and note that g is a mapping with the required properties.

There is a partial converse to theorem one which is useful in identifying some of the properties of spaces which admit to a special kind of mapping onto $[0, \infty)$.

THEOREM 2. *Let X be a metric space which admits a mapping g onto $[0, \infty)$ for which there exists a strictly monotone increasing sequence of positive real numbers a_i, $(a_i) \to \infty$, such that $g^{-1}[0, a_i]$ is a locally connected unicoherent continuum and $g^{-1}[a_i]$ is a continuum for each positive integer i. Then X has the representation $X = \bigcup_{i=1}^{\infty} A_i$, where $A_i \subset \operatorname{int} A_{i+1}$, each A_i in a locally connected unicoherent continuum, and $g^{-1}[a_i, \infty)$ is connected for $i = 1, 2, 3, \cdots$.*

Proof. Let $A_i = g^{-1}[0, a_i]$ for each positive integer i. Clearly $A_i \subset \operatorname{int} A_{i+1}$ for each positive integer i and since $g^{-1}[a_i]$ is a continuum it follows that $g^{-1}[a_i, \infty)$ is connected.

Another usuful version of theorems one and two is possible for monotone mappings. First we establish a useful lemma of which there are more general versions, i.e., Theorem 2 of [9].

LEMMA. *Let X be a locally connected generalized continuum. If g is a monotone mapping of X onto $[0, \infty)$ then f is a compact*

mapping.

Proof. Let K be a compact subset of $[0, \infty)$ with least upper bound u and greatest lower bound l. There is a continuum K_0 containing $g^{-1}(u) \cup g^{-1}(l)$. The closed interval $g(K_0)$ contains K so that $g^{-1}g(K_0)$ contains $g^{-1}(K)$. By monotoneness of g, $g^{-1}g(K_0)$ is a continuum so that $g^{-1}(K)$ is compact.

THEOREM 3. *Let X be a locally connected generalized continuum and g a monotone mapping of X onto $[0, \infty)$. If there exists a sequence of positive real numbers a_i, $(a_i) \to \infty$, such that each $g^{-1}[0, a_i]$ is embeddable in E^2 and no point inverse of g separates E^2 in some embedding, then $g^{-1}[0, a_i]$ fails to separate E^2 for each positive integer i.*

Proof. It is well known that if one embedding of a continuum in E^2 fails to separate E^2 then the same is true for all other embeddings. Suppose then that $g^{-1}[0, a_i]$ is a subset of E^2 under some embedding. Let G be the decomposition of E^2 into the points not in $g^{-1}[0, a_i]$ and the continua $g^{-1}[c]$, $0 \leq c \leq a_i$. Since this decomposition is upper semi-continuous it follows by a well-known theorem of R. L. Moore, [6], that the decomposition space Y is a topological plane. Furthermore the image of $g^{-1}[0, a_i]$ under the natural mapping ψ of the decomposition G is a simple arc. Since the arc $\psi(g^{-1}[0, a_i])$ does not separate Y and ψ is a monotone compact mapping, $g^{-1}[0, a_i]$ does not separate E^2.

It will be useful for further discussion and statements of results to refer to a locally connected generalized continuum which admits to a mapping g of the type in Theorems 1, 2, or 3 as a g-unicoherent space. To give some indication of a class of spaces which satisfy the conditions of Theorem 1 recall that a locally connected continuum each true cyclic element of which is homeomorphic to a 2-sphere is called a cactoid. It is well known that a monotone image of a 2-sphere is a cactoid and that any cactoid can be realized by a monotone mapping defined on a 2-sphere. Moreover, any monotone image of a cactoid is also a cactoid. The space obtained by removing a noncut point from a cactoid satisfies the conditions of Theorem 1. (See Theorem 4.)

A base set is a locally connected continuum which is homeomorphic to a continuum in E^2 every true cyclic element of which is a 2-cell. A hemi-cactoid is a locally connected continuum which is the union of a base set and disjoint sequence of cactoids each intersecting the base set in a single point and subject to the condition that for any $\varepsilon > 0$ at most a finite number of the cactoids have diameter greater than ε. It is well known that any monotone image of a hemi-cactoid is also a hemi-cactoid and any hemi-cactoid can be realized by a monotone mapping defined on a 2-cell. The space obtained by removing a noncut

point from a hemi-cactoid will satisfy the conditions of Theorem 1 provided the noncut point is not in an open 2-cell of a true cyclic element which is a 2-cell. (See Theorem 4).

We generalize the notion of cactoid and hemi-cactoid by considering a locally connected continuum X in which each true cyclic element is either a 2-sphere or a closed 2-cell. Before establishing the main theorem concerning such spaces consider the following remarks. For a 2-sphere S and any point $p \in S$, $S - p = \bigcup_{i=1}^{\infty} A_i$, where A_i is a closed 2-cell, $A_i \subset \operatorname{int} A_{i+1}$, and $(S - p) - A_i$ is connected for each positive integer. For a closed 2-cell C and any point p on the boundary of C, $C - p = \bigcup_{i=1}^{\infty} A_i$, where A_i is a closed 2-cell, $A_i \subset \operatorname{int} A_{i+1}$, and $(C - p) - A_i$ is connected for each positive integer i.

Much of the preceding discussion of known results can be found in [10].

THEOREM 4. *Let X be a locally connected continuum every true cyclic element of which is a closed 2-cell or a 2-sphere. If p is a noncut point, p not in an open 2-cell of a true cyclic element which is a closed 2-cell, then $X - p = \bigcup_{i=1}^{\infty} A_i$, where A_i is a unicoherent locally connected continuum, $A_i \subset \operatorname{int} A_{i+1}$, and $X - A_i$ is connected for each positive integer i.*

Proof. Suppose p is in a true cyclic element S which is a 2-sphere. Let $S - p = \bigcup_{i=1}^{\infty} A_i'$, where A_i' is a closed 2-cell, $A_i' \subset \operatorname{int} A_{i+1}'$, and $(S - p) - A_i'$ is connected for each positive integer i. For each positive integer i, let A_i be A_i' union all of the components of $X - S$ whose closure meets A_i'. This gives the required representation for $X - p$.

In the case where p is in a closed 2-cell which is a true cyclic element, the argument is similar.

Suppose now that p is a noncut point which does not belong to a true cyclic element. For $q \in X$, $q \neq p$, there is a simple arc $[q, p]$ joining q to p. Let K be true cyclic element which has a nondegenerate intersection with $[q, p]$. The set $K \cap [q, p]$ is a simple arc and hence has a last point in the order q, p. Let H be the set of such points obtained by considering all nondegenerate intersections of true cyclic elements with $[q, p]$ and let b be the least upper bound of H on $[q, p]$, (take $b = q$ if $H = \phi$). If $b = p$, then H contains a sequence (p_i) converging to p and if $b \neq p$, then choose (p_i) to be any sequence on $[b, q]$ converging to p. It can be assumed that the sequence (p_i) is monotonic increasing in the order q, p. For each positive integer i define A_i to be p_i union all the components of $X - p_i$ except the one that contains $(p_i, p]$. For each positive integer A_i is a unicoherent locally connected continuum, $A_i \subset \operatorname{int} A_{i+1}$, and $X - A_i$ is connected.

In order to make the statements of some theorems somewhat briefer, it is convenient to make the following definition. A space X is said to be a D-cactoid if X is a locally connected generalized continuum which is obtained by removing a noncut point p, p not in an open two cell of a true cyclic element which is a 2-cell, from a locally connected continuum each true cyclic element of which is a 2-sphere or a 2-cell.

COROLLARY 1. *Every D-cactoid is a g-unicoherent space.*

THEOREM 5. *If $f(X) = Y$ is a compact monotone mapping and X is a D-cactoid then so is Y.*

Proof. Let X' be the locally connected continuum and p the noncut point for which $X' - p = X$. The space Y, since f is a compact mapping, is known to be a locally connected generalized continuum. Denote by Y' the one point compactification of Y and denote the compactifying point by q, $q \notin Y$. There is a natural extension of f, call it f', which takes p onto q. Let K be a true cyclic element of Y' and note by the monotoneness of f', $f'^{-1}(K)$ is a continuum. There is a true cyclic element C of X' with $f'(C) \supset K$. The mapping f'/C is monotone hence $f'(C)$ is a cactoid or a hemi-cactoid. Clearly K is a true cyclic element of $f'(C)$ and as such is a 2-sphere or a 2-cell. Thus the space $Y' - q = Y$ is a D-cactoid.

4. **Main theorem.** In order to make use of a theorem in [11] and portions of the proof of the theorem in [3] the following notation is useful. Let $f(X) = Y$ be a one to one mapping of one locally compact separable metric space X onto another such space Y. Let S be the set of points in X at which f is a local homeomorphism and let T be its complement. From a result in [9] the set S is open, T is closed, and $f(S)$ is an open dense set in Y. The sets S and T and the following theorem of [11] will be used in proving the main theorems.

THEOREM A. *Let X be a locally compact arcwise connected separable metric space, let Y be a locally connected generalized continuum. If $f(X) = Y$ is a one to one continuous function which is not a homeomorphism, then there exists a topological ray R in X, R closed in X, with $f(R)$ a simple closed curve in Y. Moreover, if r is the initial point of R, there is a subray R' of R such that $f(R' \cup r)$ is a simple arc and $R' \subset S$.*

THEOREM 6. *Let X be a g-unicoherent space. If $f(X) = Y$ is a*

one-to-one mapping, where $Y \subset E^2$ *is locally compact, locally connected, and* $Y = \bigcup_{i=1}^{\infty} B_i$, $B_i \subset \operatorname{int} B_{i+1}$ (rel Y), $i = 1, 2, 3, \cdots$ *and no* B_i *separates* E^2, *then* f *is a homeomorphism.*

Proof. The proof is made by proving a series of statements which determine the cyclic structure of the space X if f is supposed not to be a homeomorphism and then reaching a contradiction. Let a_i be a sequence of positive real numbers for which $g^{-1}[0, a_i]$ is a locally connected unicoherent continuum and hence $fg^{-1}[0, a_i]$ does not separate E^2 for any i or as in Theorem 3 each $fg^{-1}[c]$, $0 \leq c < \infty$, fails to separate E^2 so that each $fg^{-1}[0, a_i]$ fails to separate E^2 for any sequence a_i, $(a_i) \to \infty$.

We first show that every simple closed curve J in X bounds an open two cell which is an open subset of X. To this end, let a_n be an element of the sequence a_i for which $g(J) \subset [0, a_n]$. The continuum $fg^{-1}[0, a_n]$ fails to separate E^2 and contains the simple closed curve $f(J)$ so it contains the open two cell bounded by $f(J)$. Since $f/g^{-1}[0,a_n]$ is a homeomorphism onto its image, $f^{-1}(E)$ is an open two cell which is open in X and is bounded by J.

To show the space X has exactly one nonconditionally compact cyclic element suppose first that all cyclic elements of X are compact. By Theorem A there exists a ray R, R closed in X, such that $f(R)$ is a simple closed curve and a subray $R' \subset R$ and $R' \subset S$. That is, $R' \subset S$ implies f is a local homeomorphism at each point of R'. The closed ray R' cannot be a subset of any compact cyclic element so that R' contains a point p which separates X. Let X_0 be R union all the components of $X - R$ that map into the open two cell bounded by $f(R)$. There is an integer n such that $B_n \supset f(R)$ consequently $f(X_0)$ is a closed two cell. A contradiction is obtained by nothing that p locally separates X_0 but does not locally separate $f(X_0)$. Thus X has at least one nonconditionally compact cyclic element. We show that X has exactly one nonconditionally compact cyclic element by showing that the complement of any compact set K has at most one nonconditionally compact component. For a compact set K there is an integer n for which $g(K) \subset [0, a_n)$. The set $g^{-1}[0, a_n]$ is a continuum containing and $g^{-1}[a_n, \infty)$ is connected in all of the cases being considered so that $X - K$ has at most one nonconditionally compact component.

That each nondegenerate compact cyclic element of X is topologically a closed 2-cell and $B = M \cap T$, where M is the noncompact cyclic element of X, is a topological line follow from the proofs of (iii) and (v) in [3]. Finally the proof can be completed by using the argument (vi) of [3].

COROLLARY 1. *Let* X *be a g-unicoherent space and* $f(X) = Y$ *a*

one-to-one mapping, where $Y \subset E^2$ is locally compact and locally connected. If f is not a homeomorphism then Y separates E^2.

COROLLARY 2. *Let X be a D-cactoid and $f(X) = Y$ a one-to-one mapping, where $Y \subset E^2$. If Y is a D-cactoid, then f is a homeomorphism.*

5. **Applications.** In [9] it was proved that any monotone mapping of a plane onto a plane is a compact mapping and in [4] this result was generalized by proving that any reflexive compact mapping of a plane onto a plane is a compact mapping. These results are useful and interesting but suffer from the unnatural condition that the mapping must be an onto mapping. It is the purpose of this section to show that by virtue of the theorems in section four much stronger results can be established. These results generalize corresponding theorems of *K.* Borsuk [2] and K. A. Sitnikov [7].

THEOREM 8. *If X is a D-cactoid and $f: X \to E^2$ is monotone, where $f(X)$ is locally connected, locally compact, and $f(X) = \bigcup_{i=1}^{\infty} B_i$, $B_i \subset \operatorname{int} B_{i+1}$, and $E^2 - B_i$ is connected for $i = 1, 2, 3, \cdots$, then f is a compact mapping.*

Proof. Let M be the decomposition space generated by f, π the natural mapping of X onto M and h the unique one-to-one mapping of M into E^2. The mapping π is compact and monotone so that M is a D-coctoid. By Theorem 6 the mapping h is a homeomorphism. Since f is the composition of two compact mappings, it is also compact. See [1], [8].

COROLLARY 1. *Let X be a D-cactoid. If $f(X) = Y$ is a monotone mapping, where $Y \subset E^2$ is locally compact, locally connected and Y does not separate E^2, then f is a compact mapping and Y is a D-coatoid.*

COROLLARY 2. *Let X be a D-cactoid. If $f(X) = Y$ is a monotone mapping which is not a compact mapping, where $Y \subset E^2$ is locally compact and locally connected, then Y separates E^2.*

COROLLARY 3. *Let $f: E^2 \to E^2$ be a monotone mapping, where $f(E^2)$ is locally compact and locally connected. If $f(E^2)$ satisfies the conditions of Theorem 8 or $E^2 - f(E^2)$ is connected or $f(E^2) = E^2$ then f is a compact mapping.*

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UNIVERSITY OF MIAMI
CORAL GABLES, FLORIDA

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ON PRIME DIVISORS OF THE BINOMIAL COEFFICIENT

E. F. ECKLUND, JR.

A classical theorem discovered independently by J. Sylvester and I. Schur states that in a set of k consecutive integers, each of which is greater than k, there is a number having a prime divisor greater than k. In giving an elementary proof, P. Erdös expressed the theorem in the following form:

If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p > k$.

Recently, P. Erdös suggested a problem of a complementary nature:

If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leqq \frac{n}{2}$

The problem is solved by the following

THEOREM. **If $n \geqq 2k$, then $\binom{n}{k}$ has a prime divisor $p \leqq \max\left\{\frac{n}{k}, \frac{n}{2}\right\}$, with the exception $\binom{7}{3}$.**

Throughout the paper, p denotes a prime. J. Rosser and L. Schoenfeld [2] have obtained fairly precise estimates for $\theta(x) = \sum_{p \leqq x} \log(p)$, and $\pi(x) = \sum_{p \leqq x} 1$.

$$\frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) < \pi(x) \qquad \text{for } x \geqq 59 . \tag{1}$$

$$\pi(x) < \frac{x}{\log x}\left(1 + \frac{3}{2\log x}\right) \qquad \text{for } x > 1 . \tag{2}$$

$$\pi(x) < \frac{1.25506x}{\log x} \qquad \text{for } x > 1 . \tag{3}$$

$$\theta(x) < 1.01624x \qquad \text{for } x > 0 . \tag{4}$$

$$x - 2.05282\sqrt{x} < \theta(x) < x \qquad \text{for } 0 < x \leqq 10^8 . \tag{5}$$

Using these results, we are able to prove the theorem.

First we establish the following lemmas.

LEMMA 1. *If $\binom{n}{k}$ has no prime divisors $p \leqq n/2$, then*

$$\binom{n}{k} \leqq e^{\theta(n)-\theta(n-k)} \leqq n^{\pi(n)-\pi(n-k)} . \tag{6}$$

LEMMA 2. *For $k \geqq 59$,*

$$n^{\pi(n)-\pi(n-k)} < e^{(n/\log n+k+k/2\log n)} . \tag{7}$$

LEMMA 3.

$$\frac{2^{(n+1)k-1}}{\sqrt{k}} \leqq \binom{2^n k}{k} . \tag{8}$$

Proof of Lemma 1. $\binom{n}{k} \leqq \prod_{n-k<p\leqq n} p \leqq \prod_{n-k<p\leqq n} n$. Hence

$$\binom{n}{k} \leqq e^{\theta(n)-\theta(n-k)} \leqq n^{\pi(n)-\pi(n-k)} .$$

Proof of Lemma 2. From (1) and (2), we have

$$\begin{aligned} n^{\pi(n)-\pi(n-k)} &< n^{\{n/\log n[1+3/(2\log n)]-(n-k)/\log(n-k)[1+1/(2\log(n-k))]\}} \\ &< n^{\{n/\log n[1+3/2\log n]-(n-k)/\log n[1+1/2\log n]\}} \\ &< e^{\{n[1+3/2\log n]-(n-k)[1+1/2\log n]\}} \\ &< e^{(n//\log n+k+k/2\log n)} . \end{aligned}$$

Lemma 3 is proved by induction on n for all values of k.

The proof of the theorem is by contradiction. Three cases are considered. The general case is a Sylvester-Schur type argument. The other cases involve deducing contradictions from appropriate upper and lower bounds on the inequalities, (6), of Lemma 1.

Proof of the theorem. Assume $\binom{n}{k}$ has no prime divisors

$$p \leqq \max\left\{\frac{n}{k}, \frac{n}{2}\right\} .$$

1. $k < n^{2/3}$. $\binom{n}{k} = \dfrac{n\cdot(n-1)\cdots(n-k+1)}{k\cdot(k-1)\cdots 1} \geqq \left(\dfrac{n}{k}\right)^k .$

By sieving all multiples of 2, and 3, we have

$$\pi(n) - \pi(n-k) \leqq \frac{k}{2} \qquad \text{for } k \geqq 4 .$$

Therefore from (6), we have $(n/k)^k \leqq n^{k/2}$. Thus the assumption is false if $4 \leqq k < n^{1/2}$. By sieving all multiples of 2, 3, and 5, we have

$$\pi(n) - \pi(n-k) \leqq \frac{k}{3} \qquad \text{for } k \geqq 60 .$$

Thus from (6), we have $(n/k)^k \leqq n^{k/3}$. Hence the assumption is false if $60 \leqq k < n^{2/3}$.

2. $n^{2/3} \leqq k \leqq n/16$. Let $\tilde{n} = [n/2]$, and $\tilde{k} = [k/2]$; where $[x]$ denotes the integral part of x. If $p > k$ and p divides $\binom{\tilde{n}}{\tilde{k}}$, then p divides $\binom{n}{k}$ and $p \leqq n/2$. By assumption, there are no such primes. Therefore, $\binom{\tilde{n}}{\tilde{k}}$ has no prime divisors $p > 2\tilde{k} + 1$. Thus $\binom{\tilde{n}}{\tilde{k}} < \tilde{n}^{\pi(\sqrt{\tilde{n}})} \cdot e^{\theta(2\tilde{k}+1)}$ (see paper of M. Faulkner [1]). From (3), (4), and (8), we have

$$\frac{2^{5\tilde{k}-1}}{\sqrt{\tilde{k}}} < \tilde{n}^{(1.26\sqrt{\tilde{n}}/\log\sqrt{\tilde{n}})} \cdot e^{1.02(2\tilde{k}+1)} .$$

Taking logarithms, we obtain

$$3.45\tilde{k} - 0.70 - \frac{1}{2}\log(\tilde{k}) < 2.52\sqrt{\tilde{n}} + 1.02(2\tilde{k} + 1) ,$$

which is a contradiction for $\tilde{k} > 32$. Therefore the assumption is false if $n^{2/3} \leqq k \leqq n/16$ when $k \geqq 65$.

3. $n/16 < k \leqq n/2$. Consider $n/16 < k \leqq n/8$. By (6), (7) and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(n/\log n + k + k/2\log n)} .$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2}\log(k) < \frac{n}{\log n} + k + \frac{k}{2\log n} ;$$

which is false for $k \geqq 1901$. By (5), (6), and (8), we have

$$\frac{2^{4k-1}}{\sqrt{k}} < e^{(k+2.06\sqrt{15k})} .$$

Taking logarithms, we obtain

$$2.76k - 0.70 - \frac{1}{2}\log(k) < k + 2.6\sqrt{15k} ;$$

which is false for $k \geqq 25$. Thus the assumption is false if $n/16 < k \leqq n/8$ when $k \geqq 25$. By similar arguments, we show the assumption is false is $n/8 < k \leqq n/4$ when $k \geqq 32$; and if $n/4 < k \leqq n/2$ when $k > 105$.

We have proved the theorem for $k \geqq 4$ with the exception of a finite number of cases. The cases $k = 1, 2$, and 3, are easily resolved; and the remaining cases have been checked with the aid of an IBM 1620 computer in the following manner:

The values which were checked are $4 \leqq k \leqq 60$ with $2k \leqq n \leqq k^2$, and $61 \leqq k \leqq 105$ with $2k \leqq n \leqq 4k$.

For the i-th prime, p_i, the exponent to which p_i occurred in the "numerator", $n(n-1)\cdots(n-k+1)$, and in the "denominator", $k!$,

of $\binom{n}{k}$, α_i and β_i respectively, were determined; and the values of p_i, n, and k, were reported if the difference, $\alpha_i - \beta_i$, was positive. Cross-checking was done manually. The first ten primes proved sufficient to verify the theorem in these cases.

This concludes the proof of the theorem.

In closing, I would like to thank Professor M. Faulkner for her gracious assistance.

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WESTERN WASHINGTON STATE COLLEGE

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A MODULAR TOPOLOGICAL LATTICE

DON E. EDMONDSON

The purpose of this paper is to present a construction of a compact connected topological lattice which is modular and not distributive. As a special case there will result the example which is a two dimensional subset of R^3, not embeddable in R^2.

The existence of such an example is related to structure questions in topological lattices considered by Dyer and Shields [3], Anderson [1], and others.

The first step is to present a general method for constructing a class of modular lattices. Let D denote a distributive lattice which is a chain, S a nonempty set, and L the S-fold product lattice of D. That is $L = \{f \mid f: S \to D\}$ and $f \leqq g$ if and only if $f(s) \leqq g(s)$ for every $s \in S$. It is known that $(L, \leqq)$ is a distributive lattice with its operations $\vee$ and $\wedge$ characterized by $[f \vee g](s) = f(s) \vee g(s)$ and

$$[f \wedge g](s) = f(s) \wedge g(s)$$

for every $s \in S$. Define

$M = \{f \in L \mid$ there exists $r \in S$ such that $s, t \in S - \{r\}$ implies

$$f(s) \leqq f(r) \quad \text{and} \quad f(s) = f(t)\} .$$

For intuition about M and the arguments that follow, note that M simply consists of all of the constant functions of L and the functions of L which are essentially constant in the sense that they assume but two values — the larger value at exactly one point.

If the order of L, $\leqq$, is restricted to M, it will be established through a sequence of lemmas that $(M, \leqq)$ is a modular lattice. Recall a lattice $(M, \vee, \wedge)$ is modular if and only if for every $a, b, c \in M$, $b \leqq a$ implies that $a \wedge (b \vee c) = b \vee (a \wedge c)$.

LEMMA 1. *If $f \in M$ and f is not constant, there exists a unique $r \in S$ such that $s, t \in S - \{r\}$ implies $f(s) < f(r)$ and $f(s) = f(t)$.*

The proof of the lemma is immediate from the definition of M, and consequently for $f \in M$ and not constant, define *index* f to be the unique element described in Lemma 1.

LEMMA 2. *$(M, \leqq)$ is a sub $\wedge$-semilattice of $(L, \leqq)$.*

It suffices to show that if $f, g \in M$, then $f \wedge g \in M$. If f and g are

constant then $f \wedge g$ is constant and therefore in M. If f is constant and g is not, let $b = \text{index } g$. Then $s, t \in S - \{b\}$ implies

$$[f \wedge g](s) = f(s) \wedge g(s) \leqq f(s) \wedge g(b) = [f \wedge g](b)$$

and likewise $[f \wedge g](s) = [f \wedge g](t)$ and thus $f \wedge g \in M$. If f and g g are both not constant, let $a = \text{index } f$ and $b = \text{index } g$. If $a = b$, then $s, t \in S - \{a\}$ implies $[f \wedge g](s) \leqq [f \wedge g](a)$ and $[f \wedge g](s) = [f \wedge g](t)$. If $a \neq b$, $[f \wedge g](a) = f(a) \wedge g(a)$, $[f \wedge g](b) = f(b) \wedge g(b)$, and for

$$x \in S - \{a, b\}\ [f \wedge g](x) = f(b) \wedge g(a) = [f \wedge g](a) \wedge [f \wedge g](b)\ .$$

Then since D is a chain, $[f \wedge g](x) = [f \wedge g](a)$ or $[f \wedge g](x) = [f \wedge g](b)$ depending upon which is minimal and therefore $f \wedge g \in M$.

LEMMA 3. *If a, b, c are distinct elements of S and $f \in M$, then* $f(a) \wedge f(b) = f(b) \wedge f(c) = f(c) \wedge f(a)$.

The facts of the lemma are an immediate consequence of the definition and is stated as a lemma for convenient reference.

DEFINITION. For $f, g \in M$ define $f \bar{\vee} g: S \to D$ by the following

(i) if f is constant or g is constant, or if f and g are both not constant and index f = index g, then $f \bar{\vee} g = f \vee g$,

(ii) if f and g are both not constant and index $f \neq$ index g, let $a = \text{index } f$ and $b = \text{index } g$, then

$$[f \bar{\vee} g](x) = f(x) \vee g(x) \quad \text{for} \quad x \in \{a, b\}$$
$$[f \bar{\vee} g](x) = [f(a) \vee g(a)] \wedge [f(b) \vee g(b)] \quad \text{for} \quad x \in S - \{a, b\}\ .$$

LEMMA 4. *If $f, g \in M$, then*

(1) *$f \bar{\vee} g \in M$ and $f \vee g \leqq f \bar{\vee} g$, and*

(2) *$h \in M$, $f \leqq h$, $g \leqq h$ implies $f \bar{\vee} g \leqq h$.*

In case (i) of the definition of $f \bar{\vee} g$, easily $f \bar{\vee} g \in M$ and the other results are immediate from $f \bar{\vee} g = f \vee g$. In case (ii) let $a = \text{index } f$ and $b = \text{index } g$, then since D is a chain $[f \bar{\vee} g](x) = [f \bar{\vee} g](a)$ or $[f \bar{\vee} g](x) = [f \bar{\vee} g](b)$ for $x \in S - \{a, b\}$. So in this case also $f \bar{\vee} g \in M$ and $f \vee g \bar{\vee} g$. Also relative to this case, if $h \in M$, $f \leqq h$, and $g \leqq h$, then $f(x) \vee g(x) = [f \bar{\vee} g](x)$ for $x = a$ or $x = b$. But from Lemma 3, $x \in S - \{a, b\}$ implies $h(a) \wedge h(b) \leqq h(x)$ and thus for

$$x \in S - \{a, b\}\ [f \bar{\vee} g](x) = [f \bar{\vee} g](a) \wedge [f \bar{\vee}](b) \leqq h(a) \wedge h(b) \leqq h(x)\ .$$

Therefore $f \bar{\vee} g \leqq h$.

LEMMA 5. *If $f, g, h \in M$, $a, b \in S$, $a \neq b$, $[f \vee g](x) = h(x)$ for $x \in \{a, b\}$*

and $[f \vee g](x) \leqq h(a) \wedge h(b) = h(x)$ *for* $x \in S - \{a, b\}$, *then* $h = f \bar{\vee} g$.

From the hypothesis $f \vee g \leqq h$ and therefore from Lemma 4 $f \bar{\vee} g \leqq h$. But $h(a) = [f \vee g](a) \leqq [f \bar{\vee} g](a)$ and $h(b) \leqq [f \bar{\vee} g](b)$. Then from Lemma 3, for $x \neq a$ and $x \neq b$

$$h(x) = h(a) \wedge h(b) \leqq [f \bar{\vee} g](a) \wedge [f \bar{\vee} g](b) \leqq [f \bar{\vee} g](x)$$

and $h \leqq f \bar{\vee} g$.

THEOREM 1. $(M, \leqq)$ *is a modular lattice with operations* $\bar{\vee}$ *and* $\wedge$.

Lemmas 2 and 4 establish that $(M, \leqq)$ is a lattice with operations $\bar{\vee}$ and $\wedge$, it remains to establish that it is modular. Let $f, g, h \in M$ and $f \leqq g$. It suffices to establish $g \wedge (f \bar{\vee} h) \leqq f \bar{\vee} (g \wedge h)$ since in any lattice $f \bar{\vee} (g \wedge h) \leqq g \wedge (f \bar{\vee})$. The argument will be a case argument.

If $f \bar{\vee} h = f \vee h$, then

$$g \wedge (f \bar{\vee} h) = g \wedge (f \vee h) = f \vee (g \wedge h) \leqq f \bar{\vee} (g \wedge h)$$

since L is itself modular and $g \wedge h \in M$ allows Lemma 4 to apply.

If $h \leqq g$, then $f \bar{\vee} h \leqq g$ and $g \wedge (f \bar{\vee} h) = f \bar{\vee} h = f \bar{\vee} (g \wedge h)$. If $f \leqq h$, then $f \leqq g \wedge h$ and $g \wedge (f \bar{\vee} h) = g \wedge h = f \bar{\vee} (g \wedge h)$.

If f and h are not constant, $a = \text{index } f$, $b = \text{index } g$, $a \neq b$, $h \neq g$, and $f \not\leqq h$. Then $f(b) < f(a)$ and $h(a) < h(b)$. Further, $f(a) \leqq h(a)$ implies $f \leqq h$ and therefore $h(a) < f(a)$. Also $h(b) \leqq g(b)$ and $h(a) < f(a) \leqq g(a)$ implies $h \leqq g$ and therefore $h \not\leqq g$ implies $g(b) < h(b)$. Therefore in this case $h(a) < f(a) \leqq g(a)$ and $f(b) \leqq g(b) < h(b)$. Hence

$$\begin{aligned}[g \wedge (f \bar{\vee} h)](a) &= g(a) \wedge [f(a) \vee h(a)] = f(a) \\ &= f(a) \vee [g(a) \wedge h(a)] = [f \vee (g \wedge h)](a) .\end{aligned}$$

Likewise

$$[g \wedge (f \bar{\vee} h)](b) = g(b) = [f \vee (g \wedge h)](b) .$$

If $x \in S - \{a, b\}$, then

$$\begin{aligned}[g \wedge (f \bar{\vee} h)](x) &= g(x) \wedge [f(a) \vee h(a)] \wedge [f(b) \vee h(b)] \\ &= g(x) \wedge f(a) \wedge h(b) = g(x) \wedge g(a) \wedge f(a) \wedge h(b) \\ &= g(b) \wedge g(a) \wedge f(a) \wedge h(b) = f(a) \wedge g(b) \\ &= [g \wedge (f \bar{\vee} h)](a) \wedge [g \wedge (f \bar{\vee} h)](b) .\end{aligned}$$

But $[f \vee (g \wedge h)](x) \leqq [g \wedge (f \bar{\vee} h)](x)$ and $g \wedge (f \bar{\vee} h) \in M$, therefore by Lemma 5 $g \wedge (f \bar{\vee} h) = f \bar{\vee} (g \wedge h)$.

COROLLARY. *If* card $S<3$, M *is a distributive lattice. If* $3\leqq$card S *and* $2 \leqq$ card D, *then* M *is a modular nondistributive lattice.*

If card $S < 3$, then $M = L$ and M is distributive. If $3 \leqq$ card S and $2 \leqq$ card D, let s_1, s_2, s_3 be three distinct elements of S and $c<d$ be two elements of D. Define f_1, f_2, f_3 by $f_i(x) = d$ if $x = s_i$ and $f(x) = c$ for $x \in S - \{s_i\}$. Also define g and k by $g(s) = d$ for every $s \in S$ and $k(s) = c$ for every $s \in S$. Then $f_1 \wedge f_2 = f_2 \wedge f_3 \wedge f_1 = k$ and $f_1 \vee f_2 = f_2 \vee f_3 = f_3 \vee f_1 = g$ and $\{f_1, f_2, f_3, g, k\}$ is a modular five sublattice of M. Therefore M is not distributive [2].

At this stage the algebraic nature of M has been established, in the section that follows the topological nature of M will be studied. It will be assumed in the following that D is topological chain, that is D is a Hausdorff topological space with the operations $\vee$ and $\wedge$ continuous [3]. If L is considered with the product topology, it is as usual a topological lattice and M may be considered as a topological space in the relative topology that it inherits from L. In this context, the following theorem results.

THEOREM 2. *If* D *is a topological chain, then*
(1) M *is a closed subset of* L,
(2) M *is compact if* D *is compact, and*
(3) M *is connected if* D *is connected.*

Since with card $S \leqq 2$, $M = L$, it suffices to consider $3 \leqq$ card S and to establish (1) and (3).

(1) $L-M$ is open for if $f \notin M$, then f is not constant and there exist distinct $a, b, c \in S$ such that $f(b) < f(a)$ and $f(b) < f(c)$. Then since D is a chain $f(b) < f(a) \wedge f(c)$. If there exists $z \in D$ such that $f(b)<z<f(a)\wedge f(c)$, define $W = \{g \in L \mid z < g(a),\ z < g(c),$ and $g(b)<z\}$ and define $W = \{g \in L \mid f(b) < g(a),\ f(b) < g(c),$ and $g(b) < f(a) \wedge f(c)\}$ if no such z exists. In either case, $f \in W$, W is open, and $W \cap M=\varnothing$.

(3) If D is connected, consider the map $T: D \to M$ where for each $d \in D$ $T(d) = k_d$ and k_d is the constant function generated by d. Clearly T is continuous and K the set of all constant functions is a connected subset of M. If $f \in M - K$, let $a =$ index f, $m = \max f$, and $r = \min f$ define the map H from $[r, m]=\{x \in D \mid r \leqq x \leqq m\}$ into M by $H(x) = f_x$ where $f_x(a) = x$ and $f_x(s) = r$ for $s \in S - \{a\}$. Again H is continuous and since $[r, m]$ is connected then the range of H is a connected subset of M containing f and intersecting K. Therefore M is connected.

Note. It is clear that $\wedge$ will be continuous as an operation on

M since it is continuous on L. Thus when D is a topological chain M is a closed topological sub-$\wedge$-semilattice of L. In order to study the operation $\overline{\vee}$ relative to continuity, it is necessary to restrict S to being finite, in view of the following lemma.

LEMMA 6. *If D is a topological chain and $2 \leqq \text{card } D$ and S is infinite, then $\overline{\vee}$ is not continuous.*

Let $c < d$ in D and define $k: S \to D$ by $k(s) = c$ for every $s \in S$. Then $k \,\overline{\vee}\, k = k$. Let $r \in S$ and define $W_r = \{f \in M \mid f(r) < d\}$; then W_r is an open subset of M containing k. Let U be any open set of M containing k, then there exist $s_1, s_2, \cdots, s_n$ distinct elements of S and $U_1, U_2, \cdots, U_n$ open sets of D such that if $W = \{f \in M \mid f(s_i) \in U_i$ for $i = 1, 2, \cdots, n\}$, $k \in W \subset D$. Now $k \in W$ implies

$$c \in \cap \{U_i \mid i = 1, 2, \cdots, n\} .$$

Since S is infinite there exist $a, b \in S - \{s_1, s_2, \cdots, s_n\}$ such that $a \neq b$. Define h and g by $h(a) = d$ and $h(x) = c$ if $x \in S - \{a\}$, and $g(b) = d$ and $g(x) = c$ if $x \in S - \{b\}$. Therefore $h, g \in W$ and $h \,\overline{\vee}\, g \notin W_r$ since $h \,\overline{\vee}\, g$ is the constant function defined by d. Therefore $U \,\overline{\vee}\, U \not\subset W_r$ and $\overline{\vee}$ is not continuous.

DEFINITION. For S finite and $2 \leqq \text{card } S$, denote max

$$f = \max \{f(s) \mid s \in S\} , \qquad I(f) = \{s \in S \mid f(s) = \max f\} .$$

Then define $f^-: S \to D$ by

(1) if $I(f)$ is not a unit set, $f^-(s) = \max f$ for every $s \in S$, and
(2) if $I(f)$ is a unit set, $f^-(s) = \max f$ for $s \in I(f)$, and

$$f^-(s) = \max \{f(t) \mid t \in S - I(f)\} \quad \text{for} \quad s \in S - I(f) .$$

LEMMA 7. *If S is finite and $2 \leqq \text{card } S$, then*
(1) *$f \in L$ implies $f^- M$ and $f \in M$ if and only if $f = f^-$,*
(2) *$f \leqq f^-$, $f^{--} = f^-$, and $f \leqq g$ implies $f^- \leqq g^-$,*
(3) *$f, g \in M$ implies $f \,\overline{\vee}\, g = (f \vee g)^-$.*

The lemma is a straight forward catalog of the properties following from the definition directly.

LEMMA 8. *If S is finite, $2 \leqq \text{card } S$ and D is a topological chain, then the function $J: L \to M$ defined by $J(f) = f^-$ is a retraction of L onto M.*

From Lemma 7 it suffices to show that J is continuous. This is done by letting U be an open set in D and $r \in S$, defining $W = \{f \in M \mid f(r) \in U\}$ and showing that $J^{-1}(W)$ is open. It is shown to be open by case argument. Let $g \in L$ and $g^- \in W$. If $I(g)$ is not a unit set, then g^- is constant and max $g \in U$. Define $V_1 = \{f \in L \mid f(s) \in U$ for every $s \in S\}$. Since S is finite, V_1 is open and contains g. Further $h \in V_1$ implies $h^-(r) \in U$. If $I(g) = \{r\}$, let b be an element of $S - \{r\}$. Define $V_2 = \{f \in L \mid f(r) \in U_1$, and $f(s) \in U_2$ for $s \in S - \{r\}\}$ where $U_1 = U \cap \{x \in D \mid z < x\}$ and $U_2 = \{x \in D \mid x < z\}$ if there exist $z \in D$ such that $g^-(b) < z < g^-(r)$, and if there does not exist such an element z, $U_1 = U \cap \{x \in D \mid g^-(b) < x\}$ and $U_2 = \{x \in D \mid x < g^-(r)\}$. Then $g \in V_2$, V_2 is open and $f \in V_2$ implies $f^-(r) \in U$. The other case is handled in a similar fashion.

THEOREM 3. *If D is a topological chain and S is finite, then $(M, \leqq)$ is a modular topological lattice which is nondistributive if* card $S > 2$ *and* card $D > 1$.

If card $S = 1$, $M = L$ and therefore $(M, \leqq)$ is a topological distributive lattice. If card $S \geqq 2$, then Lemma 7 and 8 establish that $\bar{\vee}$ is continuous since it is the composition of continuous maps. Therefore $(M, \leqq)$ is a topological lattice since $\wedge$ is continuous for every S. Theorem 1 establishes the modularity of M while its corollary the nondistributive nature of M when $3 \leqq$ card S and $2 \leqq$ card D.

DEFINITION. Let n be a positive integer and $3 \leqq n$, then let M_n denote the lattice constructed as the M above in the case where $S = \{1, 2, \cdots, n\}$ and $D = \{x \in R \mid 0 \leqq x \leqq 1\}$ with its usual order and the operations of D being $x \vee y = \max\{x, y\}$ and $x \wedge y = \min\{x, y\}$. When S is the set of positive integers and D as previously described let M_∞ denote the lattice M constructed. Then the following results are immediate.

THEOREM 4. *For each positive integer $n \geqq 3$, M_n is a compact connected topological lattice which is modular and not distributive.*

COROLLARY 1. *M_3 is a compact connected topological lattice, modular and not distributive, which is a two dimensional subset of R^3 that cannot be embedded in R^2.*

COROLLARY 2. *M_∞ is a compact connected topological semilattice, which is a modular lattice, and not a topological lattice.*

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UNIVERSITY OF TEXAS AT AUSTIN

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A NOTE ON A THEOREM OF HILL

PHILLIP GRIFFITH

Recently Hill has shown the existence of an abelian p-group with the property that each infinite subgroup can be embedded in a direct summand of the same cardinality but the group is not a direct sum of countable groups. Megibben has since observed that this phenomenon occurs in a larger class of abelian groups. In this note we show that such pathology is present in modules for a rather wide class of rings. In fact, the lack of such phenomena for a particular class of modules serves as a characterization for left perfect rings. Our results also yield some facts concerning pure injective modules.

All rings in this paper are associative with identity and all modules are unital.

2. A characterization of left perfect rings. Bass [1] calls a ring R left perfect if each left R-module has a projective cover (projective cover is the dual of injective envelope). Among several other characterizations of left perfect rings, Bass proves that R is left perfect if and only if R has the descending chain condition on principal right ideals. Hence, assuming that R is not left perfect, we can obtain a strictly decreasing sequence of principal right ideals of the form

$$a_1R \supset a_1a_2R \supset \cdots \supset a_1 \cdots a_nR \supset \cdots .$$

We set $P = \Pi_{n<\omega}Re_n$, where $Re_n \cong R$ for each n, and we denote by S the submodule of finitely nonzero sequences in P. We shall use the notation $\sum_{i=m}^{n} r_ie_i$, for $m \leqq n$, to denote a vector in P whose ith coordinate is zero for $i > n$ and $i < m$ and whose ith coordinate is r_ie_i for $m \leqq i \leqq n$. We define elements

$$c^{(m)} = \sum_{i \geqq m} (a_m \cdots a_i)e_i \in P \quad \text{for} \quad m = 1, 2, \cdots .$$

Let A be the submodule of P generated by S and the elements $c^{(m)}$ for $m = 1, 2, \cdots$. With this notation established, we prove the following lemma.

LEMMA 2.1. *Let R be a ring that is not left perfect and let A and S be defined as above. Then A is free and S is not a direct summand of A.*

Proof. First we note that if $n < m$, then

$$(*) \qquad c^{(n)} = \sum_{i \geqq n}^{m-1} (a_n \cdots a_i)e_i + (a_n \cdots a_{m-1})c^{(m)}$$

and in particular

$$c^{(n)} = a_n(e_n + c^{(n+1)}) \quad \text{for} \quad n = 1, 2, \cdots .$$

Now suppose that $A = S \oplus B$. Then $c^{(n)} = s_n + b_n$ where $s_n \in S$ and $b_n \neq 0 \in B$. From property $(*)$ above, we have that, for $n > 1$,

$$c^{(1)} = s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \quad \text{where} \quad s_{1n} \in S .$$

Therefore

$$\begin{aligned} s_1 + b_1 = c^{(1)} &= s_{1n} + (a_1 \cdots a_{n-1})c^{(n)} \\ &= s_{1n} + (a_1 \cdots a_{n-1})(s_n + b_n) \\ &= s_{1n} + (a_1 \cdots a_{n-1})s_n + (a_1 \cdots a_{n-1})b_n . \end{aligned}$$

Hence $s_1 = s_{1n} + (a_1 \cdots a_{n-1})s_n$ and $b_1 = (a_1 \cdots a_{n-1})b_n$ for each $n > 1$. Therefore $c^{(1)} = s_1 + (a_1 \cdots a_{n-1})b_n$ for $n = 2, 3, \cdots$. Since s_1 has only finitely many nonzero coordinates, it follows that there is a positive integer r such that $a_1 \cdots a_r = a_1 \cdots a_r a_{r+1} y$. But this implies that $a_1 \cdots a_r R = a_1 \cdots a_{r+1} R$ which is a contradiction. Thus S is not a summand of A.

To show that A is free, let $y_n = e_n + c^{(n+1)}$ for $n = 1, 2, \cdots$. Since $c^{(n)} = a_n y_n$ by property $(*)$ above, it follows that A is generated by $\{y_n\}_{n<\omega}$. Suppose that $r_1 y_1 + \cdots + r_n y_n = 0$ where $r_i \in R$. Then

$$r_1 c^{(2)} + r_2 c^{(3)} + \cdots + r_n c^{(n+1)} = -r_1 e_1 - r_2 e_2 - \cdots - r_n e_n .$$

Since the first coordinate of the left hand side is zero, it follows that $r_1 = 0$. A repetition of the preceding argument shows that $r_1 = r_2 = \cdots = r_n = 0$. This implies that A is free with $\{y_n\}_{n<\omega}$ for a basis.

We observe from [1] that a left R-module is torsionless if and only if it can be embedded as a submodule of a direct product of copies of R. We shall call a left R-module G $\aleph_1$-separable provided G is flat, torsionless and that each countably generated submodule of G is contained in a countably generated direct summand of G (this definition parallels the definition given by L. Fuchs [4] in the context of $\aleph_1$-free groups). We now prove the main result of this section. The proof is modeled after that of Hill's [5].

THEOREM 2.2. *A ring R is left perfect if and only if each $\aleph_1$-separable left R-module is a direct sum of countably generated modules.*

Proof. If R is left perfect, then by Theorem 3.2 [2] any flat left module is projective. Since an $\aleph_1$-separable left module is flat, it follows from Kaplansky's theorem [6] that each $\aleph_1$-separable left R-module is a direct sum of countably generated modules.

Now suppose that R is not left perfect. This implies by Theorem P [1] that R has a strictly decreasing sequence

$$a_1R \supset a_1a_2R \supset \cdots \supset a_1 \cdots a_nR \supset \cdots$$

of principal right ideals. Set $P^* = \Pi_{\alpha<\Omega} Re_\alpha$ where $Re_\alpha \cong R$ for each $\alpha < \Omega$ (Ω denotes the first uncountable ordinal). We construct a left submodule G of P^* such that $G = \bigcup_{\alpha<\Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha<\Omega}$ is a monotone increasing chain defined as follows: $G_0 = 0$, $G_1 = Re_1$ and suppose that G_α has been defined for each $\alpha < \beta$ such that the following conditions hold:

(i) If α is a limit ordinal, $\alpha < \beta$, $G_\alpha = \bigcup_{\gamma<\alpha} G_\gamma$.

(ii) If $\alpha - 1$ and $\alpha - 2$ exist, $G_\alpha = G_{\alpha-1} \oplus Re_{\alpha-1}$.

(iii) If $\alpha - 1$ exists and is a limit, there is a monotone increasing sequence $\sigma_\alpha(n)$ of ordinals less than $\alpha - 1$ such that $\sigma_\alpha(n) - 2$ is defined for each n and such that $\sigma_\alpha(n)$ converges to $\alpha - 1$. Then $c_\alpha^{(m)} = \sum_{i\geq m} (a_m \cdots a_i)e_{\sigma_\alpha(i)}$ for $m = 1, 2, \cdots$ and G_α is generated by $G_{\alpha-1}$ and $\{c_\alpha^{(n)}\}_{n<\omega}$.

(iv) If $\rho_{\gamma+1}$ denotes the natural projection of P^* onto $\Pi_{\lambda<\gamma} Re_\lambda$ and if $\gamma + 1 < \alpha < \beta$, then $\rho_{\gamma+1}(G_\alpha) = G_{\gamma+1}$.

(v) G_α is not a direct summand of $G_{\alpha+1}$ if α is a limit ordinal.

(vi) G_α is flat for $\alpha < \beta$.

If β is a limit ordinal we set $G_\beta = \bigcup_{\alpha<\beta} G_\alpha$ and if both $\beta - 1$ and $\beta - 2$ exist we set $G_\beta = G_{\beta-1} \oplus Re_{\beta-1}$. It is straightforward in either of the above two cases to show that (i)-(vi) hold for the collection $[G_\alpha]_{\alpha<\beta}$. Now suppose that $\beta - 1$ is a limit ordinal. Define $\sigma_\beta(n)$ and $c_\beta^{(m)}$ so that (iii) is satisfied and define G_β to be the submodule generated by $G_{\beta-1}$ and $\{c_\beta^{(c)}\}_{n<\omega}$. Suppose that $\gamma + 1 < \beta$ and consider $\rho_{\gamma+1}(G_\beta)$. To show that (iv) is satisfied, it clearly suffices to show that $\rho_{\gamma+1}(c_\beta^{(m)}) \in G_{\gamma+1}$. But this is a direct consequence of the fact that $c_\beta^{(m)} = \sum_{i\geq m} (a_m \cdots a_i)e_{\sigma_\beta(i)}$ and that $\sigma_\beta(i) > \gamma + 1$ for all i larger than some integer i_0. To see that (v) holds, let A_β be the set of ordinals $\{\sigma_\beta(1), \sigma_\beta(2), \cdots\}$ and let I_β be the ordinals less than β that are not in A_β. Let $B = G_\beta \cap \Pi_{I_\beta} Re_\lambda$, $A = G_\beta \cap \Pi_{A_\beta} Re_\lambda$ and let S denote the finite sequences in $\Pi_{I_\beta} Re_\lambda$. It is routine to show that $G_\beta = B \oplus A$ and that $G_{\beta-1} = B \oplus S$. We observe that (up to isomorphism) our A and S here are the same as the A and S, respectively, in Lemma 2.1. It follows that $G_{\beta-1}$ is not a direct summand of G_β. We also see that $G_{\beta-1}$ is flat since B is necessarily flat and since A is free. Thus the collection $[G_\alpha]_{\alpha\leq\beta}$ satisfies (i)-(vi) and hence we obtain $G = \bigcup_{\alpha<\Omega} G_\alpha$ where $\{G_\alpha\}_{\alpha<\Omega}$ satisfies (i)-(vi). Note that G is torsionless since G is a submodule of P^*. G is flat

from (vi) since a direct limit of flat modules is flat. Property (v) implies that G is not a direct sum of countably generated modules. Finally, property (iv) guarantees that $\rho_{\gamma+1}$, when restricted to G, is a projection of G onto $G_{\gamma+1}$. Thus G is $\aleph_1$-separable.

From the above proof, we obtain the following corollary.

COROLLARY 2.3. *A ring R is left perfect if and only if each $\aleph_1$-separable left R-module is projective.*

3. **Some remarks on pure injective modules over artinian rings.** An interesting consequence of our Lemma 2.1 is that the direct sum of $\aleph_0$ copies of a ring R (as a left R-module) is not a direct summand of the corresponding direct product of $\aleph_0$ copies of R if R is not left perfect. In this section we wish to consider in part the question of when the direct sum of infinitely many copies of R (as a left R-module) is a direct summand of the corresponding direct product of copies of R. More generally, we consider the problem of determining when projective modules are pure injective modules in the sense of Warfield [7]. For commutative Noetherian rings we obtain a complete answer to both of the above questions. A submodule A of a left R-module B is called a pure submodule provided, for any right module M, the natural homomorphism $M \otimes A \to M \otimes B$ is injective. A module Q is called pure injective, if for every module B and pure submodule A, each homomorphism of A into Q extends to a homomorphism of B into Q. Hence, if a pure injective module Q is a pure submodule of a module B, then Q is a direct summand of B. Our main theorems of this section follow the next lemma.

LEMMA 3.1. *If R is a left artinian ring, then any pure submodule of a left projective R-module is a direct summand.*

Proof. Suppose that A is a pure submodule of a left projective module P and suppose that M is an arbitrary right R-module. From the exact sequence

$$0 = \operatorname{Tor}_1^R(M, P) \to \operatorname{Tor}_1^R(M, P/A) \to M \otimes A \to M \otimes P,$$

we obtain that $\operatorname{Tor}_1^R(M, P/A) = 0$ since the homomorphism $M \otimes A \to M \otimes P$ is injective. Hence P/A is a flat left R-module. By Theorem P [1], P/A is projective and thus A is a direct summand of P.

In what follows, $\sum A_i$ will denote the finitely nonzero vectors in the direct product ΠA_i.

THEOREM 3.2. *If R is a commutative artinian ring, then each projective R-module is pure injective. Moreover, if R is a commutative Noetherian ring and if each projective R-module is pure injective, then R is artinian.*

Proof. First suppose that R is a commutative artinian ring. It suffices to show that each free R-module is pure projective. By Proposition 9 [7], R is pure injective as a module over itself. Let $F = \sum_\alpha R$ be an arbitrary free R-module and let P denote the direct product $P = \Pi_\alpha R$ containing F. It is elementary to see that F is a pure submodule of P and that P is pure injective since R is pure injective. By Theorem 3.4 [2], P is also a projective R-module. Hence, by Lemma 3.1, F is a direct summand of P and therefore is pure injective.

Now suppose that R is a commutative Noetherian ring for which each projective module is pure injective. Let S and A be as in Lemma 2.1. Note that $S = \sum_{\aleph_0} R$ and that $S \subseteqq A \subseteqq \Pi_{\aleph_0} R$. Therefore S is pure in A and is therefore a direct summand of A. Hence Lemma 2.1 yields that R is a perfect ring. Since R is also Noetherian, we have that R is artinian.

COROLLARY 3.3. *If R is a commutative artinian ring, then the direct sum $\sum_\alpha R$ is a direct summand of the direct product $\Pi_\alpha R$ for each cardinal number α. Moreover, if R is a commutative Noetherian ring and if $\sum_{\aleph_0} R$ is a direct summand of $\Pi_{\aleph_0} R$, then R is artinian.*

We conclude our consideration of pure injective modules with an answer to the converse problem answered in Theorem 3.2, that is, we classify those rings for which every pure injective R-module is projective. Our solution here needs no initial assumptions on the ring.

THEOREM 3.4. *A ring R has the property that each pure injective left R-module is projective if and only if R is semi-simple and artinian.*

Proof. The sufficiency is clear. Hence suppose that R has the property that each pure injective left R-module is projective. Since each injective left module is pure injective, it follows that each injective left R-module is also projective. By Theorem 5.3 [3] of Faith and Walker, we have that R is quasi-Frobenius. Since each left R-module can be embedded as a pure submodule of a pure injective left R-module by Corollary 6 [7], we have that any left R-module is isomorphic to a pure submodule of a projective module. Since a quasi-Frobenius ring is left artinian, it follows by Lemma 3.1 that each

left R-module is projective. It is well-known that such a ring is a semi-simple artinian ring.

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UNIVERSITY OF HOUSTON
HOUSTON, TEXAS

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ON FINITE GROUPS WITH INDEPENDENT CYCLIC SYLOW SUBGROUPS

MARCEL HERZOG

The purpose of this paper is to classify all perfect groups with cyclic Sylow p-subgroups which satisfy the condition

(TI) two different Sylow p-subgroups of G contain only the unit element in common

and such that

$$o(G) < o(P)^3$$

where P is a Sylow p-subgroup of G.

The main result of this paper is the following

THEOREM 1. **Let G be a perfect finite group with a cyclic Sylow p-subgroup P of order p^a and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Assume, furthermore, that**

$$o(G) < p^{3a}\ .$$

Then one of the following statements holds.

(I) $a = 1$, $G \cong PSL(2, p)$, **where $p > 3$ is a prime.**

(II) $a = 1$, $G \cong PSL(2, p-1)$, **where $p = 2^m + 1 > 5$ is a Fermat prime.**

(III) $a = 1$, $G \cong SL(2, p)$, **where $p > 3$ is a prime.**

(IV) $a = 2$, $p = 3$, $G \cong PSL(2, 8)$.

Ten years ago E. Artin raised the following problem: what are the simple finite groups G of order g which are divisible by a prime $p > g^{1/3}$? This question was answered by R. Brauer and W. F. Reynolds in [1]. They found that the only groups satisfying the above conditions are $PSL(2, p)$, where $p > 3$ is a prime, and $PSL(2, p-1)$ where $p > 3$ is a Fermat prime, $p = 2^m + 1$. In particular, the Sylow p-subgroups of these groups are of order p and therefore they are cyclic and satisfy the (TI) condition. Theorem 1 thus generalizes these results.

As a matter of fact we will prove a more general statement than Theorem 1.

THEOREM 1*. *Let G be a finite group with a cyclic Sylow p-subgroup P of order p^a and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Assume, furthermore, that*

$$o(G) < p^{3a}$$

and no homomorphic image of G is isomorphic to $N_G(P)/W$, where

W is the normal complement of P in $C_G(P)$. Then one of the following statements holds.

(I)* $a = 1$, $G \cong PSL(2, p)$, *where* $p > 3$ *is a prime.*

(II)* $a = 1$, $G \cong PSL(2, p-1)$, *where* $p = 2^m + 1 > 5$ *is a Fermat prime.*

(III)* $a = 1$, $G \cong SL(2, p)$, *where* $p > 3$ *is a prime.*

(IV)* $a = 2$, $p = 3$, $G \cong PSL(2, 8)$.

(V)* $a = 1$, $G \cong PGL(2, p)$, *where* $p > 3$ *is a prime.*

(VI)* $a = 1$, $G \cong PSL(2, p) \times M$, *where* $p > 3$ *is a prime* and $o(M) = 2$.

Since $G = G'$ implies the last condition of Theorem 1*, Theorem 1 follows immediately from Theorem 1*. In this paper the group $N_G(P)/W$ will be referred to as the p-metacyclic group of order qp^a.

Theorem 1* follows from the following more general result:

THEOREM 2. *Let G be a finite group with a cyclic Sylow p-subgroup P of order $p^a > 1$ and assume that the Sylow p-subgroups of G satisfy the (TI) condition. Suppose that no homomorphic image of G is isomorphic to the p-metacyclic group of order p^aq. Then*

$$o(G) = qwp^a(1 + np^a)$$

where $wp^a = o(C_G(P))$, $q = [N_G(P) : C_G(P)] > 1$ and n is a positive integer.

Furthermore, let G_0 be the minimal normal subgroup of G for which G/G_0 is solvable, and let M be the maximal normal subgroup of G_0 of order prime to p. Denote G_0/M by G^*. Then one of the following statements holds.

(A) $n = (hvp^a + h + v^2 + v)/(v + 1)$
where h and v are positive integers and $v + 1 \mid h(p^a - 1)$.

(B) $a = 1$, $n = 1$, $G^* \cong PSL(2, p)$ *where* $p > 3$ *is a prime.*

(C) $a = 1$, $n = (p-3)/2$, $G^* \cong PSL(2, p-1)$ *where* $p = 2^m + 1 > 5$ *is a Fermat prime.*

(D) $a = 2$, $p = 3$, $n = (p^2 - 3)/2$, $G^* \cong PSL(2, 8)$.

Theorem 2 immediately yields

COROLLARY. *Let G satisfy the assumptions of Theorem 2 and suppose that $n < (p^a + 3)/2$. Then G^* is of type* (B), (C) *or* (D).

In § 2 some basic properties of groups with a Sylow subgroup satisfying the TI-property are derived. Section 3 contains the proof of Theorem 2, from which Theorem 1* is deduced in § 4.

We use the standard notation $C_G(T)$, $N_G(T)$, $o(T)$, $T^\sharp$, and $\langle T \rangle$,

where T is a subset of the group G, to denote respectively: the centralizer, normalizer, number of elements, the nonunit elements and the group generated by T. We will say that $N_G(T)/C_G(T)$ acts frobeniusly on T if $\theta^\eta = \theta$ for $\theta \in T^\#$ and $\eta \in N_G(T)$ implies that $\eta \in C_G(T)$. An element of G is called a p'-element, where p is a prime number, if p does not divide its order. The principal character and the commutator subgroup of G will be denoted by 1_G and G' respectively. Finally, if a and b are integers, then (a, b) denotes their greatest common divisor and $a \mid b$ means: a divides b.

2. *TIP*-groups. A finite group will be called a *TIP*-group if its Sylow p-subgroups are nontrivial and satisfy the *TI*-property. This section deals with properties of *TIP*-groups in general, followed by a study of *TIP*-groups with a cyclic Sylow p-subgroup.

PROPOSITION 2.1. *Let G be a TIP-group with a Sylow p-subgroup P of order p^a. Then the following statements hold.*

(a) $C_G(P) = W \times P$
where $o(W) = w$ and $(w, p) = 1$.

(b) $o(G) = qwp^a(1 + np^a)$
where $q = [N_G(P) : C_G(P)]$ and n is a nonnegative integer.

(c) *Any normal subgroup L of G of order divisible exactly by $p^b > 1$ is a TIP-group of order $q_L w_L p^b(1 + np^a)$.*

(d) *If H is a normal subgroups of G of order prime to p, then G/H is a TIP-group.*

Proof. Let $C = C_G(P)$, $N = N_G(P)$.

(a) Since P is a normal Hall-subgroup of C, it has a complement W and $(w, p) = 1$. Since elements of W commute with elements of P, $C = W \times P$.

(b) Consider the conjugates $\{P_i\}$ of P, other than P. If $\sigma \in P$ and $P_i^\sigma = P_i$, where $P_i = P^\tau$, $\tau \in G$, then $P^{\tau\sigma\tau^{-1}} = P$, $\tau\sigma\tau^{-1} \in N_G(P)$ and $\sigma \in N_G(P^\tau)$, $\sigma \in P \cap P^\tau = \{1\}$. Thus P acts by conjugation fixed point free on $\{P_i\}$ and therefore $o\{P_i\} = np^a$ for some nonnegative integer n. Hence $[G : N] = 1 + np^a$ and $o(G) = qwp^a(1 + np^a)$.

(c)—(d) The proof of Lemma 1 in [6] obviously holds also for general *TIP*-groups, with $p \neq 2$. Thus any subgroup of G of order divisible by p is a *TIP*-group and (d) holds. Let $o(L) = q_L x_L p^b(1 + n_L p^b)$. Since L and G have the same number of Sylow p-subgroups $1 + n_L p^b = 1 + np^a$ proving (c).

PROPOSITION 2.2. *Let G be a TIP-group with a cyclic Sylow p-subgroup P of order p^a. Then in addition to properties* (a)—(d) *of Proposition* 2.1, *and using the same notation, the following state-*

ments hold.

(e) $C_G(P) = C_G(\sigma)$ *and* $N_G(P) = N_G(\langle\sigma\rangle)$ *for all* $\sigma \in P^\sharp$.

(f) q *divides* $p - 1$.

(g) $o(G/H) = q\bar{w}p^a(1 + \bar{n}p^a)$ *and there exists a nonnegative integer* z *such that*

$$n = z + \bar{n} + z\bar{n}p^a .$$

If $z = 0$ *then* $H \subset W$.

(h) *If* K *is a normal subgroup of* G *and* K *does not contain* P *then*

$$N_K(P) = C_K(P) .$$

(i) *If also* $o(K \cap P) > 1$, *then* G *can be mapped homomorphically on the* p*-metacyclic group of order* p^aq.

Proof. (e) Let $\sigma \in P^\sharp$; then by Lemma 2.1.b in [3] $C_G(\sigma) \cap N_G(P) = C_G(P)$. It follows from the *TI*-property that $C_G(\sigma) \subset N_G(P)$ and $N_G(\langle\sigma\rangle) \subset N_G(P)$. Thus $C_G(\sigma) = C_G(P)$ and since P is cyclic $N_G(\langle\sigma\rangle) = N_G(P)$.

(f) By Lemma 2.1.d of [3] N/C acts frobeniusly on P and P is cyclic. Therefore $q = [N : C]$ divides $p - 1$.

(g) The proof of Proposition 2 in [1] holds, with the obvious changes, also in the present case. It is clear from the proof in [1] that if $z = 0$ then $H \subset C_G(P)$.

(h) Suppose that $K \cap N \not\subset C$ and let $\sigma \in K \cap N - C$. Since N/C acts frobeniusly on P, it follows that the elements $\sigma\rho^{-1}\sigma^{-1}\rho$, $\rho \in P$, are distinct and belong to $P \cap K$. Thus P is contained in K, a contradiction. Consequently $K \cap N \subset C$ and $K \cap N = K \cap C$, as required.

(i) Let p^b be the exact power of p dividing $o(K)$. Then $1 < p^b < p^a$ and by Proposition 2.1.c and (h) $o(K) = w_K p^b(1 + np^a)$, where $w_K p^b = o(C_K(P \cap K)) = o(N_K(P \cap K))$. By the Burnside Theorem K has a characteristic subgroup T of order $w_K(1 + np^a)$. T is normal in G and $G = NT$. Consequently WT is a normal subgroup of G and G/WT is isomorphic to the p-metacyclic group of order p^aq.

3. **Proof of Theorem 2.** If either $p = 2$ or $q = 1$, then $C_G(P) = N_G(P)$ and by the Burnside Theorem P has a normal complement in G, in contradiction to our assumption. Thus $p > 2$ and $q > 1$.

If P is normal in G and $C_G(P) = W \times P$, then W is normal in G, again a contradiction. Thus P is not normal in G and the first statement of Theorem 2 follows from Proposition 2.1.b.

It follows from Proposition 3 in [1] and Proposition 2.2.i that

$P \subset G_0$. Indeed, if $P \not\subset G_0$ then either $a = 1$ or $a > 1$ and G contains a normal subgroup U such that $1 < o(U \cap P) < p^a$. In both cases the above mentioned propositions yield a contradiction to our assumptions.

The definition of G_0 forces it to be its own commutator subgroup and the same is true for G^*. Moreover, G^* does not have nontrivial normal subgroups of order prime to p.

From now on we will assume that (A) is not satisfied and will show that then one of the statements (B), (C), or (D) holds.

Let $o(G_0) = q_0 w_0 p^a(1 + np^a)$, $o(G^*) = q_0 w^* p^a(1 + n^* p^a)$. Since $G^* = (G^*)'$, $n^* \neq 0$. By Proposition 2.2.g there exists a nonnegative integer z such that

$$n = z + n^* + zn^* p^a .$$

If $z \neq 0$, let $h = (z + 1)n^*$, $v = z$. Then:

$$n = v + h/(v + 1) + vhp^a/(v + 1)$$

in contradiction to our assumptions. Thus $z = 0$ and $n^* = n$.

Consequently, it suffices to show that if G satisfies the assumptions of Theorem 2 and in addition, $G = G'$, G has no nontrivial normal subgroup of order prime to p and n does not satisfy (A), then G is isomorphic to one of the simple groups described in (B), (C), and (D).

We will use the following notation: $N = N_G(P)$, $C = C_G(P) = W \times P$ where $o(W) = w$ and $(w, p) = 1$.

Let B be the principal p-block of G. Then by Proposition 2.1 of [3] B contains $t = (p^a - 1)/q$ exceptional characters X_λ of degree x_0, $\lambda = 1, \cdots, t$ and q nonexceptional characters X_i of degree x_i, $i = 1, \cdots, q$. If $\sigma \in P^\sharp$ and π is a p'-element of $C_G(\sigma) = C$ then:

$$(1) \qquad \begin{aligned} X_\lambda(\sigma\pi) &= -\varepsilon_0 \sum_{\rho \in R} \zeta_\lambda^\rho(\sigma) && \text{for } \lambda = 1, \cdots, t \\ X_j(\sigma\pi) &= \varepsilon_j && \text{for } j = 1, \cdots, q \end{aligned}$$

where R is a set of coset representatives of C in N, $\{\zeta_\lambda \mid \lambda = 1, \cdots, t\}$ is a set of representatives of the t transitivity classes of characters of P under conjugation by N (see [3], Lemma 2.2), and $\varepsilon_j = \pm 1$ for $j = 0, 1, \cdots, q$. It follows also by Corollary 2.1 of [3] that the following relations hold:

$$(2) \qquad \begin{aligned} x_i &\equiv \varepsilon_i \pmod{p^a} && \text{for } i = 1, \cdots, q \\ tx_0 &\equiv \varepsilon_0 \pmod{p^a} \end{aligned}$$

and

$$(3) \qquad \sum_{i=0}^{q} \varepsilon_i x_i = 0 .$$

We are now ready to state

LEMMA 3.1.

$$(4) \qquad tx_j \mid (p^a - 1)(1 + np^a) \qquad \text{for } j = 0, \cdots, q .$$

Proof. If $\sigma \in P^\sharp$, then $C = C_G(\sigma)$ and it is well-known that the expression

$$\frac{o(G) \cdot X_j(\sigma)}{o(C) \cdot x_j}$$

is an algebraic integer for all j. It follows, from Proposition 2.1 and (1), that for $j = 1, \cdots, q$

$$qwp^a(1 + np^a)/wp^a x_j$$

is an algebraic integer and consequently

$$tx_j \mid tq(1 + np^a) = (p^a - 1)(1 + np^a) .$$

For $j = 0$, it follows from (1), Proposition 2.1 and Lemma 2.2 of [3] that

$$\sum_{\lambda=1}^{t} \frac{o(G)X_\lambda(\sigma)}{o(C)x_0} = \frac{qwp^a(1 + np^a)\varepsilon_0}{wp^a x_0}$$

is an algebraic integer and therefore $tx_0 \mid (p^a - 1)(1 + np^a)$.

Since the block B contains 1_G as a nonexceptional character, we may assume that $X_1 = 1_G$. We have then

LEMMA 3.2. *For* $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j = \begin{cases} 1 + np^a & \text{if } \varepsilon_j = 1 \\ p^a - 1 & \text{if } \varepsilon_j = -1 \end{cases}$$

where $\bar{x}_j = x_j$ *for* $j = 2, \cdots, q$ *and* $\bar{x}_0 = tx_0$.

Proof. We will show first that if

$$up^a + \varepsilon \mid (p^a - 1)(1 + np^a) , \qquad \varepsilon = \pm 1$$

then either n satisfies statement (A) or one of the following relations holds:

$$\begin{array}{llll} up^a + \varepsilon = 1 & \text{or} \quad np^a + 1 & \text{if} & \varepsilon = 1 \\ up^a + \varepsilon = p^a - 1 & \text{or} \quad (p^a - 1)(np^a + 1) & \text{if} & \varepsilon = -1 . \end{array}$$

To do so, it suffices to show that if n does not satisfy (A) then the only solutions of

$$(5) \qquad (vp^a + 1)(wp^a - 1) = (p^a - 1)(1 + np^a)$$

in nonnegative integers v and w are: $v = 0$, $wp^a - 1 = (p^a - 1)(1 + np^a)$ and $v = n$, $w = 1$.

Suppose that $v \neq 0$ and $w > 1$. Then $vp^a + 1 < 1 + np^a$, $v < n$. By multiplying out equation (5), adding 1 to both sides and dividing by p^a we get

$$(6) \qquad wvp^a + w - v = np^a - n + 1 .$$

Now by (6):

$$\begin{aligned}(vp^a + 1)(n - wv) &= vp^a n - v(wvp^a) + n - wv \\ &= vp^a n + vw - v^2 - vnp^a + vn - v + n - wv \\ &= (n - v)(v + 1) .\end{aligned}$$

Since $n > v$, the left hand side of the equation is positive and so we may put $h = n - wv$, where h is a positive integer. Solving for n we get a contradiction to the assumption that n does not satisfy (A). Thus either $v = 0$ or $w = 1$ and the above assertion follows.

Now we have seen that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j \equiv \varepsilon_j \pmod{p^a} \text{ and } \bar{x}_j \mid (p^a - 1)(1 + np^a) .$$

Since X_1 is the only character of G of degree 1, it follows that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j = \begin{cases} 1 + np^a & \text{if } \varepsilon_j = 1 \\ p^a - 1 \quad \text{or} \quad (p^a - 1)(1 + np^a) & \text{if } \varepsilon_j = -1 . \end{cases}$$

Thus it suffices to show that for $j = 0, 2, 3, \cdots, q$

$$\bar{x}_j \neq (p^a - 1)(1 + np^a) .$$

Indeed, if the equality holds, then by (3):

$$0 = \sum_{i=0}^{q} \varepsilon_i x_i \leqq 1 + (q - 1)(1 + np^a) - (p^a - 1)(1 + np^a)/t = - np^a$$

a contradiction. The proof of Lemma 3.2 is complete.

We will proceed with the proof of Theorem 2. It follows from (3) that at least one of the ε_j's, $j = 0, 1, \cdots, q$, is negative. If $\varepsilon_0 = -1$, let $X = \sum_{\lambda=1}^{t} X_\lambda$ and if $\varepsilon_i = -1$ for some $i \geqq 2$ let $X = X_i$. In either case, by Lemma 3.2 X is a character of G of degree $p^a - 1$ and by (1) and Lemma 2.2 of [3]

$$X(\sigma\pi) = -1$$

for $\sigma \in P^\sharp$, $\pi \in W$, where $C = P \times W$. Denote the restriction of X to

C also by X; then X is a character of $P \times W$ and therefore for $\rho \in P$ and $\pi \in W$ we have:

$$(7) \qquad X(\rho\pi) = \sum_{i=1}^{r} \psi_i(\pi)\varphi_i(\rho)$$

where ψ_i, $i = 1, \cdots, r$ are distinct irreducible characters of W and φ_i, $i = 1, \cdots, r$ are characters of P. Let $\sigma \in P^\sharp$, $\pi \in W$; as $X(\sigma\pi) = -1$, it follows from (7) and from the linear independence of the irreducible characters of W, that the principal character appears among the ψ_i, say $\psi_1 = 1_W$, and

$$\varphi_1(\sigma) = -1, \ \varphi_2(\sigma) = \cdots = \varphi_r(\sigma) = 0 .$$

Suppose that $r > 1$. Then φ_2 vanishes on $P^\sharp$ and therefore p^a divides $\varphi_2(1)$, in contradiction to (7) and the fact that $X(1) = p^a - 1$. Thus $r = 1$ and

$$X(\rho\pi) = \varphi_1(\rho) \qquad \text{for all } \rho \in P, \ \pi \in W .$$

In particular $X(\pi) = \varphi_1(1) = X(1)$ for all $\pi \in W$. Let V denote the kernel of X; then V is a normal subgroup G and $W \subset V$. Suppose that $W \neq \{1\}$. Then it follows from the assumption that G has no nontrivial subgroups of order prime to p and from Proposition 2.2.i that $P \subset V$, in contradiction to the fact that $X(\sigma) = -1$ for $\sigma \in P^\sharp$. Consequently $W = \{1\}$, P contains the centralizer of each of its nonunit elements and by Theorem 2 of [2] G is either of type (B), or of type C, or $G \cong PSL(2, p^a - 1)$, where $a > 1$ and $p^a - 1 = 2^b$. In view of Lemma 3.1 of [3], the only solution of the above equation with $a > 1$ is: $p = 3$, $a = 2$ and $b = 3$. Thus if $a > 1$, $G \cong PSL(2, 8)$. Since the groups of types (B), (C) and (D) satisfy the conditions of Theorem 2, the proof is complete.

4. **Proof of Theorem 1*.** It follows from Theorem 2 that one of the statements (B), (C) and (D) holds. Statement (A) could not occur, since then

$$n \geqq (p^a + 3)/2 , \qquad o(G) \geqq 2p^a(p^a + 3)p^a/2 > p^{3a}$$

a contradiction.

In cases (C) and (D) $o(G^*) > p^{3a}/2$ and therefore $G \cong G^*$, yielding statements (II)* or (IV)*. In case (B), $o(G^*) = (p^{3a} - p^a)/2$ and therefore either $[G : G_0] = 2$, $G_0 \cong G^*$, or $G = G_0$, $o(M) = 1$ or 2. If $[G : G_0] = 2$, then $o(M) = 1$, G is isomorphic to a subgroup of the automorphism group of $PSL(2, \mathrm{p})$ and by [5, Lemma 2] $G \cong PGL(2, p)$, yielding $(V)^*$. If $G = G_0$ and $o(M) = 1$, then $G \cong PSL(2, p)$, $p > 3$, and $(I)^*$ holds. Finally, if $G = G_0$ and $o(M) = 2$, then it follows from a theorem of

Schur [4, p. 120] that G is either isomorphic to $SL(2, p)$ and (III)* holds, or it is isomorphic to $PSL(2, p) \times M$ and (VI)* holds. Since the groups mentioned in Theorem 1* satisfy the conditions of that theorem, the proof is complete.

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UNIVERSITY OF CALIFORNIA, SANTA BARBARA

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EXTENSIONS OF PSEUDO-VALUATIONS

JAMES A. HUCKABA

Let w be a pseudo-valuation defined on a commutative ring R and let S be an overring of R. This paper investigates conditions needed to imply that w can be extended to S. These conditions are given in terms of a particular sequence of ideals $\{A_i\}_{i=0}^{\infty}$ in R which is called the best filtration for w. The main theorem states that if w is a pseudo-valuation on R with best filtration $\{A_i\}$ and each A_i is a contracted ideal with respect to S, then w can be extended to S. The converse of this result is then proved.

By using our main theorem and some recent results by Gilmer [1], we show in several important cases that if S is an overring of R and w is any pseudo-valuation on R possessing a best filtration, then w can be extended to S. In particular, if R is a Prüfer domain with quotient field K and if S is an overring of R such that $S \cap K = R$, then w can be extended from R to S.

We begin in §1 by defining and developing properties of a best filtration and determining classes of pseudo-valuations which have best filtrations. The main results and applications are then proved in §2.

1. Filtrations. All rings are commutative, associative, and have identity. If S is an overring of R, we assume that S and R have the same identity. A *pseudo-valuation* on the ring R is a mapping w from R into the extended real number system such that:

(i) $w(0) = \infty$, $w(1) = 0$,

(ii) $w(x - y) \geqq \min\{w(x), w(y)\}$,

(iii) $w(xy) \geqq w(x) + w(y)$, for each $x, y \in R$.

w is called a *homogeneous pseudo-valuation* in case:

(iv) $w(x^2) = 2w(x)$ for each $x \in R$.

w is called a *valuation* in case:

(v) $w(xy) = w(x) + w(y)$ for each $x, y \in R$.

Pseudo-valuations were first introduced by Rees [3]. Rees proved in [3] that (iv) is equivalent to the condition that $w(x^n) = nw(x)$ for each positive integer n and for each $x \in R$. These functions arise quite naturally in ring theory. If A is a proper ideal of R, define $v_A(x) = n$ if $x \in A^n$, $x \notin A^{n+1}$ and $v_A(x) = \infty$ if $x \in A^n$ for all n. Then v_A is a pseudo-valuation. We say that v_A is *associated with the ideal* A. A sequence of ideals $\{A_i\}_{i=0}^{\infty}$ of R such that $A_0 = R$, $A_{i+1} \subset A_i$, and $A_iA_j \subset A_{i+j}$ for all i and j is called a *filtration* on R. Notice that the nonnegative integral powers of an ideal A of R forms a filtration,

where A° is defined to be R. Also note that any filtration $\{A_i\}$ determines a pseudo-valuation in exactly the same manner that the powers of an ideal A determines v_A. For an arbitrary pseudo-valuation w on R and a subset T of R, define $w(T) = \inf\{w(t): t \in T\}$.

DEFINITION 1. If w is a pseudo-valuation on R, define

$$\begin{aligned} A_0 &= R \\ A_i &= \{x \in R: w(x) > w(A_{i-1})\}\ , \\ &\qquad \text{if } w(A_{i-1}) < \infty\ . \\ A_i &= A_{i-1}, \text{ if } w(A_{i-1}) = \infty\ . \end{aligned} \tag{1.1}$$

Each member of the sequence $\{A_i\}$ is an ideal of R. The sequence defined by (1.1) has the property that $A_0 \supset A_1 \supset A_2 \supset \cdots$. A_{i+1} is not necessarily a proper subset of A_i, as will be shown in Examples 1 and 2. Also note that for a given pseudo-valuation w, $\{x \in R: w(x) = \infty\} \subset \cap A_i$. The following example shows that there exists pseudo-valuations such that the sequence defined by (1.1) is not a filtration.

EXAMPLE 1. Let $\mathfrak{A}$ be an ideal of a ring R in which $\mathfrak{A}^i \supsetneqq \mathfrak{A}^{i+1}$ for all i. Define a sequence of ideals $\{B_i\}$ as follows: $B_0 = R$, $B_1 = \mathfrak{A}^2$, $B_2 = \mathfrak{A}^3$, $B_3 = B_4 = \mathfrak{A}^5$, and $B_i = \mathfrak{A}^7 (i \geqq 5)$. Then $\{B_i\}$ is a filtration in R and determines some pseudo-valuation w, where $w(x) = n$ if $x \in \mathfrak{s}B_n$, $x \notin B_{n+1}$ and $w(x) = \infty$ if $x \in \cap B_n$. Now use Definition 1 to define A_i with respect to w. We obtain $A_i = B_i (i = 0, 1, 2, 3)$ and $A_i = B_{i+1} (i = 4, 5, \cdots)$. But $\{A_i\}$ is not a filtration, since $(A_2)^2 \not\subset A_4$.

DEFINITION 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be defined by (1.1). If $\{A_i\}$ is a filtration in R such that $x \in \cap A_i$ if and only if $w(x) = \infty$, then $\{A_i\}$ is called a *best filtration* for w. Let $B(R)$ denote the class of all pseudo-valuations on R which have a best filtration.

Example 1 then implies that there are pseudo-valuations which do not have best filtrations. It is clear from the definition that if w has a best filtration, then it is unique. From now on we will talk about *the* best filtration for w.

EXAMPLE 2. Let w be a pseudo-valuation on R and let $\{A_i\}$ be the sequence defined by (1.1). It is possible for $\{A_i\}$ to be a filtration in R, yet not be the best filtration for w. Let v be a real valued nondiscrete valuation on a field K and consider v as a pseudo-valuation on its valuation ring R_v. Since the value group of v has no smallest positive element, $v(A_1) = 0$. Then $A_2 = \{x \in R: v(x) > v(A_1) = 0\} = A_1$.

By induction, we see that $A_i = A_1$ for each $i \geqq 1$. Hence the sequence $\{A_i\}$ is such that $A_0 \supsetneqq A_1 = A_2 = \cdots$. Therefore $v \notin B(R)$. However, it is clear that $\{A_i\}$ is a filtration.

We will be interested only when the sequence defined by (1.1) is a filtration. This always happens in one important case.

REMARK 1. If v is a valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by (1.1), then $\{A_i\}$ is a filtration in R.

Proof. It is clear that $A_i \supset A_{i+1}$ for each i. Hence, to complete the proof we need to show that $A_iA_j \subset A_{i+j}$ for all nonnegative integers i and j. We fix j and use induction on i. Clearly $A_0A_j \subset A_{0+j}$. Assume that $A_{i-1}A_j \subset A_{i+j-1}$ for $i \geqq 1$. Let $x \in A_iA_j$, then $x = \sum_{k=1}^{n} a_kb_k$ where $a_k \in A_i$ and $b_k \in A_j$. We may assume without loss of generality that $v(a_1) + v(b_1) = \min_{k=1}^{n} (v(a_k) + v(b_k))$. Then $v(x) \geqq v(a_1) + v(b_1)$. Case 1: If $v(A_{i-1}) < \infty$, then $v(a_1) > v(A_{i-1})$, and thus $v(x) > v(A_{i-1}) + v(A_j) = v(A_{i-1}A_j) \geqq v(A_{i+j-1})$. By Definition 1, $x \in A_{i+j}$. Case 2: If $v(A_{i-1}) = \infty$, then $v(x) = \infty$, which implies that $x \in A_{i+j}$. Therefore $A_iA_j \subset A_{i+j}$.

LEMMA 1. *Let $w \in B(R)$ and let $\{A_i\}$ be the best filtration for w. Then*:

(1) *$A_i = A_{i+1}$ if and only if $w(A_i) = \infty$.*

(2) *Let $x \in A_i$ and $x \notin A_{i+1}$. Then $y \in A_i$ and $y \notin A_{i+1}$ if and only if $w(x) = w(y)$. In fact, $w(x) = w(A_i)$.*

(3) *If $y \in A_i$ and $z \notin A_i$, then $w(y) > w(z)$.*

(4) *If $w(x) < \infty$, then there exists an integer i such that $x \in A_i$ and $x \notin A_{i+1}$.*

Proof. (1) Suppose $A_i = A_{i+1}$. By induction we see that $A_i = A_{i+t}$ for each positive integer t. If $w(A_i) < \infty$, then there is an element $x \in A_i$ such that $w(x) < \infty$. But $x \in \cap A_i$ which implies that $w(x) = \infty$, a contradiction. Conversely, if $w(A_i) = \infty$, then $A_i = A_{i+1}$ by definition of the best filtration.

(2) First note that $x \in A_i$, $x \notin A_{i+1}$ implies that $w(x) = w(A_i)$. If $y \in A_i$, $y \notin A_{i+1}$, then clearly $w(x) = w(y)$. Conversely, assume $w(x) = w(y)$. If $i = 0$, then $w(y) \leqq w(A_1)$ and hence y is in A_0, but not in A_1. If $i > 0$, then $w(A_{i-1}) \leqq w(A_i)$. If equality holds, then $A_i = A_{i+1}$, which implies that $x \in A_{i+1}$. Therefore $w(A_{i-1}) < w(A_i)$, which implies that $y \in A_i$. Also $y \notin A_{i+1}$, for if so, then $w(y) > w(A_i)$.

(3) and (4) are clear.

The converse of the above result is also true.

LEMMA 2. *Let w be a pseudo-valuation on R and let $\{B_i\}$ be a filtration in R satisfying properties* (1)-(4). *Then $\{B_i\}$ is the best filtration for w.*

Proof. Clearly $x \in \cap B_i$ if and only if $w(x) = \infty$. Suppose that $w(B_{i-1}) < \infty$. By properties (2) and (3) $B_i = \{x \in R: w(x) \geqq w(B_i)\}$. Thus $B_i \subset \{x \in R: w(x) > w(B_{i-1})\}$. On the other hand, suppose that $w(x) > w(B_{i-1})$. If $w(x) = \infty$, then $x \in \cap B_j$ and hence $x \in B_i$. If $w(x) < \infty$, choose k such that $x \in B_k$ and $x \notin B_{k+1}$. Suppose that $k \leqq i - 1$, then $B_k \supset B_{i-1}$, so $w(x) = w(B_k) \leqq w(B_{i-1})$, a contradiction. So we must have $k > i - 1$ and hence $x \in B_i$. Therefore $B_i = \{x \in R: w(x) > w(B_{i-1})\}$.

By (1), if $w(B_{i-1}) = \infty$, then $B_{i-1} = B_i$.

We assume from now on that all pseudo-valuations w which are considered have the property that there exists at least one x such that $0 < w(x) < \infty$.

LEMMA 3. (a) *If w is a homogeneous pseudo-valuation on R and if $\{A_i\}$ is the sequence of ideals defined by* (1.1), *then $w(A_i) < \infty$ for each i.*

(b) *If w is a pseudo-valuation on a ring R and if $\{A_i\}$ is the sequence of ideals defined by* (1.1) *such that each A_i is finitely generated, then $w(A_{i-1}) < \infty$ implies that $w(A_i) > w(A_{i-1})$.*

Proof. (a) Suppose, to the contrary, that i is the smallest positive integer such that $w(A_i) = \infty$. Since w is nontrivial, $i \geqq 2$. Choose $x \in A_{i-1}$, $x \notin A_i$. Then $0 < w(x) < \infty$, and $w(x^2) > w(x) \geqq w(A_{i-1})$, so $x^2 \in A_i$. But, $w(A_i) \leqq w(x^2) = 2w(x) < \infty$, contradicting the assumption that $w(A_i) = \infty$.

(b) Let $a_1, \cdots, a_n$ be a basis for A_i. Choose a_k such that $w(a_k) = \min\{w(a_1), \cdots, w(a_n)\}$. Then $w(A_i) = w(a_k)$. Since $a_k \in A_i$, $w(a_k) > w(A_{i-1})$ and therefore $w(A_i) > w(A_{i-1})$.

The following theorem shows that there are many pseudo-valuations with best filtrations.

THEOREM 1. (1) *Any pseudo-valuation associated with an ideal is in $B(R)$. More generally, any pseudo-valuation determined by a filtration $\{B_i\}$, where $B_i = B_{i+1}$ implies that $B_i = B_{i+k}$ for each positive integer k, is in $B(R)$.*

(2) *If the sequence $\{A_i\}$ of ideals defined by* (1.1) *is a filtration and if $\lim_{i\to\infty} w(A_i) = \infty$, then $w \in B(R)$. Both of these conditions are satisfied if w is a valuation on R and R is noetherian.*

(3) *A pseudo-valuation w on R such that the range of w is equal to the set of all multiples of some positive real number $t > 0$ is in $B(R)$. This includes all integrally valued homogeneous pseudo-valuations w such that there is an $x \in R$ for which $w(x) = 1$.*

(4) *All integrally valued pseudo-valuations and pseudo-valuations on a noetherian ring such that* (1.1) *forms a filtrations are in $B(R)$.*

Proof. (1) Clear.

(2) Let w and $\{A_i\}$ satisfy the hypothesis of the first statement of (2). Clearly $w(x) = \infty$ implies that $x \in \cap A_i$. Let $x \in \cap A_i$, then $w(x) \geqq w(A_{i-1})$ for each i. Since $\lim_{i\to\infty} w(A_i) = \infty$, $w(x) = \infty$.

We will now prove the second statement of (2). Let v be a valuation on a noetherian ring R. By Remark 1, the sequence of ideals $\{A_i\}$ defined by (1.1) is a filtration in R. We need to show that $\lim_{i\to\infty} v(A_i) = \infty$. Consider a basis $\{y_1, \cdots, y_r\}$ for the ideal A_1. Let $v(y_1) = \min\{v(y_1), \cdots, v(y_r)\}$. Then y_1 is an element of R with the property that $v(y_1) = \varepsilon$ is a minimal positive element in the range of v. Assume that $\lim_{i\to\infty} v(A_i) = t < \infty$. By Lemma 3 (b), $v(A_i) > v(A_{i-1})$ for each i. Thus we can choose a sequence $\{x_j\} \in R$ so that $v(x_j) = a_j$ where $(t-\varepsilon) < a_1 < a_2 < \cdots$, and each $a_j < t$. Let B be the ideal generated by $\{x_j\}$. Since R is noetherian there exists a positive integer n so that $\{x_1, \cdots, x_n\}$ is a basis of B. Let $p > n$, then $x_p \in B$ and so $x_p = \sum_{i=1}^{n} \alpha_i x_i$, $\alpha_i \in R$. Then $v(x_p) \geqq \min\{v(\alpha_1 x_1), \cdots, v(\alpha_n x_n)\}$. Let $v(\alpha_j x_j)$ be this minimum. Case 1: If $v(\alpha_j) \neq 0$, then $v(x_p) \geqq v(\alpha_j) + a_j \geqq \varepsilon + a_j \geqq t$, which is a contradiction. Case 2: If $v(\alpha_j) = 0$, then $v(x_p) = a_p > a_j = v(x_j) = v(\alpha_j x_j)$. By properties of a valuation, $v(\alpha_j x_j) = v(\alpha_k x_k)$ for some $k \leqq n$, $k \neq j$. Since $v(x_j) \neq v(x_k)$, $v(\alpha_k) \neq 0$. Hence, $v(x_p) \geqq v(\alpha_k) + a_k \geqq t$, a contradiction. This proves that $\lim_{i\to\infty} v(A_i) = \infty$.

(3) Define $B_0 = R$ and inductively, $B_i = \{x \in R: w(x) \geqq i \cdot t\}$. The sequence $\{B_i\}$ satisfies the hypothesis of Lemma 2 and is a best filtration for w.

(4) The first part is clear. For the second part use the same technique as in (2).

2. **The main results.** The following notation will be used in this section. Let S be an overring of R. If A is an ideal of R, then the extension of A to S, $A \cdot S$, will be denoted by A^e. If B is an ideal of S, then the contraction of B to R, $B \cap R$, will be denoted by B^c.

THEOREM 2. *Suppose that S is an overring of R, $w_0 \in B(R)$, and $\{A_i\}$ is the best filtration for w_0. If each A_i is a contracted ideal with respect to S, then w_0 can be extended to S.*

Proof. Define $B_i = A_i^e$ for each i. Then $\{B_i\}$ is a filtration on S.

Define a mapping w on S as follows: $w(x) = w_0(A_i)$ if $x \in B_i$, $x \notin B_{i+1}$ and $w(x) = \infty$ if $x \in \cap B_i$. We will show that w is a pseudo-valuation on S which extends w_0 to S. Property (i) of the definition of pseudo-valuation is obviously satisfied. Suppose that $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Without loss of generality, assume that $i \leqq j$. Then $x - y \in B_i$ and hence, $w(x - y) \geqq w_0(A_i) = \min\{w_0(A_i), w_0(A_j)\} = \min\{w(x), w(y)\}$. Similarly if either $x \in B_i$ for all i or $y \in B_j$ for all j, then $w(x - y) \geqq \min\{w(x), w(y)\}$. This proves property (ii).

Finally, we wish to show $w(xy) \geqq w(x) + w(y)$. Again let $x \in B_i$, $x \notin B_{i+1}$ and $y \in B_j$, $y \notin B_{j+1}$. Then $xy \in B_iB_j \subset B_{i+j}$, so that

$$w(xy) \geqq w_0(A_{i+j}) .$$

If $w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j})$, then $w(xy) \geqq w(x) + w(y)$. On the other hand, if $w_0(A_i) + w_0(A_j) > w_0(A_{i+j})$, there are two cases to consider. Case 1: Suppose there is a positive integer t such that

$$w_0(A_i) + w_0(A_j) \leqq w_0(A_{i+j+t}) ,$$

but $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t-1})$. Then $A_iA_j \subset A_{i+j+t}$, and hence $B_iB_j \subset B_{i+j+t}$. Since $xy \in B_iB_j \subset B_{i+j+t}$, $w(xy) \geqq w(x) + w(y)$. Case 2: Suppose that $w_0(A_i) + w_0(A_j) > w_0(A_{i+j+t})$ for all t. Then $w_0(A_iA_j) > w_0(A_k)$ for all k, which implies that $A_iA_j \subset A_k$ for all k. Hence,

$$xy \in A_i^eA_j^e \subset (\bigcap_{k=1}^{\infty} A_k)^e \subset \bigcap_{k=1}^{\infty} (A_k^e) = \bigcap_{k=1}^{\infty} B_k .$$

Therefore $w(xy) = \infty$ and $w(xy) \geqq w(x) + w(y)$. When either $x \in \cap B_k$ or $y \in \cap B_k$, clearly $w(xy) = w(x) + w(y) = \infty$. This proves property (iii), showing that w is a pseudo-valuation on S.

It is easy to see that w extends w_0. Take $z \in R$. If $z \in A_i$, $z \notin A_{i+1}$ then by Lemma 1, $w_0(z) = w_0(A_i)$. Clearly $z \in B_i$. Suppose $z \in B_{i+1}$, since z is also in R, $z \in {A_{i+1}}^{ec} = A_{i+1}$ a contradiction. Therefore $z \notin B_{i+1}$, and hence $w(z) = w_0(A_i)$. If $z \in \cap A_i$, then $z \in \cap B_i$ which implies that $w_0(z) = w(z) = \infty$.

A subring R of a ring S is said to have *property C* with respect to S in case each ideal of R is a contraction of an ideal in S. In [1], Gilmer shows that in several cases, if S is an overring of R which is integrally dependent on R, then R has property C with respect to S. Using Gilmer's theory we obtain several applications of Theorem 2, which are listed in the corollaries below. A *Prüfer domain* is a domain R with identity in which each finitely generated ideal is invertible, or equivalently, in which R_P is a valuation ring for each prime ideal P in R. An ideal A of a ring R is called a *valuation ideal* in case there exists a valuation ring R_v containing R and an ideal B of R_v such that $B \cap R = A$.

COROLLARY 1. *Suppose that R is a Prüfer domain with quotient field K and that R is a subdomain of R_1. If $R_1 \cap K = R$, then every $w \in B(R)$ can be extended to R_1.*

Proof. By [1; p. 563, Corollary 2], R has property C with respect to R_1. Then each ideal in a best filtration for w is a contracted ideal with respect to R_1. By Theorem 2, w can be extended to R_1.

COROLLARY 2. *Let R be a domain, let $w \in B(R)$, suppose that R_1 is integral over R, and let $\{A_i\}$ be the best filtration for w. If each A_i is an intersection of valuation ideals of R, then w can be extended to R_1.*

Proof. Apply [1; p. 564, Th. 2] and Theorem 2.

It is known that if R is an integrally closed domain, A is a complete ideal in R if and only if A is the intersection of valuation ideals. Now let R be an integrally closed domain with quotient field K, L a finite algebraic extension of K, and R' the integral closure of R in L. By [1; p. 569, Th. 6] and Theorem 2, we have:

COROLLARY 3. *If R' has an integral basis over R and if $w \in B(R)$, then w can be extended to R'.*

THEOREM 3. *Suppose that R is a subring of the ring S and suppose that w_0 is a pseudo-valuation on R which has an extension to a pseudo-valuation w on S. If α belongs to the set of extended reals, then the ideals $A_\alpha = \{x \in R: w_0(x) > \alpha\}$ and $B_\alpha = \{x \in R: w_0(x) \geqq \alpha\}$ are contractions of ideals of S.*

Proof. A_α is the contraction of $A'_\alpha = \{x \in S: w(x) > \alpha\}$ and B_α is the contraction of $B'_\alpha = \{x \in S: w(x) \geqq \alpha\}$.

The converse of Theorem 2 is also true.

THEOREM 4. *Let S be an overring of R, let $w_0 \in B(R)$, and let $\{A_i\}$ be the best filtration for w_0. If w_0 can be extended to S, then each A_i is a contracted ideal with respect to S.*

Proof. Apply Theorem 3.

COROLLARY 4. *Suppose $w_0 \in B(R)$ can be extended to some w on R_v, where R_v is a valuation ring. Then each ideal in the best filtration of w_0 is a valuation ideal.*

REMARK 2. Let R be a domain with quotient field K and w_0 a

nonnegative pseudo-valuation on R. (Nonnegative pseudo-valuations were the most important types of pseudo-valuations studied in [2] and [3]). w_0 can always be extended to a nonnegative pseudo-valuation w on a subring R', where $R \subset R' \subset K$, in the following way. Let M be the set of $y \in R$ such that $w_0(y) < \infty$ and $w_0(xy) = w_0(x) + w_0(y)$ for all $x \in R$. Then M is a multiplicative subset of R not containing zero. Hence we can form the quotient ring of R with respect to M, R_M. A function w' can be defined on R_M by $w'(x/y) = w_0(x) - w_0(y)$. w' is not necessarily nonnegative. However, if $R' = \{z \in R_M : w'(z) \geqq 0\}$ and w is the restriction of w' to R', then R' is a ring and w is an extension of w_0 to R'. R' is called the *natural domain* of w_0. This type of extension was discussed and used in [2].

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University of Missouri
Columbia, Missouri

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SEMIVARIETIES AND SUBFUNCTORS OF THE IDENTITY FUNCTOR

S. A. Huq

We study certain subcategories called semivarieties and obtain Kaplanasky's theorem on the decomposition of Abelian groups into a divisible group and a reduced group under the frame-work of category theory; We also investigate the connection of these epicoreflective subcategories with varieties.

Semivarieties are subcategories of a category with cartain axioms; such subcategories play an important role in Abelian categories and a description of these by means of coreflection, appears in the general context in the work of Mitchell [8, § 5, § 6, III]. Broader classes than these have also been studied by Amitsur [1], Carreau [2], under the title of $HI - RI$ properties of radicals and their classes. Our aim in this note is to give a categorical proof of Kaplanasky's Theorem 3 [6, § 5] and while so doing we generalize the concepts of varieties and variety functors of Fröhlich [4] under abstract frame work utilizing Maranda's [9] concept of a radical.

$\mathscr{C}$ is a category equipped with the following axioms:

I. $\mathscr{C}$ has a null object.

II. Every morphism α in $\mathscr{C}$, admits a factorization $\alpha = \nu\mu$, where ν is a normal epimorphism and μ is a monomorphism; we are writing the composition in the precise way $\cdot \overset{\alpha}{\longrightarrow} \cdot \overset{\beta}{\longrightarrow} \cdot$ where the dots are the unnamed objects.

III. Every family of objects has a direct and a free product.

IV. The subobjects and factor objects of any objects form a set.

It is immediate that $\mathscr{C}$ admits null morphisms. The image of a morphism in $\mathscr{C}$, defined in axiom II and some time written as (ν, L, μ) in the factorization $\cdot \overset{\nu}{\longrightarrow} L \overset{\mu}{\longrightarrow} \cdot$, is uniquely determined to within equivalence. Every family of subobjects of an object A in $\mathscr{C}$ has a union and as such every morphism has a kernel. Dual consideration holds for factor objects and cokernel. A map admitting null subobject as the kernel is a monomorphism and a sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact if and only if α is a monomorphism, β is a normal epimorphism and the subobject (A, α) serves as the kernel of β. For details on the notation used and results mentioned in this section the reader

is referred to [7], [10]. In view of axiom II, one can check that every retraction is a normal epimorphism as such a monomorphism which is a retraction is invertible, [7] and further the composition of two normal epimorphisms is again normal. Also if μ is a normal monomorphism admitting cokernel ν, then μ is the kernel of ν and dual result holds for the normal epimorphism ν, admitting kernel μ.

2. Subfunctors of I. We shall consider covariant functors from $\mathscr{C} \to \mathscr{C}$, and such subfunctors of the identity functor I.

DEFINITION. A subfunctor (F, μ) of I will be called a *normal subfunctor* if for each $A \in \mathscr{C}$, the natural map $\mu_A : F(A) \to A$ is a normal monomorphism; and (F, μ) is called *injective* if μ_A is a coretraction.

A subfunctor (F, μ) of $\mathscr{C}$ is said to be a *radical subfunctor*, (dually *coradical subfunctor*) in the sense of Maranda [9] if for any A, $(F(A), \mu_A)$ has cokernel (ε, C) then $1_{F(C)} \sim \omega[\omega = \omega_{AB}$ always stands for the null morphism of $H(A, B)]$, so that $F(\varepsilon) : F(A) \to F(C)$ is the null normal epimorphism $\omega_{F(A),0}$ (dually for the inclusion $\mu_A : F(A) \to A$; $F(\mu_A) : FF(A) \to F(A)$ is an equivalence).

3. Semivarieties and cosemivarieties. A *semivariety* $\mathscr{B}$ is a full subcategory of $\mathscr{C}$, satisfying the following axioms:

I. $f : A \to B$ is a monomorphism and $B \in B \Rightarrow A \in B$.

II. If $(A_i)_{i \in I}$ is an indexed set of objects of $\mathscr{B}$, then their direct product $\prod A_i \in \mathscr{B}$.

EXAMPLES. (i) All varieties [4].

(ii) Torsion free groups (respectively Abelian groups) form a semivariety of all groups (respectively of Abelian groups)

(iii) Reduced groups form a semivariety of all Abelian groups.

(iv) Nilpotent free rings (i.e., a ring in which $a^2 = 0 \Rightarrow a = 0$) in the category of rings.

PROPOSITION 3.1. (1) *If (V, μ) is a subfunctor of I, then the objects A for which $(V(A), \mu_A) \sim (0, \omega)$ form a semivariety $\mathscr{B}_V$.*

(2) *If $\mathscr{B}$ is a semivariety then every object A has a minimal normal subobject $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ whose cokernel lies in $\mathscr{B}$. These $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ determine a normal radical subfunctor $(V_{\mathscr{B}}, \mu_{\mathscr{B}})$ of I.*

(3) *If $V \sim V_{\mathscr{B}}$ then $\mathscr{B} = \mathscr{B}_V$. Conversely $\mathscr{B} = \mathscr{B}_V$ implies V is a subfunctor of $V_{\mathscr{B}}$; but if (V, μ) is normal and a radical subfunctor then $\mathscr{B} = \mathscr{B}_V \Rightarrow V \sim V_{\mathscr{B}}$.*

Proof. (1) Let $\mathscr{B}_V = \{A \mid (V(A), \mu_A) \sim (0, \omega)\}$ and all their mor-

phisms. Since (V, μ) is a subfunctor of I, $0 \in \mathscr{B}_V$. Thus $\mathscr{B}_V$ is a full subcategory.

Next if $A \xrightarrow{f} B$ is a monomorphism, then $V(f): V(A) \to V(B)$ is a monomorphism. Thus $B \in \mathscr{B}_V \Rightarrow V(B) \sim 0$ and as such $1_{V(A)} = \omega$ which implies $A \in \mathscr{B}_V$.

Next if $(A_i)_{i \in I}$ is a family of objects of $\mathscr{B}_V$, then we have a commutative diagram

$$\begin{array}{ccc} \prod A_i & \xrightarrow{\pi_i} & A_i \\ {\scriptstyle \mu_{\prod A_i}}\uparrow & & \uparrow{\scriptstyle \mu_{A_i}} \\ V(\prod A_i) & \xrightarrow[V(\mu_i)]{} & V(A_i) \end{array}$$

where the top horizontal is the natural projection for each i which is a normal epimorphism. Now $A_i \in \mathscr{B}_V \Rightarrow (V(A_i), \mu_{A_i}) \sim (0, \omega)$ that is $\mu_{\prod A_i}\pi_i = \omega$, i.e., $\mu_{\prod A_i} = \omega$ so $1_{V(\prod A_i)} = \omega$ giving $\prod A_i \in \mathscr{B}_V$.

(2) Suppose $\mathscr{B}$ is a semivariety. We consider all normal subobjects $(V_i(A), \mu_i(A))$ of A whose cokernel (ε_i, C_i) lies in $\mathscr{B}$. Then $\varepsilon_i: A \to C_i$ determine a unique map $\varepsilon: A \to \prod C_i$ such that $\varepsilon\pi_i = \varepsilon_i$ (where $\prod C_i \xrightarrow{\pi i} C_i$, stands for the direct product). Now $\prod C_i \in \mathscr{B}$. We denote by $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ the kernel of ε; then this determines a normal subfunctor $(V_{\mathscr{B}}, \mu_{\mathscr{B}})$ of I. We note $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ is also the kernel of the epimorphism ν, in the image (ν, C, μ) of ε. Then $(V_{\mathscr{B}}(A), \mu_{\mathscr{B}}(A))$ is minimal normal subobject (in the sense of partial ordering of subobjects, see [7]) whose cokernel lies in $\mathscr{B}$ is obvious from the construction. Evidently $V_{\mathscr{B}}$ has the radical property.

(3) Suppose $\mathscr{B}$ is a semivariety and $V \sim V_{\mathscr{B}}$, then $\mathscr{B}_V$ consists of all objects A for which $(V(A), \mu_A) \sim (0, \omega)$. Now $\mathscr{B}$ consists of all objects A whose cokernel is $(1_A, A)$; thus $\mathscr{B} = \mathscr{B}_V$.

Now suppose $\mathscr{B} = \mathscr{B}_V$. If for any object A, $V(A) \xrightarrow{\mu_A} A$ and $V_{\mathscr{B}}(A) \xrightarrow{\mu_{\mathscr{B}}} A$ admit the cokernels (ε^*, D^*) and (ε, C) respectively then $C \in \mathscr{B} = \mathscr{B}_V$, i.e., $(V(C), \mu_C) \sim (0, \omega)$ so $\mu_A\varepsilon = \omega$, therefore there exists a monomorphism $\lambda_A: V(A) \to V_{\mathscr{B}}(A)$ such that $\lambda_A\mu_{\mathscr{B}} = \mu_A$. That this λ is a functor monomorphism is easy to check. If further V is normal and a radical subfunctor, then for the cokernel (ε^*, D^*), $1_{V(D^*)} \sim \omega$, i.e., $D^* \in \mathscr{B}_V = \mathscr{B}$. Thus $(\varepsilon^*, D^*) \leqq (\varepsilon, C)$, i.e., $\varepsilon^* \sim \varepsilon$, giving $(V, \mu) \sim (V_{\mathscr{B}}, \mu_{\mathscr{B}})$.

Dual to the concept of a semivariety is that of a cosemivariety.

DEFINITION. A full subcategory $\mathscr{T}$ of C is called a *cosemivariety* if

I. $f: A \to Q$ a normal[1] epimorphism $A \in \mathscr{T}, \Rightarrow Q \in \mathscr{T}$.

[1] This is required since axiom II, is not self dual.

II. $(A_i)_{i\in I}$ is a family of objects of $\mathscr{F}$, then their free product $\Pi^* A_i \in \mathscr{F}$.

EXAMPLES. (i) All covarieties. In particular in the category of Abelian groups, torsion groups from a covariety.

(ii) Divisible groups form a cosemivariety of all Abelian groups. Mirror image of Proposition 3.1. will be

PROPOSITION 3.1*. (1) *If F is subfunctor of the identity functor I of $\mathscr{C}$, then the collection of all objects A for whjch $(F(A), \mu_A) \sim (A, 1_A)$ from a cosemivariety $\mathscr{F}^F$.*

(2) *If $\mathscr{F}$ is a cosemivariety, then every object A has a maximal subobject $(F^{\mathscr{F}}(A), \mu^{\mathscr{F}}(A) \in \mathscr{F}$. This $(F^{\mathscr{F}}, \mu^{\mathscr{F}})$ is a coradical subfunctor of I.*

(3) *$F \sim F^{\mathscr{F}} \Rightarrow \mathscr{F} = \mathscr{F}^F$. Conversely $\mathscr{F} = \mathscr{F}^F$ implies $F^{\mathscr{F}}$ is a subfunctor of F; further if F is a coradical subfunctor then $\mathscr{F} = \mathscr{F}^F \Rightarrow F \sim F^{\mathscr{F}}$.*

We exhibit the proof for (1), leaving the second and third part for the reader to dualize their counter parts in Proposition 3.1 in the same way.

Let $\mathscr{F}^F = \{A \mid (F(A), \mu_A) \sim (A, 1_A)\}$ and all their morphisms; the null object is obviously $\in \mathscr{F}^F$.

Next if $A \xrightarrow{\alpha} B$ is a normal epimorphism and $A \in \mathscr{F}^F$, then we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle \mu_A}\uparrow & & \uparrow{\scriptstyle \mu_B} \\ F(A) & \xrightarrow[F(\alpha)]{} & F(B) \end{array}$$

in which μ_A is an equivalence. Now $\mu_A\alpha$ is a normal epimorphism as observed in § 1. Let δ be its kernel. Then $\delta F(\alpha) = \omega$, i.e., $F(\alpha) = \mu_A\alpha\, \acute{\mu}(\mu_A\alpha$ is normal and therefore cokernal of $\delta)$. Now $\mu_A\alpha\acute{\mu}\mu_B = F(\alpha)\mu_B = \mu_A\alpha$. Thus $\acute{\mu}\mu_B = 1$, i.e., μ_B is a retraction and hence invertible, i.e., $B \in \mathscr{F}^F$.

Next suppose $(A_i)_{i\in I}$ be a family of objects of $\mathscr{C}$ belonging to $\mathscr{F}$, then we have for each i, a commutative diagram

$$\begin{array}{ccc} A_i & \xrightarrow{\tau_i} & \Pi^* A_i \\ {\scriptstyle \mu_{A_i}}\uparrow & & \uparrow{\scriptstyle \mu_{\Pi^* A_i}} \\ F(A_i) & \xrightarrow[F(\tau_i)]{} & F(\Pi^* A_i) \end{array}$$

where τ_i are the natural monomorphisms associated with the free

product. Now μ_{A_i} is invertible, let $\mu_{A_i}^{-1}$ be its inverse. Then $\mu_{A_i}^{-1}F(\tau_i) : A_i \to F(\prod^* A_i)$ determines a unique $\lambda : \prod^* A_i \to F(\prod^* A_i)$ such that $\tau_i\lambda = \mu_{A_i}^{-1}F(\tau_i)$.

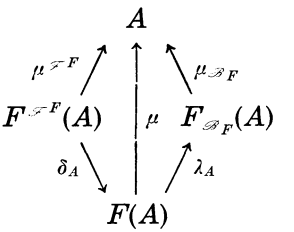
Now $\tau_i(\lambda\mu_{\Pi^* A_i}) = \mu_{A_i}^{-1}F(\tau_i)\mu_{\Pi^* A_i} = \mu_{A_i}^{-1}\mu_{A_i}\tau_i = \tau_i$.

Thus $\lambda\mu_{\Pi^* A_i} = 1$ showing that $\mu_{\Pi^* A_i}$ is a retraction and hence invertibe, i.e., $\prod^* A_i \in \mathscr{T}^F$.

Thus if (F, μ) is any subfunctor of I, determining a semivariety $\mathscr{B}_F$ and a cosemivariety $\mathscr{T}^F$, then we have a functor transformation

$$F^{\mathscr{T}^F} \xrightarrow{\delta} F \xrightarrow{\lambda} F_{\mathscr{B}_F}, \quad \text{and}$$

for any object A, we have a commutative diagram



where $\mu_{\mathscr{B}_F}$ is a normal monomorphism.

It is clear from our construction that if $\mathscr{T}$ is a cosemivariety determining a normal radical subfunctor $(F^{\mathscr{T}}, \mu^{\mathscr{T}})$ of I, then for any $A \in \mathscr{C}$ the cokernel of $\mu_A : F^{\mathscr{T}}(A) \to A \in \mathscr{B}_{F^{\mathscr{T}}}$, and by Proposition 3.1 $F^{\mathscr{T}} \sim F_{\mathscr{B}_{F^{\mathscr{T}}}}$, and we collect these informations in the following.

PROPOSITION 3.2. *If $\mathscr{T}$ is a cosemivariety, such that the functor $(F^{\mathscr{T}}, \mu^{\mathscr{T}})$ determined by it is a normal radical subfunctor, then for any object A, we have the exact sequence*

$$0 \longrightarrow F^{\mathscr{T}}(A) \xrightarrow{\mu_A} A \xrightarrow{\varepsilon} C \longrightarrow 0$$

where $F^{\mathscr{T}}(A) \in \mathscr{T}$ and $C \in \mathscr{B}_{F^{\mathscr{T}}}$.

We are content to leave the reader to mirror the images of the above observations.

If $\mathscr{C}$ is the category of Abelian groups, and $\mathscr{T}$ is the cosemivariety of torsion groups (or divisible groups) then if to each group A, we assign the torsion subgroup $F^{\mathscr{T}}(A)$ (or the mixiмal divisible subgroup $F^{\mathscr{T}}(A)$), then $A/F^{\mathscr{T}}(A)$ lies in the semivariety of torsion free groups (or reduced groups).

PROPOSITION 3.3. *If $\mathscr{A}$ is an Abelian category in which a cosemivariety $\mathscr{T}$ determines a radical which is injective, then every object A is the direct sum of an object in $\mathscr{T}$ and $\mathscr{B}_{F^{\mathscr{T}}}$, and as*

such the identity functor splits having components in the cosemivariety and the semivariety.

Proof. We have an exact sequence $0 \to F^{\mathscr{F}}(A) \xrightarrow{\mu} A \to A/_{F^{\mathscr{F}}(A)} \to 0$ in which the natural injection is a coretraction. Thus there exists a $\delta : A \to F^{\mathscr{F}}(A)$ such that $\mu\delta = 1$. So the theorem is obvious from Theorem 2.68 [5]. Thus for the category of abelian groups, if $F^{\mathscr{F}}$ is the functor determined by the cosemivariety of divisible groups, then we have Theorem 3. [6, § 5].

It is easy to see that the variety functors (covariety functors or torsion radicals) [[4], [9]] are indeed radical functors (coradical functors) and a coradical subfunctor (F, μ) determines a cotriple $[(F\mu)^{-1}, \mu, F]$ in the sense of Eilenberg and Moore [3] where $F\mu$ is the equivalence $F^2 \to F$ which appears in the definition of coradical subfunctor given by $F\mu_A = F(\mu_A)$ for any object A of $\mathscr{C}$, and this is cogenerated by the adjoint F of the inclusion of the associated cosemivariety $\mathscr{F} = \mathscr{F}^F$ into $\mathscr{C}$, where F can be considered as well a functor from $\mathscr{C} \to \mathscr{C}$ dropping the inclusion completely.

We notice if $V_{\mathscr{B}}$ the normal radical functor associated with the semivariety $\mathscr{B}$ is a variety functor, then $\mathscr{B}$ is a variety; converse association holds, i.e, the normal radical functor associated with a variety will be a variety functor (11) in categories with the additional axiom.

(v) *If α is a monomorphism and β is a normal epimorphisms such that $\alpha\beta$ admits image $\nu\mu$, then α normal $\Rightarrow \mu$ is normal.*

Added in proof. While this work was in press, the author was given to understand by P. Lecouturier that certain generalization of Fröhlich's work (a weaker version of Proposition 3.1) has also been obtained by him in more *restricted class of categories* (in which every epimorphism is normal, etc). However he does not obtain the characterization of semivarieties by normal radical functors.

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UNIVERSITY OF MONTREAL

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FINITE GROUPS WITH SMALL CHARACTER DEGREES AND LARGE PRIME DIVISORS II

I. M. ISAACS AND D. S. PASSMAN

In a previous paper one of the authors considered groups G with r. b. n (representation bound n) and $n < p^2$ for some prime p. Here we continue this study. We first offer a new proof of the fact that if $n = p - 1$ then G has a normal Sylow p-subgroup. Next we show that if $n = p^{3/2}$ then $p^2 \nmid |G/O_p(G)|$. Finally we consider $n = 2p - 1$ and with the help of the modular theory we obtain a fairly precise description of the structure of G. In particular we show that its composition factors are either p-solvable or isomorphic to $PSL(2, p)$, $PSL(2, p - 1)$ for p a Fermat prime or $PSL(2, p + 1)$ for p a Mersenne prime.

Now the irreducible characters of $PSL(2, p)$ have degrees (see [10] p. 128) $1, p, p \pm 1, (p \pm 1)/2$ for p odd and those of $PSL(2, 2^a)$ have degrees (see [10] p. 134) $1, 2^a, 2^a \pm 1$. Thus for $p > 2$ the linear groups of the preceding paragraph do in fact have r.b. $(2p - 1)$.

The notation here is standard. In addition, if χ is a character of G we let $\det \chi$ denote the linear character which is the determinant of the representation associated with χ. Also $n_p(G)$ denotes the number of Sylow p-subgroups of G.

LEMMA 1. *Let G be a group with* r.b.n. *and let $N \neq G$* be a *subgroup. Suppose $G = \bigcup_{i=0}^{t} Nx_iN$ is the (N, N)-double coset decomposition of G with $x_0 = 1$. Set $a_i = |Nx_iN|/|N| = [N: N \cap N^{x_i}]$. Then $n \geqq (a_1 + a_2 + \cdots + a_t)/t$.*

Proof. Let $\theta = (1_N)^G$ be the character of the permutation representation of G on the cosets of N. Then $\theta(1) = [G: N]$, $[\theta, 1_G] = 1$ and $\|\theta\|^2 = 1 + t$. Since $[\theta, 1_G] = 1$ we can write

$$\theta = 1_G + b_1\chi_1 + \cdots + b_s\chi_s$$

where the χ_i are distinct nonprincipal irreducible characters of G. Thus since G has r.b.n we have

$$\begin{aligned} 1 + nt &= 1 + n(\|\theta\|^2 - 1) = 1 + n(b_1^2 + \cdots + b_s^2) \\ &\geqq 1 + n(b_1 + \cdots + b_s) \geqq 1 + b_1\chi_1(1) + \cdots + b_s\chi_s(1) \\ &= \theta(1) = 1 + (a_1 + a_2 + \cdots + a_t) \end{aligned}$$

and the result follows.

LEMMA 2. *Let G be a group with* r.b.n.
(i) Let $N \neq G$ *be a subgroup. Then*

$$n \geqq \min \{[N: N \cap N^x] \mid x \in G - N\} .$$

(ii) *Let π be a set of primes and let H be a maximal π-subgroup of G. Then either $H \triangle G$ or $n \geqq \min \{[H: H \cap H^x] \mid x \in G - \boldsymbol{N}(H)\}$.*

Proof. (i) follows immediately from Lemma 1. Now let H be as in (ii) and suppose H is not normal in G. Set $N = \boldsymbol{N}(H) \neq G$. Since H is a maximal π-subgroup it follows that $H = \boldsymbol{O}_\pi(N)$. Thus if $x \in G$ then $H^x = \boldsymbol{O}_\pi(N^x)$ so $H \cap N^x = H \cap H^x$ and

$$[N: N \cap N^x] \geqq [H: H \cap N^x] = [H: H \cap H^x] .$$

Thus the result follows from (i).

Applying Lemma 2(ii) with $\pi = \{p\}$ and H a Sylow p-subgroup of G yields

THEOREM 3. *Let p be a prime and let G be a group with* r.b.$(p-1)$. *Then $n_p(G) = 1$.*

This result was originally proved in [7] (Theorem E) in a much more complicated way.

LEMMA 4. *Let G have* r.b.(p^2-1) *and let Q_1 and Q_2 be p-subgroups of G with $\langle Q_1, Q_2 \rangle$ not a p-group. Then $n_p(\boldsymbol{C}(Q_1) \cap \boldsymbol{C}(Q_2)) = 1$. If further the Sylow p-subgroups of G are abelian, then*

$$n_p(\boldsymbol{N}(Q_1) \cap \boldsymbol{N}(Q_2)) = 1$$

Proof. Set $W = \langle Q_1, Q_2 \rangle$. Since W is not a p-group we see that $n_p(W) > 1$. We assume now that $n_p(C) > 1$ where $C = \boldsymbol{C}(Q_1) \cap \boldsymbol{C}(Q_2) = \boldsymbol{C}(W)$ and derive a contradiction. Set $Z = W \cap C$ so that Z is central in W and C and let $\bar{W} = W/Z$, $\bar{C} = C/Z$. Since Z is central we have easily $n_p(\bar{W}) > 1$, $n_p(\bar{C}) > 1$ and $(WC)/Z = \bar{W} \times \bar{C}$. By Theorem 3 both $\bar{W}$ and $\bar{C}$ have irreducible characters of degree $\geqq p$ and hence $\bar{W} \times \bar{C}$ has an irreducible character of degree $\geqq p^2$. This is a contradiction since G has r.b.(p^2-1) and this property is inherited by subgroups and quotient groups. If the Sylow p-subgroups of G are abelian then any p-group normalizing Q_i centralizes it. Thus the second result follows from the first.

THEOREM 5. *Let p be a prime and let G be a group with* r.b.$p^{3/2}$. *Then $p^2 \nmid |G/\boldsymbol{O}_p(G)|$.*

Proof. If $p = 2$ then G has r.b.2 and the result follows from Theorem C of [7]. Thus we can assume that $p \geqq 3$ and clearly also that $\boldsymbol{O}_p(G) = \langle 1 \rangle$. Since $p^2 - p - 1 \geqq [p^{3/2}]$ for $p \geqq 3$, Proposition 1.3 of [6] implies that a Sylow p-subgroup P of G is abelian. We assume that $|P| \geqq p^2$ and derive a contradiction. Set $n = p^{3/2}$.

Let $N = \boldsymbol{N}(P)$ so that $N \neq G$. By Lemma 2(i) there exists $w \in G - N$ with $n \geqq [N: N \cap N^w] \geqq [P: P \cap P^w]$. Set $Q = P \cap P^w$ so since $p^2 > n$ and $w \notin \boldsymbol{N}$ we see that $[P: Q] = p$ and hence $Q \neq \langle 1 \rangle$. Let $M = \boldsymbol{N}(Q)$. Since $P \triangle N$, $P^w \triangle N^w$ we have $Q \triangle (N \cap N^w)$. Also $Q \triangle P$ and $P \nsubseteqq N^w$. Hence $M \cap N \supseteqq \langle P, N \cap N^w \rangle$ so $[N: N \cap M] \leqq n/p = p^{1/2}$.

We now make the following crucial observation. If $[M: M \cap M^x] < p^2$ for some $x \notin M$ then Q and Q^x commute elementwise and $x \in MNM$. To see this suppose that Q and Q^x do not commute. Then since the Sylow p-subgroups of G are abelian, $\langle Q, Q^x \rangle$ is not a p-group. By Lemma 4, $n_p(M \cap M^x) = 1$ so if $U = \boldsymbol{O}_p(M \cap M^x)$ then U is also a Sylow p-subgroup of $M \cap M^x$. Now $p^2 \nmid [M: M \cap M^x]$ and $Q \nsubseteqq M^x$ clearly so QU is a Sylow p-subgroup of M. Since $N_M(QU) \supseteqq \langle Q, M \cap M^x \rangle$ we have $[M: N_M(QU)] < p$ and hence by Sylow's theorem $QU \triangle M$ and $n_p(M) = 1$. This is a contradiction since $Q = P \cap P^w$ and $P, P^w \subseteqq M$. Thus Q and Q^x commute and since $Q \neq Q^x$ and $[P: Q] = p$ it follows that $QQ^x = P^{y^{-1}}$ is a Sylow p-subgroup of G. Thus Q, Q^y and Q^{xy} are all contained in P. By Burnside's lemma these three groups are conjugate in N. Thus $Q^y = Q^h$, $Q^{xy} = Q^k$ for some $h, k \in N$. This yields $yh^{-1} \in M$, $xyk^{-1} \in M$ so

$$x = (xyk^{-1})kh^{-1}(yh^{-1})^{-1} \in MNM .$$

Since $Q \neq \langle 1 \rangle$ we have $M \neq G$. Let $G = \bigcup_{i=0}^{t} Mx_iM$ be the (M, M)-double coset decomposition of G with $x_0 = 1$. Set $a_i = |Mx_iM|/|M|$ and suppose that there are precisely r such $i \neq 0$ with $a_i < p^2$ and s with $a_i \geqq p^2$. Then by Lemma 1, $p^{3/2} = n \geqq (r + p^2s)/(r + s)$. Clearly $r \neq 0$ here so

$$p^{3/2} \geqq (r + p^2s)/(r + s) > p^2/(1 + r/s) .$$

If $s = 0$ then by the preceding paragraph Q commutes with all its conjugates. This implies that $\langle Q^x \mid x \in G \rangle$ is a nontrivial normal p-subgroup of G, a contradiction. Thus $s \geqq 1$. Also if $a_i < p^2$ then $Mx_iM \subseteqq MNM$ by the above. Since we have seen that $[N: N \cap M] \leqq p^{1/2}$ we have $r \leqq p^{1/2} - 1$ since the double coset M itself is not counted. Thus $r/s \leqq p^{1/2} - 1$ and

$$p^{3/2} > p^2/(1 + r/s) \geqq p^{3/2}$$

a contradiction and the result follows.

We now turn to the main result of this paper.

THEOREM 6. *Let p be a prime and let G be a nonabelian simple group with* r.b.$(2p-1)$. *Then $p>2$ and we have one of the following.*

(i) *G is a p'-group.*

(ii) *$G \cong PSL(2,p)$ for $p>3$.*

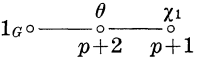
(iii) *$G \cong PSL(2,p-1)$ for p a Fermat prime, $p>3$.*

(iv) *$G \cong PSL(2,p+1)$ for p a Mersenne prime.*

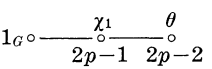
Proof. Since groups with r.b.3 are solvable (Corollary 6.5 of [8]) we have $p>2$. By Theorem 5 since $p^{3/2} \geqq 2p-1$ for $p>2$ we have $p^2 \nmid |G|$. If $p \nmid |G|$ then G satisfies (i) above. Thus we can assume now that G has a Sylow p-subgroup P of order p. Let $B_1(p)$ denote the principal p-block of G. We will use freely the structure of $B_1(p)$ and its associated tree as described in [1] and [2]. Since $B_1(p)$ contains a nonprincipal irreducible character χ and $\chi(1)<2p$, Lemma 1 of [3] implies that P is self centralizing. Let $N=N(P)$ and let $e=|N/P|$ so that $e \mid (p-1)$. By Burnside's transfer theorem $e>1$. We assume now that $G \ncong PSL(2,p)$ and $G \ncong PSL(2,p-1)$ for p a Fermat prime.

Step 1. Let θ denote an exceptional character in $B_1(p)$. Then the tree of $B_1(p)$ must be one of the following.

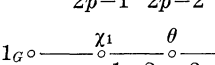
(1) $e=2$



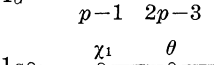
(2) $e=2$



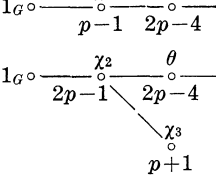
(3) $e=3$



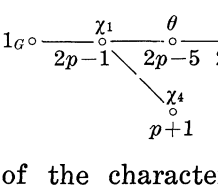
(4) $e=4$



(5) $e=4$



(6) $e=5$



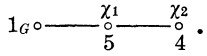
.

Here the degree of the character is written below the character designation.

Since G is nonabelian and simple it follows that every nonprincipal irreducible representation of G is faithful. Thus by Theorem 2 of [5] the degrees of the nonprincipal ordinary irreducible characters of G are all $\geqq p-1$ and by Theorem 1 of [4] the degrees of the nonprincipal

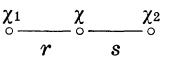
irreducible Brauer characters of G are $\geqq 2(p-1)/3$. Now G has r.b.$(2p-1)$ and the degrees of the ordinary irreducible characters of $B_1(p)$ satisfy $\chi(1) \equiv \pm 1(p)$ if χ is nonexceptional and $\theta(1) \equiv \pm e(p)$. This yields easily $\chi(1) = p-1, p+1, 2p-1$ or $\chi = 1_G$ and $\theta(1) = 2p-e, p+e$ or $\theta(1) = e$ if $e = p-1$.

Suppose first that $p = 3$. Since $e \mid (p-1)$ we have $e = 2$ and the tree is a line with three vertices. Now the center degree is maximal and the principal character must occur so we have clearly



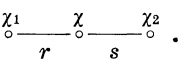
This is either tree (1) or tree (2) according to which of χ_1 of χ_2 we consider exceptional.

Now assume that $p \geqq 5$. Let $\chi \in B_1(p)$ with $\chi(1) = p-1$ and suppose that χ is modular reducible. We can denote this latter fact graphically by

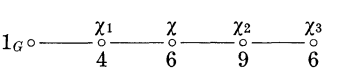


where r and s are degrees of modular constituents. Since $r + s \leqq p-1$ we cannot have both $r, s \geqq 2(p-1)/3$. This implies that one of the modular constituents is the principal character and the other has degree $p-2$.

Now let $\chi \in B_1(p)$ with $\chi(1) = p+1$ and suppose χ is modular reducible. Say we have



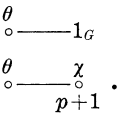
Here by the alternating nature of the tree $\chi_1, \chi_2 \neq 1_G$ so $r, s \geqq 2(p-1)/3$. Since $r + s \leqq p+1$ this yields $4(p-1)/3 \leqq p+1$ so $p = 5$ or 7. In fact either $p = 5, r = s = 3$ or $p = 7, r = s = 4$. We consider these in turn. Let $p = 5$. Since $e \mid (p-1)$ we have $e = 2$ or 4. Certainly $e \neq 2$ here since $B_1(p)$ contains $e+1$ ordinary irreducible characters including 1_G and hence $e = 4$. Thus the degrees of the nonprincipal ordinary irreducible characters of $B_1(p)$ are 4, 6 or 9 and the nonprincipal Brauer characters have degrees $\geqq 3$. The former fact implies that χ_1 and χ_2 are modular reducible so the tree having five vertices is a straight line. This yields easily



and we obtain tree (4). Now let $p = 7$ so that $e = 2, 3$ or 6. Clearly neither χ_1 nor χ_2 can be modular irreducible so the tree has at least five vertices and hence $e = 6$. Thus the degrees of the nonprincipal

ordinary irreducible characters of $B_1(p)$ are 6, 8 or 13 and the non-principal Brauer characters have degrees $\geqq 4$. This latter fact implies that neither χ_1 nor χ_2 can have degree 6 and hence $\chi_1(1) = \chi_2(1) = 13$. Now at least one of χ_1 or χ_2, say χ_1, is not adjacent to 1_G. Hence all characters adjacent to χ_1 have degree $p + 1$ and as we have seen these are either modular irreducible or have constituents of degree 4. This shows that all modular constituents of χ_1 have degree divisible by 4, a contradiction since $\chi_1(1) = 13$. Thus this case does not occur. It now clearly suffices to assume for the remainder of this step that all $\chi \in B_1(p)$ with $\chi(1) = p + 1$ are modular irreducible.

Let θ denote an exceptional character of $B_1(p)$ and we consider the possible branches leaving the vertex associated with θ. Suppose first that $\theta(1) \equiv +e(p)$. The above implies easily that we can only have

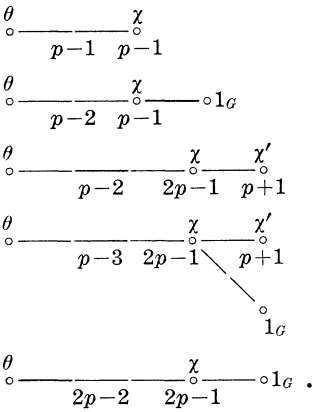


Now the first branch must occur precisely once and let the second branch occur a times. Since the tree has e edges we have

$$1 + a = e$$
$$1 + a(p + 1) = \theta(1) .$$

Now $e \geqq 2$ so $a \geqq 1$ and hence $\theta(1) > p$. Thus $\theta(1) = p + e$ and we obtain $a = 1, e = 2$ and this is tree (1).

Now let $\theta(1) \equiv -e(p)$ so that $\theta(1) = 2p - e$. Using the above information and the alternating nature of the tree we see easily that the only possible branches leaving the vertex associated with θ are



If the last branch occurs then since $2p - e = \theta(1) \geqq 2p - 2$ we have $e = 2$ and this is tree (2). Thus we can assume that only the first

four branches occur say with multiplicities a, b, c, d respectively. Since there are precisely e edges in the tree we have

$$a + 2b + 2c + 3d = e$$
$$a(p-1) + b(p-2) + c(p-2) + d(p-3) = \theta(1) = 2p - e .$$

Adding these two and dividing by p yields

$$a + b + c + d = 2 .$$

In addition the vertex of 1_G occurs precisely once so $b + d = 1$. Thus $a + c = 1$ and there are four possibilities which are easily seen to be trees (3), (4), (5) and (6).

Step 2. Let $N = N(P)$. We consider the restriction of the ordinary irreducible characters of G to N.

Now $N = PE$ is a Frobenius group of order pe with $E = \langle x \rangle$ cyclic of order e. N has precisely e linear characters, namely those of $N/P \cong E$, and the remaining irreducible characters have degree e. Let Δ denote any sum of irreducible characters of N of degree e. Clearly $\Delta_E = \Delta(1)/e \cdot \rho_E$ where ρ_E is the regular character of E. This yields easily

$$\Delta(x) = 0$$
$$(1) \qquad \det \Delta(x) = \begin{cases} 1 & \text{if } e \text{ is odd} \\ (-1)^{\Delta(1)/e} & \text{if } e \text{ is even} . \end{cases}$$

If e is even let δ denote the linear character of N given by $\delta(x) = -1$.

Let ψ be an ordinary irreducible character of G with $\psi \notin B_1(p)$. Since P is self centralizing it follows that $B_1(p)$ is the unique p-block of positive defect and hence ψ belongs to a block of defect 0. Thus $p \mid \psi(1)$. Since G has r.b.$(2p-1)$ this yields $\psi(1) = p$ and clearly $\psi_P = \rho_P$. Thus $\psi_N = \lambda + \Delta$ where λ is linear and $\Delta(1) = p - 1$. Now G is simple so the linear character $\det \psi$ must be principal and hence $1 = \det \psi(x) = \lambda(x) \det \Delta(x)$. This yields by the above

$$(2) \qquad \psi_N = \begin{cases} 1_N + \Delta & \text{if } e \text{ is odd} \\ 1_N + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is even} \\ \delta + \Delta & \text{if } e \text{ is even and } (p-1)/e \text{ is odd} . \end{cases}$$

Now let $\chi \in B_1(p)$ and let $m(\chi)$ denote the number of linear characters counting multiplicities which occur in χ_N. Obviously $m(\chi)$ is the multiplicity of 1_P in χ_P. Suppose first that χ_i, χ_j are nonexceptional ordinary irreducible characters of $B_1(p)$ which are adjacent in the tree. Then $\chi_i + \chi_j = \Phi$ is a projective character and since $\Phi_P = \Phi(1)/p \cdot \rho_P$ we have easily

$$m(\chi_i) + m(\chi_j) = [\chi_i(1) + \chi_j(1)]/p .$$

Now suppose that χ_i is adjacent to the vertex of the exceptional characters $\{\theta_k\}$. Clearly $m(\theta_r) = m(\theta_s)$ for all r, s so since $\chi_i + \sum_{k=1}^{(p-1)/e} \theta_k = \Phi$ we have with $\theta = \theta_1$

$$m(\chi_i) + (p-1)/e \cdot m(\theta) = [\chi_i(1) + (p-1)/e \cdot \theta(1)]/p.$$

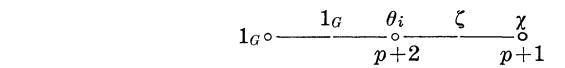
Using the above two equations, the fact that the tree of $B_1(p)$ is connected and the obvious fact that $m(1_G) = 1$ we obtain easily for irreducible χ

$$(3) \qquad m(\chi) = \begin{cases} k-1 & \text{if } \chi(1) = kp - 1 \\ k & \text{if } \chi(1) = kp \pm e \\ k+1 & \text{if } \chi(1) = kp + 1 . \end{cases}$$

There is of course additional information available, for example the fact that $\det \chi(x) = 1$ and the position of χ in the tree, which further limits the structure of χ_N.

Step 3. If tree (1) occurs in step 1, then p is a Mersenne prime and $G \cong PSL(2, p+1)$.

By assumption $e = 2$ and the tree of $B_1(p)$ has the form



where θ_i represents the $(p-1)/2$ exceptional characters with $\theta = \theta_1$. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of irreducible characters of G not in $B_1(p)$.

Let α denote the unique nonprincipal linear character of N. By (1) and (2) we have

$$(4) \qquad \begin{aligned} \psi_{jN} &= 1_N + \Delta'_j , \quad \psi_j(x) = 1 \quad \text{for } p \equiv 1 \ (4) \\ \psi_{jN} &= \alpha + \Delta'_j , \quad \psi_j(x) = -1 \quad \text{for } p \equiv 3 \ (4) . \end{aligned}$$

By (3), $\theta_{iN} = \lambda + \Delta_i$ and since the θ_i are all algebraically conjugate λ is the same for all i. Thus $\det \theta_i(x) = 1$ and equation (1) yield

$$(5) \qquad \begin{aligned} \theta_{iN} &= \alpha + \Delta_i , \quad \theta_i(x) = -1 \quad \text{for } p \equiv 1 \ (4) \\ \theta_{iN} &= 1_N + \Delta_i , \quad \theta_i(x) = 1 \quad \text{for } p \equiv 3 \ (4) . \end{aligned}$$

Now $\chi_N = a1_N + b\alpha + \Delta$ with $a + b = 2$ by (3) and since x is a p'-element $a - b = \chi(x) = \zeta(x) = \theta(x) - 1$. Thus (5) yields

$$(6) \qquad \begin{aligned} \chi_N &= 2\alpha + \Delta , \quad \chi(x) = -2 \quad \text{for } p \equiv 1 \ (4) \\ \chi_N &= 1_N + \alpha + \Delta , \quad \chi(x) = 0 \quad \text{for } p = 3 \ (4) . \end{aligned}$$

Equations (4), (5) and (6) and Frobenius reciprocity now yield

$$(7)\qquad \begin{aligned} (1_N)^G &= 1_G + \sum_1^k \psi_j\,, & \alpha^G &= 2\chi + \sum_1^{(p-1)/2} \theta_i \quad \text{for } p \equiv 1\ (4)\\ (1_N)^G &= 1_G + \chi + \sum_1^{(p-1)/2} \theta_i\,, & \alpha^G &= \chi + \sum_1^k \psi_j \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Thus since $(1_N)^G(1) = \alpha^G(1) = [G: N]$ and $|N| = 2p$ we obtain easily

$$(8)\qquad \begin{aligned} |G| &= p(p^2 + 5p + 2)\,, \quad k = (p+5)/2 \quad \text{for } p \equiv 1\ (4)\\ |G| &= p(p+1)(p+2)\,, \quad k = (p+1)/2 \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Now using $|C(x)| = \sum \eta(x)\overline{\eta(x)}$, where η runs over all ordinary irreducible characters of G, along with equations (4), (5), (6) and (8) we have

$$(9)\qquad \begin{aligned} |C(x)| &= p + 7 \quad \text{for } p \equiv 1\ (4)\\ |C(x)| &= p + 1 \quad \text{for } p \equiv 3\ (4)\,. \end{aligned}$$

Since $(p+7) \nmid p(p^2 + 5p + 2)$ the case $p \equiv 1\ (4)$ is eliminated. Thus $p \equiv 3\ (4)$.

Set $S = C(x)$ so that $|S| = p + 1$ and $[G: S] = p(p+2)$. We consider $(1_S)^G$. Since this character is rational and the θ_i are algebraically conjugate we have

$$(1_S)^G = 1_G + a\sum_1^{(p-1)/2} \theta_i + b\chi + \sum_1^k c_j\psi_j\,.$$

Set $c = \sum_1^k c_j$. By considering degrees we have

$$p(p+2) = 1 + a(p+2)(p-1)/2 + b(p+1) + cp$$

and evaluating at x yields

$$0 < (1_S)^G(x) = 1 + a(p-1)/2 - c$$

by (4), (5), (6) and the fact that $x \in S$. Certainly $a \leqq 2$. Also $b \leqq \chi(1)/2 = (p+1)/2$ by Frobenius reciprocity and the fact that $\chi(x) = 0$. Thus $a = 0$ yields a contradiction. If $a = 1$ then $b \equiv 0\ (p)$ so $b = 0$ and $c = (p+3)/2 > 1 + a(p-1)/2$ again a contradiction. Thus $a = 2$ and we have easily

$$(10)\qquad (1_S)^G = 1_G + 2\sum_1^{(p-1)/2} \theta_i + \chi$$

so $(1_S)^G(x) = p$ by (5) and (6). By definition of induced character and the fact that $S = C(x)$ this implies that S contains precisely p distinct conjugates of x. Since $|S| = p + 1$ this shows that S is an elementary abelian 2-group and therefore that S is a Sylow 2-subgroup of G and p is a Mersenne prime. By Burnside's lemma the nonidentity elements of S are all conjugate in $N(S)$ so $N(S) > S$.

Set $H = N(S) > S$. Then $(1_H)^G$ is a national constituent of $(1_S)^G$ and $(1_H)^G(1) \leqq p(p+2)/3$. Thus by (10) we have easily

$$(1_H)^G = 1_G + \chi \,. \tag{11}$$

Therefore G is a doubly transitive permutation group on the set Ω where $H = G_\infty$ for some point $\infty \in \Omega$. By (10) χ_S contains 1_S with multiplicity one so (11) implies that S has two orbits on Ω. Hence since $|S| = |\Omega| - 1$, S is in fact a regular normal subgroup of G_∞. Now $|H| = p(p+1)$ so if $\tilde{P}$ is a Sylow p-subgroup of H, then since $\tilde{P}$ is self centralizing and $|\tilde{P}| = |S| - 1$ we see that G is sharply 3-transitive.

With the structure of H as given above we can clearly identify Ω with $GF(p+1) \cup \{\infty\}$ in such a way that S is the set of translations $\left\{\begin{pmatrix} z \\ z+r \end{pmatrix} \middle| r \in GF(p+1)\right\}$ and $\tilde{P}$ is the set $\left\{\begin{pmatrix} z \\ sz \end{pmatrix} \middle| s \in GF(p+1), s \neq 0\right\}$. Let $\tilde{x} \in G$ with $\tilde{x} = (0\ \infty)(1) \cdots$. Then $\tilde{x}$ has order 2 and normalizes $\tilde{P} = G_{\infty 0}$ so $\tilde{x}$ acts in a dihedral manner on $\tilde{P}$. If $\tilde{x} = \begin{pmatrix} z \\ f(z) \end{pmatrix}$ then for all $s \in GF(p+1)$, $s \neq 0$

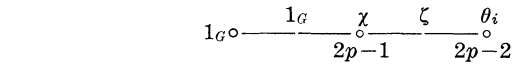
$$\begin{pmatrix} z \\ sz \end{pmatrix}\begin{pmatrix} z \\ f(z) \end{pmatrix} = \begin{pmatrix} z \\ f(z) \end{pmatrix}\begin{pmatrix} z \\ s^{-1}z \end{pmatrix}$$

so $f(sz) = s^{-1}f(z)$. Setting $z = 1$ yields $f(s) = s^{-1}$. Thus $\tilde{x} = \begin{pmatrix} z \\ 1/z \end{pmatrix}$ and since $G = \langle H, \tilde{x} \rangle$ we have clearly $G \subseteq PSL(2, p+1)$. By (8) we have in fact $G \cong PSL(2, p+1)$ and this step follows.

Step 4. Completion of the proof.

We now consider the remaining trees in turn. Let $\{\psi_j \mid j = 1, 2, \cdots, k\}$ denote the set of ordinary irreducible characters of G not in $B_1(p)$.

Suppose first that we have tree (2). If $p = 3$ this is the same as tree (1) so we assume that $p > 3$. From



and (3) and $\det \chi(x) = 1$ we have $\chi_N = 1_N + \Delta$. Let α be the unique nonprincipal linear character of N so that we have by (3) $\theta_{iN} = a1_N + b\alpha + \Delta_i$ with $a + b = 2$. Since x is a p'-element $a - b = \theta_i(x) = \zeta(x) = \chi(x) - 1 = 0$ so $\theta_{iN} = 1_N + \alpha + \Delta_i$. Now by (2) all the ψ_j occur in either $(1_N)^G$ or α^G depending on the parity of $(p-1)/2$. Since $(1_N)^G(1) = \alpha^G(1)$, the above and Frobenius reciprocity imply that the ψ_j occur in α^G and hence

$$(1_N)^G = 1_G + \chi + \sum_1^{(p-1)/2} \theta_i \,.$$

Since $|N| = 2p$ this yields $|G| = 2p(p^2 + 1)$. Now $\theta_i(1) \,|\, |G|$ so $(p-1) \,|\, (p^2+1)$ and this is easily seen to be a contradiction for $p > 3$.

Now consider tree (3)

$$1_G \circ \overset{1_G}{\text{———}} \underset{p-1}{\circ} \overset{\chi_1}{\text{———}} \underset{2p-3}{\circ} \overset{\theta_2}{\text{———}} \underset{p-1}{\circ} \overset{\zeta_2}{\text{———}} \circ \overset{\chi_2}{} .$$

By (3), $\chi_{iN} = \Delta_i$ and $\chi_i(x) = 0$ for $i = 1, 2$. This implies that $\zeta_1(x) = -1$, $\zeta_2(x) = 0$ so $\theta_i(x) = -1$. Now by (3), $\theta_{iN} = a1_N + b\alpha + c\alpha^2 + \Delta_i'$ where α is a nonprincipal linear character and $a + b + c = 2$. Since $\theta_i(x) = -1$ we have easily $\theta_{iN} = \alpha + \alpha^2 + \Delta_i'$. Applying Frobenius reciprocity to the above and (2) we have

$$(1_N)^G = 1_G + \sum_1^k \psi_j , \qquad \alpha^G = \sum_1^{(p-1)/3} \theta_i$$

and this yields easily

$$|G| = p(p-1)(2p-3) , \qquad k = (2p-5)/3 .$$

Using $|C(x)| = \sum \bar{\eta}(x)\eta(x)$ along with the above and (2) we obtain $|C(x)| = p - 1$. Now clearly x is a real element so $|C^*(x)| = 2(p-1)$ where $C^*(x) = \{g \in G \mid x^g = x \text{ or } x^{-1}\}$. Since $2(p-1)$ does not divide $|G|$ as given above, it follows that this tree does not occur.

Suppose tree (4) or (5) occurred. Since $\chi_1(1) = p - 1$ and $\det \chi_1(x) = 1$, (1) and (3) imply that $(p-1)/4$ is even. Hence by (2), $\psi_{jN} = 1_N + \Delta_j$. Now there are four linear characters of N and at most two occur in θ_N so choose $\alpha \neq 1_N$ such that α does not occur in θ_N. Thus α can occur only in χ_{2N} or χ_{3N} with multiplicity at most two. Hence

$$[G: N] = \alpha^G(1) \leqq 2\chi_2(1) + 2\chi_3(1) = 6p .$$

Now choose β so that β occurs in θ_N. Then

$$[G: N] = \beta^G(1) \geqq \sum_1^{(p-1)/4} \theta_i(1) = (2p-4)(p-1)/4 .$$

Since $(p-1)/4$ is even and $(p-1)/4 \neq 2$ we have $(p-1)/4 \geqq 4$, $p \geqq 17$ and

$$6p \geqq \alpha^G(1) = \beta^G(1) \geqq 4(2p-4) ,$$

a contradiction.

Finally consider tree (6). By (1), (2) and (3) we have easily $\chi_{1N} = 1_N + \Delta$, $\chi_{2N} = 1_N \Delta'$, and $\psi_{jN} = 1_N + \Delta_j$. Now since $e = 5$ and $m(\theta) = 2$ we can choose a linear character α of N with $\alpha \neq 1_N$ and such that α does not occur in θ_N. Hence by the above and the fact that $m(\chi_3) = m(\chi_4) = 2$ we have $\alpha^G = a\chi_3 + b\chi_4$ with $a, b \leqq 2$. Thus since

$$[G : N] = \alpha^G(1) = a\chi_3(1) + b\chi_4(1) = (a+b)(p+1)$$

and $[G:N] \equiv 1\ (p)$ we have $[G:N] = p+1$. Now choose β so that β occurs in θ_N. Then

$$p+1 = [G:N] = \beta^G(1) \geqq \sum_{1}^{(p-1)/5} \theta_i(1) = (2p-5)(p-1)/5\ ,$$

a contradiction since $5 \mid (p-1)$ implies that $p \geqq 11$. This therefore completes the proof of the theorem.

Finally we consider the remaining groups with r.b.$(2p-1)$.

THEOREM 7. *Let p be a prime and let G be a group with* r.b.$(2p-1)$. *Then we have one of the following.*

(i) *G has a normal abelian Sylow p-subgroup.*
(ii) *G is solvable and has p-length* 1.
(iii) *$G/\boldsymbol{Z}(G) \cong PSL(2,p)$ or $PGL(2,p)$ for $p > 3$.*
(iv) *$G/\boldsymbol{Z}(G) \cong PSL(2,p-1)$ for p a Fermat prime, $p > 3$.*
(v) *$G/\boldsymbol{Z}(G) \cong PSL(2,p+1)$ for p a Mersenne prime.*
(vi) *$G/\boldsymbol{Z}(G) \cong$* Sym (4) for $p = 2$.

Proof. If $p = 2$ then G has r.b.3. Thus by Corollary 6.5 of [8], G satisfies (ii) or (vi) above. Now let $p > 2$. Since $2p-1 \leqq p^{3/2}$, Theorem 5 implies that $p^2 \nmid |G/\boldsymbol{O}_p(G)|$. With this additional fact it is easy to see that the proof of the main theorem of [6] applies also to groups with r.b.$(2p-1)$ with $p > 2$ yielding the same conclusion. (The $p > 2$ assumption is used crucially in the last paragraph of the proof of Proposition 3.1 of [6].) Thus either G satisfies (i) or (ii) above or $G = P_1 \times G_1$ where P_1 is an abelian p-group and $p^2 \nmid |G_1|$. Clearly G_1 has r.b.$(2p-1)$ and if G_1 satisfies any of the above then so does G. Therefore it suffices to assume that $G = G_1$ or equivalently that $p^2 \nmid |G|$. We assume now that G does not satisfy (i). This of course implies that $p \,||\, |G|$.

Let $K = \boldsymbol{O}_{p'}(G)$ and let H/K be a minimal normal subgroup of G/K. Then $p \,||\, |H/K|$ and since $p^2 \nmid |G/K|$ this implies that H/K is the unique minimal normal subgroup. Now H/K is a product of isomorphic simple groups and $p^2 \nmid |H/K|$ so H/K is simple. If $|H/K| = p$ then G is p-solvable of p-length 1. Thus since G does not have a normal Sylow p-subgroup, Proposition 2.3 of [6] implies that G is solvable and G satisfies (ii). Hence it suffices to assume that $\bar{H} = H/K$ is a nonabelian simple group. It is convenient to first consider the possibility $p \geqq 5$.

Since $\bar{H}$ is the unique minimal normal subgroup of $\bar{G} = G/K$ we have $\boldsymbol{C}_{\bar{G}}(\bar{H}) = \langle 1 \rangle$ and thus $\bar{G} \subseteq \text{Aut}\, \bar{H}$. Suppose $\bar{T}$ is a subgroup of $\bar{H}$ with $1 < [\bar{H}:\bar{T}] < 2p$. Since $\bar{H}$ is simple and $p \,||\, |\bar{H}|$ we cannot have $[\bar{H}:\bar{T}] < p$. Thus $p \leqq [\bar{H}:\bar{T}] < 2p$ and $\bar{T}$ is maximal in $\bar{H}$ and hence self normalizing. If $\bar{T}$ were abelian it would follow easily that

$\bar{T}$ is a T.I. set and then $\bar{H}$ is a simple Frobenius group, a contradiction. Thus $\bar{T}$ is nonabelian.

Let ψ be an irreducible character of K and let χ be an irreducible constituent of ψ^H. If $e = [\chi_K, \psi]_K$ then $\chi(1) = et\psi(1)$ where $t = [H: T]$ and T is the inertial group of ψ in H. Suppose $T < H$ and set $\bar{T} = T/K$. Since $\chi(1) < 2p$ we have $t < 2p$ and thus by the remarks of the preceding paragraph $t \geqq p$ and $\bar{T}$ is nonabelian. Thus we have $2p > \chi(1) = et\psi(1) \geqq ep\psi(1)$ so $e = \psi(1) = 1$. Now there exists an irreducible character η of T with $\eta^H = \chi$ and $\eta_K = e\psi = \psi$. Since $\bar{T}$ is nonabelian we can choose a nonlinear irreducible character β of T containing K in its kernel. Thus since η is linear, $\eta_0 = \eta\beta$ is also an irreducible character of T. Let χ_0 be an irreducible constituent of η_0^H. Then $\beta(1)\psi = \eta_{0K}$ occurs in χ_{0K} and therefore $[\chi_{0K}, \psi] \geqq \beta(1) > 1$. The above reasoning applied to χ_0 now yields a contradiction. Thus $H = T$ and H fixes all irreducible characters of K. By Brauer's lemma, H fixes all conjugacy classes of K. Let P be a Sylow p-subgroup of H. Then P fixes each class of K and since K is a p'-group, P centralizes K. Thus if $C = \boldsymbol{C}_H(K)$ then $KC > K$ and since H/K is simple we have $H = KC$. Now $C/(C \cap K) \cong \bar{H}$ and $\boldsymbol{Z}(C) \supseteq C \cap K$ so $\boldsymbol{Z}(C) = C \cap K$.

Let D denote the last term in the derived series of C. Then clearly $D = D'$, $D/\boldsymbol{Z}(D) \cong \bar{H}$ and $Z = \boldsymbol{Z}(D) = D \cap K$. Thus Z is a homomorphic image of the Schur multiplier of $\bar{H}$. By Theorem 6, $\bar{H} \cong PSL(2, p)$, $PSL(2, p-1)$ for p a Fermat prime or $PSL(2, p+1)$ for p a Mersenne prime. We have by assumption $p \geqq 5$. Also for $p = 5$, $PSL(2, p) \cong PSL(2, p-1)$ and we will view this group as $PSL(2, p)$. By [10] (Satz IX, p. 119) either $Z = \langle 1 \rangle$ or $\bar{H} \cong PSL(2, p)$, $D \cong SL(2, p)$ and $|Z| = 2$.

We show now that K is central. Suppose first that $Z = \langle 1 \rangle$ so that $H \cong D \times K$. Let χ be a fixed irreducible character of D with $\chi(1) = p$ and let λ be an irreducible character of K. Then $\chi\lambda$ is an irreducible character of H so $2p > \chi(1)\lambda(1) = p\lambda(1)$ and $\lambda(1) = 1$. Thus K is abelian and central in H. If K is not central in G, then some linear character λ of K is not fixed by G. This implies easily that if θ is a constituent of $(\chi\lambda)^G$ then $\theta(1) \geqq 2\chi(1) = 2p$, a contradiction. Thus K is central in G in this case. Now let $Z \neq \langle 1 \rangle$ so that $|Z| = 2$ and $D \cong SL(2, p)$. We have an epimorphism $D \times K \to DK = H$ where the kernel is the third subgroup W of order 2 in the group generated by the copies of Z in D and K. Let λ be an irreducible character of K. Since Z is central in K and $|Z| = 2$ it is easy to see from the character table of $SL(2, p)$ ([10], p. 128) that there exists an irreducible character χ of D with $\chi(1) \geqq p$ and with W in the kernel of $\chi\lambda$, an irreducible character of $D \times K$. Thus $\chi\lambda$ is a character of H. The preceding argument now shows first that K is abelian and then that K is central. We have therefore shown that $G/\boldsymbol{Z}(G) \cong \bar{G}$ and it remains

to identify $\bar{G}$.

Now $\bar{G} \subseteq \text{Aut } \bar{H}$ and $\bar{H}$ is a 2-dimensional projective group so the possibilities for $\bar{G}$ are given by Satz 1 of [9]. Suppose first that $\bar{H} \cong PSL(2, p)$. Then either $\bar{G} \cong PSL(2, p)$ or $\bar{G} \cong PGL(2, p)$ and we have (iii). Note the fact that $PGL(2, p)$ has r.b.$(2p - 1)$ can be seen from the character table on page 136 of [10]. We consider the remaining two cases. Thus $\bar{H} \cong PSL(2, s)$ with $2^n = s = p \pm 1$ and $\bar{G}/\bar{H}$ is isomorphic to a subgroup of the Galois group of $GF(2^n)/GF(2)$, a cyclic group of order n. Suppose $\bar{G} > \bar{H}$ and let $t \in \bar{G}$ correspond to a nontrivial field automorphism $x \to x^j$. Then in the notation of page 134 of [10], but replacing upper case by lower case letters, we have $t^{-1}at = a^j \neq a$. Since $s > 4$ by our assumption for $p = 5$ it follows easily that $a^j \neq a^{-1}$ so a^j is not conjugate to a in $\bar{H}$. From the character table of $\bar{H}$ we now see easily that t moves some irreducible character of $\bar{H}$ of degree $s + 1$ and thus $\bar{G}$ has an irreducible character of degree at least $2(s + 1) \geqq 2p$, a contradiction. Hence $\bar{G} = \bar{H}$ and G satisfies (iv) or (v). This completes the proof of the theorem for $p \geqq 5$.

Finally let $p = 3$. Since $\bar{H}$ is a nonabelian simple group with r.b.$(2p - 1)$, $\bar{H} \cong PSL(2, 4) \cong PSL(2, 5)$ by Theorem 6. Certainly G is not 5-solvable and G has r.b.$(2 \cdot 5 - 1)$. Thus by the prime 5 case already proved, $G/\boldsymbol{Z}(G) \cong PSL(2, 5)$ or $PGL(2, 5)$. Since the latter group has an irreducible character of degree $6 > 2p - 1$ we have $G/\boldsymbol{Z}(G) \cong PSL(2, 5) \cong PSL(2, p + 1)$ and G satisfies (v). Thus the result follows.

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University of Chicago
Yale University

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A GREEN'S FUNCTION APPROACH TO PERTURBATIONS OF PERIODIC SOLUTIONS

CARL KALLINA

Consider the nonlinear differential system

(1.1) $$\dot{y} = A(t)\cdot y + \varepsilon q(t, y, \varepsilon)$$

where y, q are column n-vectors, q is continuous in (t, y, ε) and has continuous second partial derivatives with respect to y, ε for all values of $t, 0 \leqq y \leqq R$ for some $R>0$ and $0 \leqq \varepsilon \leqq \varepsilon_0$ for some $\varepsilon_0>0$. Further assume $A(t)$ is an $n\times n$ matrix such that $A \in C^1$, and both A and q are periodic in t with period T. Associated with system (1.1) are the general homogeneous and nonhomogeneous equations

(1.2) $$\dot{y} = A(t)\cdot y$$

(1.3) $$\dot{y} = A(t)\cdot y + f(t)$$

where $f(t)$ is an arbitrary n-vector function periodic in t of period T. In this paper we consider the classical problem of proving the existence of T-periodic solutions $y = y(t)$ of (1.1) when the homogeneous system (1.2) has nontrivial T-periodic solutions.

In his book "Oscillations in Nonlinear Systems" [5], J. K. Hale treats this problem in Chapter 11 as an extension of the general method developed in Chapter 6 for the case when $A(t)$ is a constant matrix A. Among other treatments of this problem is the work of D. C. Lewis [6] utilizing a generalized Green's matrix. It is the aim of the present paper to incorporate certain ideas from [6] into the general method employed in [5] in Chapter 6. We feel that the approach presented here brings into sharper focus the generality of the methods employed in [5] and also has certain computational advantages which will be mentioned in the body of the paper. The present paper has many points of contact with a recent paper of S. Bancroft, J. K. Hale, and D. Sweet [2]. See also [1], [7] and [8].

2. We must first consider the general nonhomogeneous equation (1.3). We will state a necessary and sufficient condition for the existence of T-periodic solutions of (1.3) when (1.2) has nontrivial T-periodic solutions. Following [6] we then construct a generalized Green's function in order to find an explicit periodic solution of (1.3).

Let $S = \{f: f(t) = f(t + T); f \in C$; and $f(t)$ is a real valued n-vector for all $t\}$. Then S is a Banach space if we define the norm $V(f)$ to

be $V(f)=\sup\{\|f(t)\|;\ 0\leqq t\leqq T\}$. Further let $X(t)$ be any real fundamental matrix solution of (1.2), and let $n-k$ be the rank of the $n\times n$ matrix $L = X(0) - X(T)$. Then there exist k linearly independent column vectors $b_1, b_2, \cdots, b_k$ such that $L\cdot b_i = 0$ for $i = 1, 2, \cdots, k$. Let B denote the $n \times k$ matrix $(b_1, b_2, \cdots, b_k)$. Then the columns $\varphi_1(t), \varphi_2(t), \cdots, \varphi_k(t)$ of the matrix $\Phi(t)$ defined by $\Phi(t) = X(t)\cdot B$ form a set of k linearly independent solutions of (1.2) which are T-periodic. By assumption $k > 0$, and of course the columns of $\Phi(t)$ span the space S_1 of all T-periodic solutions of (1.2). Explicitly we have

$$S_1 = \{f \in S: f(t) = \Phi(t)\cdot a = \sum a_i\cdot\varphi_i(t) \text{ for all } a = (a_1, a_2, \cdots, a_k) \in R^k\}\ .$$

Let A' denote the transpose of an $r\times s$ matrix A. Let $N(A)$ and $R(A)$ denote the nullity and range of A, respectively. Consider the $n \times n$ matrix $L' = X'(0) - X'(T)$. Since $\dim N(L') = \dim N(L)$, we can find k linearly independent column vectors $u_1, u_2, \cdots, u_k$ such that $L'\cdot u_i = u_i'\cdot L = 0$ for $i = 1, 2, \cdots, k$. Let U denote the $k \times n$ matrix with rows $u_1', u_2', \cdots, u_k'$. We define an $n \times k$ matrix $\Psi(t)$ to be

$$\Psi(t) = [U\cdot X(T)\cdot X^{-1}(t)]'\ .$$

Then $\Psi'(t) = U\cdot X(T)\cdot X^{-1}(t)$. Clearly, $\Psi(t)$ has rank k and each of its columns is a solution of the differential equation

$$\dot{Z} = -A'(t)\cdot Z \tag{2.1}$$

which is adjoint to (1.2).

Let $\psi_1(t), \psi_2(t), \cdots, \psi_k(t)$ denote the k linearly independent columns of $\Psi(t)$. These functions span the space S_2 of all T-periodic solutions of the adjoint equation (2.1). In fact, since $\Psi(0) = \Psi(T)$ and $\Psi(t)$ is a matrix solution of (2.1) we see that $\Psi(s) = \Psi(s + T)$ for $0 \leqq s \leqq T$. If (2.1) had l linearly independent periodic solutions with $l > k$ then the original system (1.2), which is adjoint to (2.1), would also have at least l linearly independent T-periodic solutions, contrary to assumption. Explicitly, we have

$$S_2 = \{f \in S: f(t) = \Psi(t)\cdot a = \sum a_i\cdot\psi_i(t) \text{ for all } a = (a_1, a_2, \cdots, a_k) \in R^k\}\ .$$

It is easy to see that S becomes a real Euclidean space when an inner product (f, g) is defined for $f, g \in S$ by

$$(f, g) = \frac{1}{T}\cdot\int_0^T \langle f(t), g(t)\rangle dt = \sum_{i=1}^{n} \frac{1}{T}\cdot\int_0^T f_i(t)\cdot g_i(t)dt \tag{2.2}$$

where $\langle a, b\rangle = \sum_{i=1}^{n} a_i\cdot b_i$ denotes the usual inner product for $a, b \in R^n$. Without loss of generality we may assume that both $\{\varphi_i(t)\}$ and $\{\psi_i(t)\}$ are orthonormal bases for their respective subspaces S_1 and S_2; i.e.,

$(\varphi_i, \varphi_j) = (\psi_i, \psi_j) = \delta ij$ for $1 \leqq i, j \leqq k$.

By P and Q we denote the orthogonal projections of S onto S_1 and S_2, respectively. Explicitly we have

$$P: S \to S_1 , \qquad Pf(t) = \sum_{i=1}^{n} (f, \varphi_i) \cdot \varphi_i(t) \tag{2.3}$$

$$Q: S \to S_2 , \qquad Qf(t) = \sum_{i=1}^{n} (f, \psi_i) \cdot \psi_i(t) . \tag{2.4}$$

For convenience we summarize certain well-known general properties of projections in Euclidean space.

(i) $f \in S_1 \Rightarrow Pf = f$ and $f \in S_2 \Rightarrow Qf = f$.
(ii) $P^2 f = Pf$ and $Q^2 f = Qf$ for all $f \in S$.
(iii) $Pf = 0 \Leftrightarrow (f, \varphi_i) = 0$ for $i = 1, 2, \cdots k$.
(iv) $Qf = 0 \Leftrightarrow (f, \psi_i) = 0$ for $i = 1, 2, \cdots k$.

Let us consider the condition $Qf = 0$. By (iv) we have $Qf = 0 \Leftrightarrow$

$$0 = (f, \psi_i) = (\psi_i, f) = \frac{1}{T} \int_0^T \langle \psi_i(t), f(t) \rangle dt$$

for $i = 1, 2, \cdots, k$. In matrix notation these k equations can be written as

$$0 = \int_0^T \Psi'(t) \cdot f(t) dt = \int_0^T U \cdot X(T) \cdot X^{-1}(t) \cdot f(t) dt . \tag{2.5}$$

Again using matrix notation we note explicitly that for $f \in S$ we have

$$Qf(t) = \Psi(t) \cdot \frac{1}{T} \cdot \int_0^T \Psi'(s) \cdot f(s) ds . \tag{2.6}$$

LEMMA 1. *The differential system* (1.3) *with* $f \in S$ *has a* T*-periodic solution if and only if* $Qf = 0$.

Proof. It is well known that the unique solution $y(t)$ of (1.3) satisfying $y(0) = y_0$ is given by

$$y(t) = X(t) \cdot X^{-1}(0) \cdot y_0 + \int_0^t X(t) \cdot X^{-1}(s) \cdot f(s) ds . \tag{2.7}$$

Let $\beta = X^{-1}(0) \cdot y_0$. The solution $y(t)$ is T-periodic if and only if $y(0) = y(T)$, and from (2.7) this is equivalent to the statement that the matrix equation

$$L \cdot \beta = \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s) ds \tag{2.8}$$

can be solved for β. Suppose first that such a vector β has been

found. The matrix U was defined in such a way that $U \cdot L = 0$. Multiplying (2.8) by U and making use of (2.5) we see that $Qf = 0$.

Conversely suppose $Qf = 0$. It follows from (2.5) that for each column u_i of the matrix U' we have

$$\left\langle u_i, \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s) ds \right\rangle = 0 .$$

Thus for each vector $x \in N(L')$ we have

$$\left\langle x, \int_0^T X(T) \cdot X^{-1}(s) \cdot f(s) ds \right\rangle = 0 .$$

From the compatibility theorem of linear algebra this guarantees the existence of some solution of (2.8).

For an arbitrary linear equation $L \cdot \beta = \gamma$, it is known that one can find a nonsingular $n \times n$ matrix M, called a pseudoinverse of L, such that $L \cdot M \cdot L = L$ and for each $\gamma \in R(L)$ we have $L \cdot (M \cdot \gamma) = \gamma$. Pseudo inverses are constructed from the given matrix L by means of elementary row operations (see [3], [4]). The following corollary is an immediate consequence of this observation.

COROLLARY. *If $f \in S$ and $Qf = 0$, then* (2.8) *has a solution of the form $\beta = \int_0^T P(s) \cdot f(s) ds$, where $P(s) = M \cdot X(T) \cdot X^{-1}(s)$ and M is a nonsingular pseudo inverse of L.*

LEMMA 2. *Consider* (1.3) *and suppose $f \in S$ and $Qf = 0$. Then there exists an $n \times n$ matrix $G(t, s)$ which is independent of the function f and continuous in t and s except at $t = s$ where it has a finite jump, and such that*

(i) *The function $y_0(t) = \int_0^T G(t, s) \cdot f(s) ds$ is a T-periodic solution of* (1.3) *and moreover is the only T-periodic solution satisfying $Py_0 = 0$.*

(ii) *The columns of $\Psi(s)$ are orthogonal to the rows of $G(t, s)$ for each t; i.e.,*

$$\int_0^T G(t, s)^0 \cdot \Psi(s) ds = 0 . \tag{2.9}$$

Proof. Since $Qf = 0$, (2.8) has a solution β of the form indicated in the preceding corollary. Substituting β into (2.7) we get the equation

$$y(t) = \int_0^t X(t) \cdot [P(s) + X^{-1}(s)] \cdot f(s) ds + \int_t^T X(t) \cdot P(s) \cdot f(s) ds ,$$

or

$$y(t) = \int_0^T K(t, s)\cdot f(s)ds$$

$$K(t, s) = X(t)\cdot[P(s) + X^{-1}(s)] \qquad 0 \leqq s \leqq t \tag{2.10}$$

$$= X(t)\cdot P(s) \qquad t \leqq s \leqq T$$

for the unique T-periodic solution of (1.3) satisfying $y(0) = y_0$. Thus any T-periodic solution of (1.3) will be in the form

$$y(t) = \Phi(t)\cdot c + \int_0^T K(t, s)\cdot f(s)ds \tag{2.11}$$

for a suitable choice of $c \in R^k$.

It is easy to see that at most one T-periodic solution y_0 of (1.3) can satisfy $Py_0 = 0$. For if $y^* \in S$ is another solution with $Py^* = 0$, then $y = y_0 - y^* \in S$ and solves (1.2). Thus $y \in S_1$ and so $Py = y$. On the other hand we have $Py = Py_0 - Py^* = 0$, which implies $y_0 = y^*$.

To verify (i), we first observe that the condition $Py = 0$ is equivalent to the equation

$$\int_0^T \Phi'(t)\cdot y(t)dt = 0\ . \tag{2.12}$$

We want to find $c \in R^k$ such that both (2.11) and (2.12) are satisfied. For this it is necessary and sufficient that

$$0 = \frac{1}{T}\int_0^T \Phi'(t)\cdot\Phi(t)\cdot c dt + \frac{1}{T}\int_0^T \Phi'(t)\cdot\left[\int_0^T K(t, s)\cdot f(s)ds\right]dt\ .$$

Since $\{\varphi_i(t)\}$ forms an orthonormal basis,

$$\frac{1}{T}\int_0^T \Phi'(t)\cdot\Phi(t)dt = I(k \times k)$$

and from $\Phi(t) = X(t)\cdot B$ we find

$$c = -\frac{1}{T}\int_0^T B'\cdot\left[\int_0^T X'(T)\cdot K(t, s)dt\right]\cdot f(s)ds\ . \tag{2.13}$$

Thus the vector c is completely determined by an equation of the form

$$c = \int_0^T \mathscr{L}(s)\cdot f(s)ds \tag{2.14}$$

where $\mathscr{L}(s)$ for a given $K(t, s)$ is a continuous $k \times n$ matrix function of s, independent of f, given explicitly by (2.13). Substituting (2.14) into (2.11) and introducing the function

$$G^*(t, s) = \Phi(t)\cdot\mathscr{L}(s) + K(t, s) \tag{2.15}$$

we find the T-periodic solution of (1.3) satisfying $Py_0 = 0$ to be

$$y_0(t) = \int_0^T G^*(t, s) \cdot f(s) ds \,. \tag{2.16}$$

From (2.5), however, we see that for any $n \times k$ matrix function $R(t)$ the solution $y_0(t)$ in (2.16) can also be written as

$$y_0(t) = \int_0^T [G^*(t, s) - R(t) \cdot \Psi'(s)] \cdot f(s) ds \,. \tag{2.17}$$

Choosing in particular the matrix

$$R(t) = \frac{1}{T} \int_0^T G^*(t, s) \cdot \Psi(s) ds$$

and recalling that $\{\psi_i(t)\}$ forms an orthonormal basis, we see that (2.9) is satisfied for

$$G(t, s) = G^*(t, s) - R(t) \cdot \Psi'(s) \,. \tag{2.18}$$

Finally, it is clear that $G(t, s)$ has the specified smoothness properties.

COROLLARY. *Let $f \in S$ and $Qf = 0$. The unique solution $y_0(t)$ of* (1.3) *with $Py_0 = 0$ satisfies the inequality*

$$V(y_0) \leqq K \cdot V(f) \tag{2.18}$$

for $K = T \cdot \sup \{|| G(t, s) ||: 0 \leqq s, t \leqq T\}$.

LEMMA 3. *The matrix $G(t, s)$ given above is uniquely determined by its properties specified in the statement of Lemma* 2.

Proof. Let $g(s)$ be any element of S. Set $f(s) = g(s) - Qg(s)$. Then $Qf = Qg - Q^2g = 0$ and so we may apply Lemma 1. From (2.6) and (2.9), however, we find that

$$\int_0^T G(t, s) \cdot f(s) ds = \int_0^T G(t, s) \cdot g(s) ds \tag{2.19}$$

and so the unique solution $y_0(t)$ satisfying $Py_0 = 0$ is given by

$$y_0(t) = \int_0^T G(t, s) \cdot g(s) ds \,.$$

If $G_1(t, s)$ were a second matrix with the properties given in Lemma 2, then the same argument shows that

$$y_0(t) = \int_0^T G_1(t, s) \cdot g(s) ds$$

as well. From

$$\int_0^T [G(t, s) - G_1(t, s)] \cdot g(s) ds = 0$$

it follows immediately that $G_1(t, s) = G(t, s)$.

REMARKS. The subspaces S_1 and S_2 were introduced by Hale in [5]. The proofs of Lemma 1, 2 and 3 are modeled on the treatment in [6]. A discussion of these ideas employing the Moore-Penrose notion of generalized inverse can be found in [7]. It turns out that the estimate (2.18) is the essential information required for the iteration scheme which we construct in the next section. We note, however, that the existence of the function $G(t, s)$ has been established by constructive methods which in principle make it possible to write out explicitly every T-periodic solution of (1.3) for arbitrary $f \in S$.

3. In this section we indicate how the above analysis can be applied to the question of the existence of periodic solutions of (1.1) for $\varepsilon \neq 0$. Our treatment is a modification of the analysis in Chapters 6 and 11 of [5], and the reader is referred there for details and further results.

Let $a^*(t)$ be any T-periodic solution of (1.2). For some $a \in R^k$ we have $Pa^*(t) = a^*(t) = \Phi(t) \cdot a$. For given constants b, d with $0 < b < d < R$ and $a^*(t) \in S_1$ we define $S_0 \subset S$ as

$$S_0 = \{f \in S: Pf(t) = a^*(t);\ V(f) \leqq d\} .$$

THEOREM 1. *Given a constant b, $0 < b \leqq R$, there exists an $\varepsilon_1 > 0$ such that corresponding to each k-vector function $a^*(t) \in S_1$ with $a^*(t) = \Phi(t) \cdot a$ and $V(a^*) \leqq b$ and to each ε with $|\varepsilon| \leqq \varepsilon_1$, there is a unique function $y^*(t) = y^*(t, a, \varepsilon) \in S_0$ which has a continuous first derivative with respect to t, satisfies $y^*(t, a, 0) = a^*(t)$ and*

$$\dot{y}^* = A(t) \cdot y^* + \varepsilon \cdot [q(t, y^*, \varepsilon) - Qq(t, y^*, \varepsilon)] . \tag{3.1}$$

Finally, the function $y^*(t)$ may be obtained by the iteration procedure in (3.4) below.

Proof. Since the function $G(t, s)$ constructed in Lemma 2 is independent of $f \in S$, it makes sense to define an operator $\mathscr{T}$ for $y = y(t, a, \varepsilon)$ in S_0 as

$$w = \mathscr{T} y = a^*(t) + \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y(s, a, \varepsilon), \varepsilon) ds . \tag{3.2}$$

As in the proof of Lemma 3 we see that the function $Z(t)$ defined by

$$Z(t) = \varepsilon \cdot \int_0^T G(t, s) \cdot q(s, y(s, a, \varepsilon), \varepsilon) ds$$

satisfies $\dot{Z} = A(t)\cdot Z + \varepsilon\cdot[q(t, y, \varepsilon) - Qq(t, y, \varepsilon)]$ and also $PZ = 0$. Thus we get $Pw(t) = Pa^*(t) + PZ(t) = a^*(t)$.

Without difficulty one can show that $\mathscr{T}$ is a contraction map in S_0 for $0 \leqq |\varepsilon| \leqq \varepsilon_1$ and ε_1 sufficiently small. Thus $\mathscr{T}$ has a unique fixed point $y^*(t) = y^*(t, a, \varepsilon)$ given by

$$y^*(t) = a^*(t) + \varepsilon\cdot\int_0^T G(t, s)\cdot q(s, y^*(s, a, \varepsilon), \varepsilon)ds\ . \tag{3.3}$$

Again from (2.6) and (2.9) we see that $y^*(t)$ is a solution of (3.1). The contraction mapping principle states that the solution $y^*(t)$ can be obtained as the limit of a sequence $\{y^k(t)\}$ defined by

$$y_0(t) = a^*(t) = \Phi(t)\cdot a$$

$$y^{k+1}(t) = \mathscr{T}y^k(t) = a^*(t) + \varepsilon\cdot\int_0^T G(t, s)\cdot q(s, y^k(s), \varepsilon)ds\ . \tag{3.4}$$

From (3.4) we conclude $y(t, a, 0) = a^*(t)$. To prove uniqueness, let $y_1 = y_1(t, \varepsilon)$ be another solution of (3.1) in S_0 for $|\varepsilon| \leqq \varepsilon_1$. Let us define $w(t) = y_1(t) - a^*(t)$. Then w is a T-periodic solution of

$$\dot{w} = A(t)\cdot w + \varepsilon\cdot[q(t, y_1, \varepsilon) - Qq(t, y_1, \varepsilon)]$$

satisfying $Pw = 0$. It follows now from (2.6) and Lemma 2 that y_1 is a fixed point of $\mathscr{T}$, which can only mean that $y_1 = y^*$.

REMARK. Consider the special case when $A(t)$ is a constant matrix of the form diag $(0, B)$, where B is an $(n-k)\times(n-k)$ matrix such that $\dot{X} = B\cdot X$ has no T-periodic solution $X(t)$. Thus (1.2) has precisely k linearly independent constant solutions, and so we may choose an orthonormal basis of S_1 to be the first k unit n-vectors in R^n. Then $\Phi(t)$ is the $n\times k$ matrix whose only nonzero terms are k 1's along the diagonal. Clearly we may choose the same basis for S_2, and so $\Psi(t) = \Phi(t)$. Thus $Pf = Qf$ for all $f \in S$ and the spaces S_1 and S_2 coincide. This is the case treated in Chapter 6 of [5].

THEOREM 2. *Let $y^*(t) = y^*(t, a, \varepsilon)$ be the solution of* (3.1) *defined for all $a \in R^k$ and ε such that $0 \leqq |\varepsilon| \leqq \varepsilon_1$ and $V(a^*) = V(\Phi(t)\cdot a) \leqq b$. If there exists an $\varepsilon_2 \leqq \varepsilon_1$ and a continuous function $a = a(\varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$ such that*

$$Qq(t, y^*(t, a(\varepsilon), \varepsilon), \varepsilon) = 0 \tag{3.5}$$

then $y^*(t, a(\varepsilon), \varepsilon)$ is a T-periodic solution of (1.1) *for $|\varepsilon| \leqq \varepsilon_2$. Conversely, suppose* (1.1) *has a T-periodic solution $\bar{y} = \bar{y}(t, \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2 \leqq \varepsilon_1$, continuous in ε with $V(P\bar{y}) < R$. Then for some continuous function $a(\varepsilon)$ satisfying* (3.5) *we have $\bar{y}(t, \varepsilon) = y^*(t, a(\varepsilon), \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$*

In this sense we say that the existence of a continuous function $a(\varepsilon)$ *satisfying* (3.5) *is a necessary and sufficient condition for the existence of a periodic solution of* (1.1).

Proof. The sufficiency of the condition is obvious. Conversely, given $\bar{y} = \bar{y}(t, \varepsilon)$ define $a(\varepsilon)' = ((\bar{y}, \varphi_1), (\bar{y}, \varphi_2), \cdots, (\bar{y}, \varphi_k))$. Then for $a^*(t, \varepsilon) = \Phi(t) \cdot a(\varepsilon)$ we have $P\bar{y}(t, \varepsilon) = a^*(t, \varepsilon)$. Clearly $a(\varepsilon)$ is continuous and by assumption $V(P\bar{y}) = VP(\bar{y}(\cdot, \varepsilon)) = V(a^*(\cdot, \varepsilon)) < R$. Thus for some b, d with $0 < b < d < R$ we have $V(a^*(\cdot, \varepsilon)) \leqq b < R$ for $0 \leqq |\varepsilon| \leqq \varepsilon_2$. Since $\bar{y}$ is a solution of (1.1) we obviously have

$$(I - Q)[\dot{\bar{y}} - A(t) \cdot \bar{y} - \varepsilon \cdot q(t, \bar{y}, \varepsilon)] = 0$$
$$Q[\dot{\bar{y}} - A(t) \cdot \bar{y} - \varepsilon \cdot q(t, \bar{y}, \varepsilon)] = 0 .$$

Since $\bar{y}$ is assumed known we can regard it as a solution of an equation of the form (1.3). From Lemma 1 this implies $Qq(t, \bar{y}(t, \varepsilon), \varepsilon) = 0$. Thus the above equations reduce to

$$\dot{\bar{y}} = A(t) \cdot \bar{y} + \varepsilon \cdot [q(t, \bar{y}, \varepsilon) - Qq(t, \bar{y}, \varepsilon)] \tag{3.6}$$

$$Qq(t, \bar{y}, \varepsilon) = 0 \tag{3.7}$$

for $0 \leqq |\varepsilon| \leqq \varepsilon_2$. From Theorem 1, however, the function $y^*(t, a(\varepsilon), \varepsilon)$ is the only T-periodic solution of (3.6) satisfying

$$P(y^*(t, a(\varepsilon), \varepsilon)) = a^*(t, \varepsilon)$$

for $|\varepsilon| \leqq \varepsilon_1$. Since $\varepsilon_2 \leqq \varepsilon_1$, from (3.6) we conclude $\bar{y}(t, \varepsilon) = y^*(t, a(\varepsilon), \varepsilon)$ for $|\varepsilon| \leqq \varepsilon_2$ and from (3.7) we see that $a(\varepsilon)$ satisfies (3.5).

REMARK. Condition (3.5) is most easily verified, of course, when the implicit function theorem can be applied in the manner of Chapter 6 of [5]. If this fails, then one can use (3.4) to find approximate determining equations for (1.1) involving the successive approximations $y_0, y^1, \cdots y^k$, as done in Chapter 7 of [5]. One advantage of the present methods is that these higher order equations can be explicitly written out, due to the constructive method used to define the kernel $G(t, s)$ of the integral operator $\mathscr{T}$.

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ANALYTIC TWO-DIMENSIONAL SUBCENTER MANIFOLDS FOR SYSTEMS WITH AN INTEGRAL

AL KELLEY

For a real analytic system of ordinary differential equations with an integral H

$$H = \tfrac{1}{2}\lambda(x^2 + y^2) + F(z) + G(x, y, z)$$
$$\dot{x} = \lambda y + X(x, y, z)$$
$$\dot{y} = -\lambda x + Y(x, y, z)$$
$$\dot{z} = Bz + Z(x, y, z)$$

where x and y are scalars; z is an m-vector; X, Y, Z are power series with no constant or linear terms; B is a constant matrix with eigenvalues $\mu_1, \cdots, \mu_m$ and $i\lambda^{-1}\mu_j \neq$ integer $(j = 1, \cdots, m)$; the existence of a unique, local, real analytic, two-dimensional, invariant subcenter manifold

$$M^\lambda = \{(x, y, z) \mid |x| + |y| < \delta, z = w(x, y)\}$$

is proved, where w is a real analytic function with no constant or linear terms in its expansion about the origin. The manifold M^λ is composed of a nested, one-parameter $(\rho \geqq 0)$ family of periodic orbits, and as $\rho \to 0$ the corresponding periodic orbit goes to the origin and its period goes to $2\pi\,|\,\lambda^{-1}\,|$.

In this paper we extend the result of C. L. Siegel [7] to analytic systems of ordinary differential equations with an integral (see (1) below). C. L. Siegel proved Theorem 1 below with the additional restrictions that system (1) is Hamiltonian and that the matrix B in (1) is a diagonal matrix. The removal of these two restrictions is a technical improvement; our emphasis is on the invariant manifold aspect of the theory (see Theorem 2 below). As far as proving the existence of a local, nested, one-parameter family of periodic orbits is concerned, the theorem proved by C. L. Siegel [7] is equivalent to a result of Liapounov [5] proved before the turn of the century. However, in addition C. L. Siegel proved that the invariant, two-dimensional manifold comprising the nested, one-parameter family of periodic orbits is an analytic manifold, and that by means of an analytic change of variables, the differential equations on the invariant manifold have a very simple form. Using the proof method of Liapounov, one is able to conclude only that the invariant manifold of periodic orbits is continuous. (See A. Kelley [3] for a C^1 version of these ideas.) For real analytic systems with an integral we prove in § 4 below that the invariant manifold is directly computable from the differential equations, and hence unique and real analytic. Since

our manifold is imbedded in the center manifold (see A. Kelley [4]), we call it a subcenter manifold.

The convergence proof in Theorem 1 below is based on techniques developed in C. L. Siegel [7], J. Moser [6], and A. Kelley [1], [2].

2. **Notation.** Let $z = (z_1, \cdots, z_p)$ be a complex vector and let q be a nonnegative integer. We define $\Gamma^q(z)$ to be the class of functions $\{F\}$ which satisfy (i), (ii), (iii) below.

(i) $F = F(z)$ is a complex scalar valued function, vector valued function, or matrix valued function (dim F unspecified) defined for z in some neighborhood of $z = 0$.

(ii) F has a convergent power series expansion in z (components of z) about the point $z = 0$; the coefficients in the expansion are either scalars, vectors, or matrices (as the case may be); there are no terms of degree less than q in the expansion of F.

(iii) For some $K > 0$ the components of F are majorized by

$$K(z_1 + \cdots + z_p)^q[1 - K(z_1 + \cdots + z_p)]^{-1} .$$

The constant K depends on F.

Primes in this paper either designate majorants or designate variables and functions in a complex setting as contrasted to a real setting. Given $F(z) \in \Gamma^q(z)$ there exists $F'(z)$ such that

$$F(z) \ll F'(z)$$

where the symbol $\ll$ means F and F' belong to the same Γ class, $\dim F = \dim F'$, and F is majorized componentwise by F' in the sense of Cauchy.

The norm $|\cdot|$ is the euclidean norm on vectors and the operator norm on matrices. If $F = F(z)$ is a vector valued function of the vector z, then F_z represents the usual Jacobian matrix of partial derivatives.

If $F \in \Gamma^0(x, y)$ where x and y are scalar variables, then $[F]_{\alpha,\beta}$ represents the coefficient of the $x^\alpha y^\beta$ term in the expansion of F.

Let $F \in \Gamma^0(x, y)$ be a vector valued, real analytic function of the scalar variables x and y, and let

$$F(x, y) = \sum_{\alpha,\beta=0}^{\infty} F_{\alpha,\beta} x^\alpha y^\beta .$$

In § 4 below we will use the notation

$$F_n = [F]_n = (F_{n,0}, F_{n-1,1}, \cdots, F_{0,n}) .$$

Thus F_n is a vector in $R^{(n+1)\dim F}$.

Throughout this paper $R = R(p, q)$ is used to designate an

anylytic function, although occassionally (as above) the notation $z \in R^n$ is used to indicate that z is an n-vector. The distinction will always be clear from the context.

3. **Analytic changes of variables.** Consider the complex analytic system of ordinary differential equations with an integral H

$$
\begin{aligned}
H(x, y, z) &= i\lambda xy + F(z) + G(x, y, z) \\
\dot{x} &= i\lambda x + X(x, y, z) \\
\dot{y} &= -i\lambda y + Y(x, y, z) \\
\dot{z} &= Bz + Z(x, y, z)
\end{aligned} \tag{1}
$$

where x and y are complex variables; z is a complex m-vector variable; λ is a nonzero complex number; (in the most interesting case λ is real, see (19) below for the real counterpart to system (1)); B is a constant matrix in Jordan canonical form; $B = B^0 + B^1$,

$$
\begin{aligned}
B^0 &= \operatorname{diag}(\mu_1, \cdots, \mu_m) \\
B^1 &= \operatorname{subdiag}(\delta_1, \cdots, \delta_{m-1})
\end{aligned}
$$

with $\delta_j = 0$ or $\delta, \delta > 0$ sufficiently small (to be specified below); $X, Y, Z \in \Gamma^2(x, y, z)$; $\dim X = \dim x$, etc.; F is a quadratic form in the components of z ($F \equiv 0$ is allowed); $G \in \Gamma^3(x, y, z)$ (which implies that $H \in \Gamma^2(x, y, z)$); H is an integral for system (1).

THEOREM 1. *If $i\lambda^{-1}\mu_j \neq$ integer $(j = 1, \cdots, m)$, then for system* (1) *there exists a change of variables (not unique)*

$$
\begin{aligned}
x &= p + P(p, q) \\
y &= q + Q(p, q) \\
z &= r + R(p, q)
\end{aligned} \tag{2}
$$

such that

$$
\begin{aligned}
\dot{p} &= i\lambda(1 + a(pq))p + \tilde{P}(p, q, r) \\
\dot{q} &= -i\lambda(1 + a(pq))q + \tilde{Q}(p, q, r) \\
\dot{r} &= Br + \tilde{R}(p, q, r) \\
\tilde{P}, \tilde{Q}, \tilde{R} &\equiv 0 \text{ when } r = 0 ,
\end{aligned} \tag{3}
$$

where $a = a(pq)$ is a function of the product pq; $a(\omega) \in \Gamma^1(\omega)$ (which implies $a(pq) \in \Gamma^2(p, q)$); $P, Q, R \in \Gamma^2(p, q)$; $\tilde{P}, \tilde{Q}, \tilde{R} \in \Gamma^2(p, q, r)$.

Proof. First we construct a formal change of variables (2) such that

$$
\begin{aligned}
\dot{p} &= i\lambda(1 + a(pq)) + \tilde{P}(p, q, r) \\
\dot{q} &= -i\lambda(1 + \tilde{a}(pq)) + \tilde{Q}(p, q, r) \\
\dot{r} &= Br + \tilde{R}(p, q, r) \\
\tilde{P}, \tilde{Q}, \tilde{R} &\equiv 0 \text{ when } r = 0 ,
\end{aligned} \tag{4}
$$

where a and $\tilde{a}$ are formal power series in the product pq with no constant terms. Then we will use the fact that H is an integral for system (1) to prove that $\tilde{a} = a$ as formal power series. Finally, convergence will be proved using the Cauchy method of majorants.

If such a change of variables (2) exists, then from (1), (2), (4) we obtain

$$
\begin{aligned}
\dot{P} &= -i\lambda ap + i\lambda P + X(p + P, q + Q, r + R) - \tilde{P}(p, q, r) \\
\dot{Q} &= i\lambda \tilde{a} q - i\lambda Q + Y(p + P, q + Q, r + R) - \tilde{Q}(p, q, r) \\
\dot{R} &= BR + Z(p + P, q + Q, r + R) - \tilde{R}(p, q, r)
\end{aligned} \tag{5}
$$

where $P = P(p, q)$, etc. Taking the derivative on the left side of (5) implicity and using the condition $\tilde{P}, \tilde{Q}, \tilde{R} \equiv 0$ when $r = 0$, we obtain

$$
\begin{aligned}
& pP_p - qP_q - P \\
&= -ap - apP_p + \tilde{a}qP_q - i\lambda^{-1}X(p + P, q + Q, R) \\
& pQ_p - qQ_q + Q \\
&= \tilde{a}q - apQ_p + \tilde{a}qQ_q - i\lambda^{-1}Y(p + P, q + Q, R) \\
& p\frac{\partial}{\partial p}R_j - q\frac{\partial}{\partial q}R_j + i\lambda^{-1}\mu_j R_j \\
&= -i\lambda^{-1}\delta_{j-1}R_{j-1} - ap\frac{\partial}{\partial p}R_j + \tilde{a}q\frac{\partial}{\partial q}R_j - i\lambda^{-1}Z_j(p + P, q + Q, R) \\
& \qquad\qquad (j = 1, \cdots, m)
\end{aligned} \tag{6}
$$

where $\delta_0 = 0$. From (6) the coefficients of $P, Q, R, a, \tilde{a}$ can be computed recursively. Let

$$
P(p, q) = \sum_{\alpha+\beta\geq 2} P_{\alpha,\beta}p^\alpha q^\beta ,
$$

etc. From (6) we have

$$
\begin{aligned}
\sum(\alpha - \beta - 1)P_{\alpha,\beta}p^\alpha q^\beta &= -\sum a_{\beta,\beta}p^{\beta+1}q^\beta + \cdots \\
\sum(\alpha - \beta + 1)Q_{\alpha,\beta}p^\alpha q^\beta &= \sum \tilde{a}_{\alpha,\alpha}p^\alpha q^{\alpha+1} + \cdots \\
\sum(\alpha - \beta + i\lambda^{-1}\mu_j)R_{j,\alpha,\beta} &= \cdots \qquad (j = 1, \cdots, m) .
\end{aligned} \tag{7}
$$

If we equate corresponding coefficients on the left and right side of (6) (equivalently (7)), we obtain

$$
\begin{aligned}
&(\alpha-\beta-1)P_{\alpha,\beta}\\
&=[-ap-apP_p+\tilde{a}qP_q-i\lambda^{-1}X(p+P,q+Q,R)]_{\alpha,\beta}\\
&(\alpha-\beta+1)Q_{\alpha,\beta}\\
&=[\tilde{a}q-apQ_p+\tilde{a}qQ_q-i\lambda^{-1}Y(p+P,q+Q,R)]_{\alpha,\beta}\\
&(\alpha-\beta+i\lambda^{-1}\mu_j)R_{j,\alpha,\beta}\\
&=\left[-i\lambda^{-1}\delta_{j-1}R_{j-1}-ap\frac{\partial}{\partial p}R_j+\tilde{a}q\frac{\partial}{\partial q}R_j-i\lambda^{-1}Z_j(p+P,q+Q,R)\right]_{\alpha,\beta}.
\end{aligned}
\tag{8}
$$

Clearly, the expressions on the right side of (8) involve only the coefficients of P, Q, R of order less than $\alpha+\beta$ and the coefficient of $R_1, \cdots, R_{j-1}$ of order less than or equal to $\alpha+\beta$. When $\alpha=\beta+1$, then $a_{\beta,\beta}$ must be chosen so that the right side of the first equation in (8) vanishes. Similarly when $\beta=\alpha+1$, then $\tilde{a}_{\alpha,\alpha}$ must be chosen so that the right side of the second equation in (8) vanishes. By hypothesis $i\lambda^{-1}\mu_j \neq$ integer $(j=1,\cdots,m)$. Therefore

$$
\alpha-\beta+i\lambda^{-1}\mu_j \neq 0 \qquad (j=1,\cdots,m)
$$

and all the coefficients of R can be computed recursively from (8). All the coefficients of P and Q can be computed recursively, except $P_{\beta+1,\beta}$ and $Q_{\alpha,\alpha+1}$ $(\alpha,\beta=1,2,\cdots)$ which can be chosen arbitrarily. Let

$$
\hat{P}(p,q)=\sum_{\beta=1}^{\infty}\hat{P}_{\beta+1,\beta}p^{\beta+1}q^{\beta}
$$

$$
\hat{Q}(p,q)=\sum_{\alpha=1}^{\infty}\hat{Q}_{\alpha,\alpha+1}p^{\alpha}q^{\alpha+1}
$$

by any given convergent power series (for example $\hat{P},\hat{Q}\equiv 0$), and define

$$
\begin{aligned}
P_{\beta+1,\beta}&=\hat{P}_{\beta+1,\beta}\\
Q_{\alpha,\alpha+1}&=\hat{Q}_{\alpha,\alpha+1} \qquad (\alpha,\beta=1,2,\cdots).
\end{aligned}
\tag{9}
$$

Now, from (8) and (9) the coefficients of $P, Q, R, a, \tilde{a}$ can be computed recursively.

Next we want to show that a and $\tilde{a}$ are equal as formal power series. Since H is an integral for (1),

$$
\tilde{H}(p,q)=H(p+P,q+Q,R)
$$

is a (formal) integral for system (3) restricted to the manifold given by $r=0$; namely for the system

$$
\begin{aligned}
\dot{p}&=i\lambda(1+a)p\\
\dot{q}&=-i\lambda(1+\tilde{a})q\,.
\end{aligned}
$$

Therefore, operating only in the sense of formal power series, we obtain

$$\frac{d}{dt}\tilde{H} = \tilde{H}_p i\lambda(1 + a)p - \tilde{H}_q i\lambda(1 + \tilde{a})q = 0 ,$$

or equivalently

$$p\tilde{H}_p - q\tilde{H}_q = -ap\tilde{H}_p + \tilde{a}q\tilde{H}_q . \tag{10}$$

Let

$$\tilde{H}(p, q) = \sum_{\alpha+\beta\geq 2} \tilde{H}_{\alpha,\beta} p^\alpha q^\beta .$$

Equating coefficients in (10), we have

$$(\alpha - \beta)\tilde{H}_{\alpha,\beta} = [-ap\tilde{H}_p + \tilde{a}q\tilde{H}_q]_{\alpha,\beta} . \tag{11}$$

Since $a = a(pq)$ and $\tilde{a} = \tilde{a}(pq)$ are power series in the product pq, it follows from (11) by induction that $\tilde{H}_{\alpha,\beta} = 0$ for all $\alpha \neq \beta$. Therefore $\tilde{H}$ is a (formal) power series in the product pq, and this implies

$$p\tilde{H}_p = q\tilde{H}_q .$$

Hence from (10)

$$(a - \tilde{a})p\tilde{H}_p = 0 .$$

Since

$$p\tilde{H}_p = i\lambda pq + \cdots$$

is a nontrivial power series, we conclude that as formal power series

$$a(pq) = \tilde{a}(pq) .$$

Equation (6) can now be written

$$\begin{aligned} &pP_p - qP_q - P = -ap - a\{pP_p - qP_q\} - i\lambda^{-1}X(p + P, q + Q, R) \\ &pQp - qQ_q + Q = aq - a\{pQ_p - qQ_q\} - i\lambda^{-1}Y(p + P, q + Q, R) \\ &pR_p - qR_q + i\lambda^{-1}B^0R \\ &= -i\lambda^{-1}B^1R - a\{pR_p - qR_q\} - i\lambda^{-1}Z(p + P, q + Q, R) . \end{aligned} \tag{12}$$

Let $X', Y', Z', \hat{P}', \hat{Q}', a'$ be the smallest majorants of

$$i\lambda^{-1}X, i\lambda^{-1}Y, i\lambda^{-1}Z, \hat{P}, \hat{Q}, a$$

respectively. Thus

$$X', Y', Z' \in \Gamma^2(x, y, z); \ \hat{P}', \hat{Q}' \in \Gamma^2(p, q) .$$

The power series a' must still be shown to be convergent. Define

$$
\begin{aligned}
P'(p,q) &= \sum(|\alpha-\beta|+\delta_{\alpha\beta})\,|P_{\alpha,\beta}|\,p^\alpha q^\beta \\
Q'(p,q) &= \sum(|\alpha-\beta|+\delta_{\alpha\beta})\,|Q_{\alpha,\beta}|\,p^\alpha q^\beta \\
R'_j(p,q) &= \sum(|\alpha-\beta|+\delta_{\alpha\beta})\,|R_{j,\alpha,\beta}|\,p^\alpha q^\beta \qquad (j=1,\cdots,m) \\
R' &= (R'_1,\cdots,R'_m)
\end{aligned} \tag{13}
$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. Clearly

$$P \ll P';\quad Q \ll Q';\quad R \ll R'\,.$$

Choose $K>0$ such that for all nonnegative integers α and β (except as noted)

$$
\begin{aligned}
&|\alpha-\beta-1|^{-1}(|\alpha-\beta|+\delta_{\alpha\beta}) \leqq K \qquad (\alpha-\beta-1\neq 0) \\
&|\alpha-\beta+1|^{-1}(|\alpha-\beta|+\delta_{\alpha\beta}) \leqq K \qquad (\alpha-\beta+1\neq 0) \\
&|\alpha-\beta+i\lambda^{-1}\mu_j|^{-1}(|\alpha-\beta|+\delta_{\alpha\beta}) \leqq K \qquad (j=1,\cdots,m)\,.
\end{aligned} \tag{14}
$$

From (12), (13), (14) it follows that

$$
\begin{aligned}
P' &\ll K[a'P' + X'(p+P', q+Q', R')] + \hat{P}' \\
Q' &\ll K[a'Q' + Y'(p+P', q+Q', R')] + \hat{Q}' \\
R' &\ll K[|\lambda^{-1}|\,B^1R' + a'R' + Z'(p+P', q+Q', R')] \\
a'p &\ll a'P' + X'(p+P', q+Q', R')\,.
\end{aligned} \tag{15}
$$

Since P', Q', R', a' all have real, nonnegative coefficients, it is sufficient to prove convergence when $p=q=\zeta$. Let

$$
\begin{aligned}
&P'(\zeta,\zeta)=\zeta P''(\zeta);\ Q'(\zeta,\zeta)=\zeta Q''(\zeta);\ R'(\zeta,\zeta)=\zeta R''(\zeta); \\
&\hat{P}'(\zeta,\zeta)=\zeta\hat{P}''(\zeta);\ \hat{Q}'(\zeta,\zeta)=\zeta\hat{Q}''(\zeta);\ a'(\zeta^2)=a''(\zeta)\,.
\end{aligned}
$$

From (15) we have

$$
\begin{aligned}
P'' &\ll K[a''P'' + \zeta^{-1}X'(\zeta+\zeta P'', \zeta+\zeta Q'', \zeta R'')] + \hat{P}'' \\
Q'' &\ll K[a''Q'' + \zeta^{-1}Y'(\zeta+\zeta P'', \zeta+\zeta Q'', \zeta R'')] + \hat{Q}'' \\
R'' &\ll K[|\lambda^{-1}|\,B^1R'' + a''R'' + \zeta^{-1}Z'(\zeta+\zeta P'', \zeta+\zeta Q'', \zeta R'')] \\
a'' &\ll a''P'' + \zeta^{-1}X'(\zeta+\zeta P'', \zeta+\zeta Q'', \zeta R'')\,.
\end{aligned} \tag{16}
$$

Define formal power series $P^*=P^*(\zeta)$, $P^*(0)=0$, etc., by the functional equations

$$
\begin{aligned}
P^* &= K[a^*P^* + \zeta^{-1}X'(\zeta+\zeta P^*, \zeta+\zeta Q^*, \zeta R^*)] + \hat{P}'' \\
Q^* &= K[a^*Q^* + \zeta^{-1}Y'(\zeta+\zeta P^*, \zeta+\zeta Q^*, \zeta R^*)] + \hat{Q}'' \\
R^* &= K[|\lambda^{-1}|\,B^1R^* + a^*R^* + \zeta^{-1}Z'(\zeta+\zeta P^*, \zeta+\zeta Q^*, \zeta R^*)] \\
a^* &= a^*P^* + \zeta^{-1}X'(\zeta+\zeta P^*, \zeta+\zeta Q^*, \zeta R^*)\,.
\end{aligned} \tag{17}
$$

Comparing (17) with (16) it follows that

$$P'' \ll P^*;\ Q'' \ll Q^*;\ R'' \ll R^*;\ a'' \ll a^* .$$

But the convergence of P^*, Q^*, R^*, a^* in (17) follows from the implicit function theorem provided δ in B^1 is chosen sufficiently small to insure that

$$K\,|\,\lambda^{-1}B^1\,| < 1 .$$

Thus from

$$P^*, Q^*, R^*, a^* \in \Gamma^1(\zeta)$$

we conclude that

$$P, Q, R, a \in \Gamma^2(p, q) .$$

To show that

$$\tilde{P}, \tilde{Q}, \tilde{R} \in \Gamma^2(p, q, r)$$

write equation (5) as

$$\begin{aligned}
\tilde{P} &= -i\lambda ap + i\lambda P + X(p + P, q + Q, r + R) \\
&\quad -P_p\{i\lambda(1 + a)p + \tilde{P}\} - P_q\{-i\lambda(1 + a)q + \tilde{Q}\} \\
\tilde{Q} &= i\lambda aq - i\lambda Q + Y(p + P, q + Q, r + R) \\
&\quad -Q_p\{i\lambda(1 + a)p + \tilde{P}\} - Q_q\{-i\lambda(1 + a)q + \tilde{Q}\} \\
\tilde{R} &= BR + Z(p + P, q + Q, r + R) \\
&\quad -R_p\{i\lambda(1 + a)p + \tilde{P}\} - R_q\{-i\lambda(1 + a)q + \tilde{Q}\} .
\end{aligned} \tag{18}$$

Since P, Q, R, a are known analytic functions of (p, q), we can regard (18) as an equation in the variables $p, q, r, \tilde{P}, \tilde{Q}, \tilde{R}$ and use the implicit function Theorem to solve for $\tilde{P} = \tilde{P}(p, q, r)$, etc. Clearly

$$\tilde{P}, \tilde{Q}, \tilde{R} \equiv 0 \text{ when } r = 0 .$$

This completes the proof of Theorem 1.

4. **Real analytic invariant manifolds.** For real analytic systems of equations with an integral, we want to show the existence of real analytic invariant manifolds. Theorem 1 will be the prime tool for this program.

Consider the *real* analytic system of ordinary differential equations with an integral H

$$\begin{aligned}
H(x, y, z) &= \frac{1}{2}\lambda(x^2 + y^2) + F(z) + G(x, y, z) \\
\dot{x} &= \lambda y + X(x, y, z) \\
\dot{y} &= -\lambda x + Y(x, y, z) \\
\dot{z} &= Bz + Z(x, y, z)
\end{aligned} \tag{19}$$

where x and y are real scalar variables; z is a real m-vector variable; λ is a nonzero real number; B is a real, constant matrix with eigenvalues $\mu_1, \cdots, \mu_m$; $X, Y, Z \in \Gamma^2(x, y, z)$ are real analytic; $\dim X = \dim x$, etc.; F is a real quadratic form in the components of z; $G \in \Gamma^3(x, y, z)$ is real analytic; H is a real analytic integral for (1).

THEOREM 2. *If $i\lambda^{-1}\mu_j \neq$ integer $(j = 1, \cdots, m)$, then for system* (19) *there exists a unique, local, real analytic, two-dimensional, invariant manifold*

$$M^\lambda = \{(x, y, z) \mid |x| + |y| < \delta, z = w(x, y)\}$$

where $w \in \Gamma^2(x, y)$ is real analytic and is directly computable (thus it is not necessary to reduce the differential equations to a normal form such as (38) *below in order to compute M^λ); moreover, the manifold M^λ is composed of a nested, one-parameter $(\rho \geqq 0)$ family of periodic orbits, and as $\rho \to 0$ the corresponding periodic orbit goes to the origin and its period goes to $2\pi |\lambda^{-1}|$.*

Proof. The second part of this theorem is relatively easy to prove. Suppose the real analytic function w which defines M^λ is known. Then

$$\tilde{H}(x, y) = \frac{1}{2}\lambda(x^2 + y^2) + F(w(x, y)) + G(x, y, w(x, y))$$

is a real analytic integral for (1) restricted to M^λ. Introduce

$$x = \rho(1 + b)\cos\theta$$
$$y = \rho(1 + b)\sin\theta$$

into the equation

$$\tilde{H}(x, y) = \frac{1}{2}\lambda\rho^2$$

to obtain

$$b + \frac{1}{2}b^2 + \rho^{-2}\lambda^{-1}(\tilde{F} + \tilde{G}) = 0 \tag{20}$$

where

$$\tilde{F} = \tilde{F}(\theta, \rho, b) = F(w(\rho(1 + b)\cos\theta, \rho(1 + b)\sin\theta))$$
$$\tilde{G} = \tilde{G}(\theta, \rho, b) = G(\rho(1 + b)\cos\theta, \cdots).$$

We observe that $\rho^{-2}\lambda^{-1}(\tilde{F} + \tilde{G})$ is well defined for $\rho = 0$. From (20) by means of the implicit function theorem we can solve for $b = b(\theta, \rho)$ with b having period 2π in θ and $b(\theta, 0) \equiv 0$, so that

$$x = \rho(1 + b(\theta, \rho)) \cos \theta$$
$$y = \rho(1 + b(\theta, \rho)) \sin \theta$$
$$z = w(\rho(1 + b(\theta, \rho)) \cos \theta, \rho(1 + b(\theta, \rho)) \sin \theta)$$

is a representation of the one-parameter ($\rho \geqq 0$) family of nested periodic orbits which comprise M^λ.

Continuing in this manner, it is not difficult to prove the assertion concerning the periods of these periodic orbits. See [3] for the details.

For a more complete analysis of the differential equations (19) restricted to M^λ, one can show the existence of a real analytic change of variables

$$x = p + P(p, q)$$
$$y = q + Q(p, q)$$
$$z = r + R(p, q)$$

such that on M^λ (given by $z = R(p, q)$)

$$\dot{p} = \lambda(1 + a(p^2 + q^2))q \tag{21}$$
$$\dot{q} = -\lambda(1 + a(p^2 + q^2))p$$

where $P, Q, R, a \in \Gamma^2(p, q)$ are real analytic; a is a real power series in the quadratic form $p^2 + q^2$. In this formulation the periodicity of the solutions of (21) is obvious (the circles $p^2 + q^2 = \rho^2$ in the (p, q)-plane are orbits) and the assertion concerning the periods of these solutions is immediate. The details of this analysis are given in § 5 below.

We now prove the first part of Theorem 2. Primes for the rest of this section will denote variables and functions in a complex setting as opposed to unprimed variables and functions in a real setting. Introduce the change of variables

$$x = \frac{1}{\sqrt{2}}(ix' + y')$$
$$y = \frac{1}{\sqrt{2}}(-x' - iy') \tag{22}$$
$$z = \tilde{B}z'$$

so that system (19) transforms to

$$H'(x', y', z') = i\lambda x'y' + F'(z') + G(x', y', z')$$
$$\dot{x}' = i\lambda x' + X'(x', y', z') \tag{23}$$
$$\dot{y}' = -i\lambda y' + Y'(x', y', z')$$
$$\dot{z}' = B'z' + Z'(x', y', z')$$

where

$$H'(x', y', z') = H\Big(\frac{1}{\sqrt{2}}(ix' + y'), \frac{1}{\sqrt{2}}(-x' - iy'), \tilde{B}z'\Big);$$

$B' = \tilde{B}^{-1}B\tilde{B}$ is in Jordan canonical form; X', Y', Z' are related to X, Y, Z in the usual way. Theorem 1 asserts the existence of an invariant manifold

$$\begin{aligned} M'^{\lambda} = \{(x', y', z') \mid x' &= p' + P'(p', q'), \\ y' &= q' + Q'(p', q'), \ z' = R'(p', q')\} \end{aligned} \tag{24}$$

where $P', Q', R' \in \Gamma^2(p', q')$. But the change of variables

$$\begin{aligned} x' &= p' + P'(p', q') \\ y' &= q' + Q'(p', q') \end{aligned}$$

can be inverted to give

$$\begin{aligned} p' &= p'(x', y') = x' + \cdots \\ q' &= q'(x', y') = y' + \cdots \end{aligned}$$

so that

$$M'^{\lambda} = \{(x', y', z') \mid |x'| + |y'| < \delta, z' = w'(x', y')\} \tag{25}$$

where

$$w'(x', y') = R'(p'(x', y'), q'(x', y')),$$

which implies $w' \in \Gamma^2(x', y')$. With the representation given by (24) it is not clear that M'^{λ} is unique, but with the representation (25) we can easily show uniqueness for M'^{λ}. Since the manifold is invariant,

$$\dot{w}' = B'w' + Z'(x', y', w'),$$

equivalently

$$\begin{aligned} &i\lambda x' w'_{x'} - i\lambda y' w'_{y'} - B'w' \\ &= Z'(x', y', w') - w'_{x'}X'(x', y', w') - w'_{y'}Y'(x', y', w'). \end{aligned} \tag{26}$$

Expanding w' as a power series and equating coefficients in (26), we have

$$\begin{aligned} &(i\lambda(\alpha - \beta)I - B')w'_{\alpha,\beta} \\ &= [Z'(x', y', w') - w'_{x'}X'(x', y', w') - w'_{y'}Y'(x', y', w')]_{\alpha,\beta} \end{aligned} \tag{27}$$

where I is the identity matrix. Since by hypothesis $i\lambda^{-1}\mu_j \neq$ integer $(j = 1, \cdots, m)$ where $\mu_1, \cdots, \mu_m$ are the eigenvalues of B', it follows that

$$\det(i\lambda(\alpha-\beta)I - B') \neq 0 .$$

Since the right side of (27) depends only on the coefficients of w' having order less than $\alpha+\beta$, we conclude that w' is uniquely determined by (27) and that the manifold M'^{λ} is unique. Define

$$w(x,y) = \tilde{B}w'\Big(\frac{1}{\sqrt{2}}(-ix-y), \frac{1}{\sqrt{2}}(x+iy)\Big) . \tag{28}$$

Then

$$\dot{w} = Bw + Z(x,y,w) , \tag{29}$$

equivalently

$$\begin{aligned} &\lambda y w_x - \lambda x w_y - Bw \\ &= Z(x,y,w) - w_x X(x,y,w) - w_y Y(x,y,w) . \end{aligned} \tag{30}$$

The function w given by (28) satisfies (30). Suppose $w^* \in \Gamma^2(x,y)$ is another solution to (30). Then w^* determines a function $w^{*\prime} \in \Gamma^2(x',y')$ given by

$$w^{*\prime}(x',y') = \tilde{B}^{-1}w^*\Big(\frac{1}{\sqrt{2}}(ix'+y'), \frac{1}{\sqrt{2}}(-x'-iy')\Big) . \tag{31}$$

But since w^* satisfies (30), it also satisfies (29). Then from (31) we conclude that $w^{*\prime}$ satisfies (26), hence $w^{*\prime} \equiv w'$ and because (28) and (31) are inverse relations to each other, $w^* \equiv w$. Therefore we conclude that (30) has a unique solution belonging to the class $\Gamma^2(x,y)$.

It remains to be shown that w is a *real* analytic function. If we expand the left side of (30) in a power series, we obtain

$$\begin{aligned} &\lambda y w_x - \lambda x w_y - Bw \\ &= \sum \lambda\alpha w_{\alpha,\beta} x^{\alpha-1} y^{\beta+1} - \sum \lambda\beta w_{\alpha,\beta} x^{\alpha+1} y^{\beta-1} - \sum B w_{\alpha,\beta} x^\alpha y^\beta \\ &= \sum \{\lambda(\alpha+1) w_{\alpha+1,\beta-1} - \lambda(\beta+1) w_{\alpha-1,\beta+1} - B w_{\alpha,\beta}\} x^\alpha y^\beta . \end{aligned} \tag{32}$$

For any power series

$$f = \sum f_{\alpha,\beta} x^\alpha y^\beta$$

define

$$f_n = [f]_n = [\sum f_{\alpha,\beta} x^\alpha y^\beta]_n = (f_{n,0}, f_{n-1,1}, \cdots, f_{0,n})$$

for all nonnegative integers n. Thus if f is a vector, then $\dim f_n = (n+1)\dim f$. If f and f' are two power series related by

$$f'(x',y') = f(x,y) = f\Big(\frac{1}{\sqrt{2}}(ix'+y'), \frac{1}{\sqrt{2}}(-x'-iy')\Big)$$

$$f(x,y) = f'(x',y') = f'\Big(\frac{1}{\sqrt{2}}(-ix-y), \frac{1}{\sqrt{2}}(x+iy)\Big) ,$$

then for each nonnegative integer n there exists a matrix L_n, $\det L_n \neq 0$, such that

$$f'_n = L_n f_n \,.$$

Clearly the matrix L_n depends only on $\dim f$; other than this L_n is independent of f.

Now, from (32) we obtain

$$[\lambda y w_x - \lambda x w_y - Bw]_n = B_n w_n \qquad (n = 2, 3, \cdots) \tag{33}$$

where

$$\begin{aligned} B_n = {} & \text{super-diag}\,(-\lambda I, -2\lambda I, \cdots, -n\lambda I) \\ & + \text{diag}\,(-B, -B, \cdots, -B) \\ & + \text{sub-diag}\,(n\lambda I, (n-1)\lambda I, \cdots, \lambda I)\,. \end{aligned}$$

with I = identity matrix, $\dim I = \dim B$. In a similar fashion we obtain

$$[i\lambda x' w'_{x'} - i\lambda y' w'_{y'} - B'w']_n = B'_n w'_n \qquad (n = 2, 3, \cdots)$$

where

$$B'_n = \text{diag}\,(in\lambda I - B', i(n-2)\lambda I - B', i(n-4)\lambda I - B', \cdots, -in\lambda I - B')\,.$$

Since $\det(ik\lambda I - B') \neq 0$ for any integer k, it follows that

$$\det B'_n \neq 0 \qquad (n = 2, 3, \cdots)\,.$$

A straight forward calculation shows that

$$\lambda y \frac{\partial}{\partial x} - \lambda x \frac{\partial}{\partial y} = i\lambda x' \frac{\partial}{\partial x'} - i\lambda y' \frac{\partial}{\partial y'}\,,$$

and from this fact along with (28) we conclude

$$\begin{aligned} & \lambda y w_x - \lambda x w_y - Bw = \tilde{B}\{i\lambda x' w'_{x'} - i\lambda y' w'_{y'} - B'w\} \\ & [\lambda y w_x - \lambda x w_y - Bw]_n = L_n^{-1}[\tilde{B}\{i\lambda x' w'_{x'} - i\lambda y' w'_{y'} - B'w'\}]_n \\ & B_n w_n = L_n^{-1}\tilde{B}_n B'_n w'_n = L_n^{-1}\tilde{B}_n B'_n L_n w_n\,, \end{aligned} \tag{34}$$

where

$$\tilde{B}_n = \text{diag}\,(\tilde{B}, \tilde{B}, \cdots, \tilde{B})\,.$$

Since (34) can be shown valid for all $w_n \in R^{(n+1)\dim w}$, it follows that

$$\begin{aligned} & B_n = L_n^{-1}\tilde{B}_n B'_n L_n \\ & \det B_n = \det \tilde{B}_n \cdot \det B'_n \neq 0\,. \end{aligned} \tag{35}$$

From (30) and (33) we obtain

$$B_n w_n = [Z(x, y, w) - w_x X(x, y, w) - w_y Y(x, y, w)]_n \qquad (n = 2, 3, \cdots) . \tag{36}$$

Finally, from (35) and (36) we conclude the coefficients of w are determined recursively from (30), or equivalently (36), and since B_n $(n = 2, 3, \cdots)$ is a real matrix, B_n^{-1} is also real and $w \in \Gamma^2(x, y)$ is real analytic. Thus we have shown that the function w which defines the manifold

$$M^\lambda = \{(x, y, z) \mid |x| + |y| < \delta,\ z = w(x, y)\}$$

is uniquely and recursively determined from (30), (35), (36) as a real power series. The fact that $w \in \Gamma^2(x, y)$ is guaranteed by the existence of the real analytic integral H. This completes the proof of Theorem 2.

5. **Real analytic changes of variables.** In this section for a real analytic system of equations we construct a real analytic change of variables.

THEOREM 3. *If* $i\lambda^{-1}\mu_j \neq$ *integer* $(j = 1, \cdots, m)$ *then for system* (19) *there exists a real analytic change of variables* (*not unique*)

$$\begin{aligned} x &= p + P(p, q) \\ y &= q + Q(p, q) \\ z &= r + R(p, q) \end{aligned} \tag{37}$$

such that

$$\begin{aligned} \dot{p} &= \lambda(1 + a(p^2 + q^2))q + \tilde{P}(p, q, r) \\ \dot{q} &= -\lambda(1 + a(p^2 + q^2))p + \tilde{Q}(p, q, r) \\ \dot{r} &= Br + \tilde{R}(p, q, r) \\ \tilde{P}, \tilde{Q}, \tilde{R} &\equiv 0 \ \textit{when}\ r = 0 , \end{aligned} \tag{34}$$

where $a = a(p^2 + q^2)$ *is a real analytic function of the quadratic form* $p^2 + q^2$ *with no constant term in its expansion about the origin;* $P, Q, R \in \Gamma^2(p, q)$ *are real analytic;* $\tilde{P}, \tilde{Q}, \tilde{R} \in \Gamma^2(p, q, r)$ *are real analytic.*

Proof. By means of the change of variables (22) we obtain system (23). From Theorem 1 we can find a change of variables

$$
\begin{aligned}
x' &= p' + P'(p', q') \\
y' &= q' + Q'(p', q') \\
z' &= r' + R'(p', q')
\end{aligned} \tag{39}
$$

such that

$$
\begin{aligned}
\dot{p}' &= i\lambda(1 + a'(p'q'))p' + \tilde{P}'(p', q', r') \\
\dot{q}' &= -i\lambda(1 + a'(p'q'))q' + \tilde{Q}'(p', q', r') \\
\dot{r}' &= B'r' + \tilde{R}'(p', q', r') \\
\tilde{P}', \tilde{Q}', \tilde{R}' &\equiv 0 \text{ when } r' = 0
\end{aligned} \tag{40}
$$

where $a' = a'(p'q')$ is a convergent power series in the product $p'q'$ with no constant term; $P', Q', R' \in \Gamma^2(p', q')$; $\tilde{P}, \tilde{Q}, \tilde{R} \in \Gamma^2(p', q', r')$. In the construction of the change of variables (39) one must restrict the choice of power series $\hat{P}', \hat{Q}'$ (see (8) and (9)) to be transformations via (22) of real power series. We will make this condition explicit in (43) below. Define

$$
\begin{aligned}
p &= \frac{1}{\sqrt{2}}(ip' + q') \\
q &= \frac{1}{\sqrt{2}}(-p' - iq') \\
r &= \tilde{B}r' .
\end{aligned} \tag{41}
$$

Then from (22), (39), (41) we obtain the change of variables (37) where

$$
\begin{aligned}
a(p^2 + q^2) &= a'\Big(-\frac{i}{2}(p^2 + q^2)\Big) \\
P(p, q) &= \frac{1}{\sqrt{2}}iP'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
&\quad + \frac{1}{\sqrt{2}}Q'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
Q(p, q) &= -\frac{1}{\sqrt{2}}P'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
&\quad - \frac{1}{\sqrt{2}}iQ'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
R(p, q) &= \tilde{B}R'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) .
\end{aligned} \tag{42}
$$

In particular we require $\hat{P}$, $\hat{Q}$ given by

$$
\begin{aligned}
\hat{P}(p, q) = &\frac{1}{\sqrt{2}} i\hat{P}'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
&+\frac{1}{\sqrt{2}}\hat{Q}'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
\hat{Q}(p, q) = &-\frac{1}{\sqrt{2}}\hat{P}'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big) \\
&-\frac{1}{\sqrt{2}} i\hat{Q}'\Big(\frac{1}{\sqrt{2}}(-ip - q), \frac{1}{\sqrt{2}}(p + iq)\Big)
\end{aligned} \tag{43}
$$

to be real analytic functions of (p, q). The functions $\tilde{P}, \tilde{Q}, \tilde{R}$ are given by formulas similar to those for P, Q, R in (42).

It remains to show that $a, P, Q, R, \tilde{P}, \tilde{Q}, \tilde{R}$ are *real* analytic functions. From (19), (37), (38) we obtain

$$
\begin{aligned}
&qP_p - pP_q - Q = -aq - a\{qP_p - pP_q\} + \lambda^{-1}X(p + P, q + Q, R) \\
&qQ_p - pQ_q + P = ap - a\{qQ_p - pQ_q\} + \lambda^{-1}Y(p + P, q + Q, R) \\
&qR_p - pR_q - \lambda^{-1}BR = -a\{qR_p - pR_q\} + \lambda^{-1}Z(p + P, q + Q, R)\,.
\end{aligned} \tag{44}
$$

Following a procedure analogous to the computation of w in (30) we can compute the coefficients of a, P, Q, R recursively from (44). Since X, Y, Z are real analytic and λ is a real, nonzero scalar, it follows that a, P, Q, R are real analytic. This completes the proof of Theorem 3.

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University of California, Santa Cruz

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INITIAL SEGMENTS OF ONE-ONE DEGREES

A. H. LACHLAN

Let $\mathbf{0}$ be the one-one degree consisting of the infinite recursive sets whose complements are also infinite. In this paper are studied initial segments of the one-one degrees $\geqq \mathbf{0}$. A characterization is stated of the order types of those initial segments with greatest member which are at the same time initial segments of the many-one degrees. It is shown that any finite initial segment with greatest member is a lattice, and that any finite recursively enumerable initial segment with greatest member is a distributive lattice.

According to the usual definition of one-one reducibility a set of natural numbers A is one-one reducible to a set of natural numbers B written $A \leqq_1 B$ if there is a one-one recursive function ρ such that $\rho^{-1}(B) = A$. The relation of one-one equivalence $\equiv_1$ defined by

$$A \equiv_1 B \Leftrightarrow [A \leqq_1 B \quad \text{and} \quad B \leqq_1 A]$$

is an equivalence relation, and the equivalence classes into which it partitions the sets of natural numbers are called one-one degrees. The natural ordering of these degrees induced by $\leqq_1$ is denoted by $\leqq$. In this paper we shall be wholly concerned with the one-one degrees $\geqq \mathbf{0}$. Notice that if A has one-one degree $\geqq \mathbf{0}$ then neither A nor its complement is immune. In this context it is convenient to adopt a new definition of one-one reducibility: we say that A is *one-one reducible* to a set of natural numbers B, again written $A \leqq_1 B$, if there exists a one-one partial recursive (p.r.) function ρ such that $\operatorname{dom}\rho$ (the domain of ρ) and $A \cup \operatorname{dom}\rho$ are both recursive, and such that $\rho^{-1}(B) = A \cap \operatorname{dom}\rho$. On sets whose one-one degrees in the old sense are $\geqq \mathbf{0}$ the new definition of $\leqq_1$ is the same as the old. Also, for any set A there is a set B whose one-one degree in the old sense is $\geqq \mathbf{0}$, and such that $A \equiv_1 B$ in the new sense. Thus adopting this new definition leaves the one-one degrees $\geqq \mathbf{0}$ unchanged, and suppresses the remaining one-one degrees.

We adopt the usual definition of many-one reducibility save that $\varnothing$, N (the set of all natural numbers) are both by convention in the zero many-one degree consisting of all recursive sets.

The only notations likely to be unfamiliar consist in writing $A \oplus B$ for the set $\{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}$, A' for the complement of A, and $\operatorname{rng}\rho$ for the range of ρ.

1. Common initial segments of the many-one and one-one degrees.

In [3] we obtained a characterization of the isomorphism

types of the initial segments of the many-one degrees. We can sharpen the construction in § 2 of [3] to obtain:

THEOREM 1. *An upper semilattice L is isomorphic to a common initial segment of the many-one and one-one degrees with greatest member $\neq \mathbf{0}$ if and only if there are sequences $\langle D_i \rangle$ of finite distributive lattices of cardinality >1, and $\langle \chi_i \rangle$ of maps, such that for each i, χ_i is an upper semilattice homomorphism of D_i into D_{i+1} preserving 0 and 1, and such that L is isomorphic to the direct limit of the sequence*

$$D_0 \xrightarrow{\chi_0} D_1 \xrightarrow{\chi_1} \cdots \xrightarrow{\chi_{i-1}} D_i \xrightarrow{\chi_i} \cdots .$$

(By saying that χ_i "preserves 0 and 1" we mean that χ_i maps the least element, greatest element respectively, of D_i into the least element, greatest element respectively, of D_{i+1}.)

Clearly if we consider only finite initial segments then the condition of the theorem reduces to: L is a finite distributive lattice of cardinality >1. By an argument not unlike Friedberg's maximal set construction one can establish:

THEOREM 2. *If L is a finite distributive lattice then there is a common recursively enumerable* (r.e.) *initial segment of the many-one and one-one degrees isomorphic to L.*

It follows from Theorem 1 that the elementary theory of the partial ordering of one-one degrees is undecidable (for details see [1], § 3). From Theorem 2 the same is true of the partial ordering of the r.e. one-one degrees, but for this one has to use the fact that the sets of first order sentences true of no distributive lattice, and true of some finite distributive lattice respectively, are recursively inseparable. (Unfortunately, we know no reference for this result.)

2. **Finite initial segments of the one-one degrees.** In this section we shall show that every finite initial segment of the one-one degrees is a lattice of a certain kind.

The *disjoint union* $\boldsymbol{a} \oplus \boldsymbol{b}$ of a pair of one-one degrees $\boldsymbol{a}, \boldsymbol{b}$ is defined to be the one-one degree of $A \oplus B$ where A, B are any representatives of $\boldsymbol{a}, \boldsymbol{b}$ respectively. The reader will easily verify that $\boldsymbol{a} \oplus \boldsymbol{b}$ depends only on $\boldsymbol{a}, \boldsymbol{b}$ and that the operation $\oplus$ is associative on one-one degrees. For each one-one degree $\boldsymbol{a}$ and natural number n, let $n\boldsymbol{a}$ be $\mathbf{0}$ if $n = 0$, and be

$$\underbrace{\boldsymbol{a} \oplus \boldsymbol{a} \oplus \cdots \oplus \boldsymbol{a}}_{n \text{ times}}$$

otherwise. A one-one degree $\boldsymbol{a}$ is called a *cylinder* just if $2\boldsymbol{a} = \boldsymbol{a}$; notice that $\boldsymbol{a}$ is a cylinder just if all its representatives are cylinders in the sense of Myhill [5].

A one-one degree $\boldsymbol{a}$ is called *decomposable* if there exist one-one degrees $\boldsymbol{b}, \boldsymbol{c}$ both $< \boldsymbol{a}$ such that $\boldsymbol{a} = \boldsymbol{b} \oplus \boldsymbol{c}$; otherwise $\boldsymbol{a}$ is called *indecomposable.*

A finite sequence $\{\boldsymbol{a}_1, \boldsymbol{a}_2 \cdots, \boldsymbol{a}_n\}$ of one-one degrees is called a *canonical decomposition* of a one-one degree $\boldsymbol{a}$ if $\boldsymbol{a} = \boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \oplus \cdots \oplus \boldsymbol{a}_n$ where each $\boldsymbol{a}_i$ is indecomposable and where deleting any of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ in the last equation destroys the equality. We shall first show that when canonical decompositions exist they are unique. To this end we prove:

LEMMA 1. *If*

$$\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_{m-1} = \boldsymbol{b}_0 \oplus \boldsymbol{b}_1 \oplus \cdots \oplus \boldsymbol{b}_{n-1}$$

with m, n *both* $\geqq 1$ *then there exist one-one degrees* $\{\boldsymbol{a}_{i,j} \mid i < m, j < n\}$ *such that*

$$\boldsymbol{a}_i = \boldsymbol{a}_{i,0} \oplus \boldsymbol{a}_{i,1} \oplus \cdots \oplus \boldsymbol{a}_{i,n-1},\ i < m\ ;$$
$$\boldsymbol{b}_j = \boldsymbol{a}_{0,j} \oplus \boldsymbol{a}_{1,j} \oplus \cdots \oplus \boldsymbol{a}_{m-1,j},\ j < n\ .$$

Proof. For $i < m$ let A_i be a representative of $\boldsymbol{a}_i$, for $j < n$ let B_j be a representative of $\boldsymbol{b}_j$. Let

$$A = \{mx + i \mid x \in A_i \quad \text{and} \quad i < m\}\ ,$$
$$B = \{nx + j \mid x \in B_j \quad \text{and} \quad j < n\}$$

then A, B have one-one degrees

$$\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \cdots \oplus \boldsymbol{a}_{m-1}\ , \qquad \boldsymbol{b}_0 \oplus \boldsymbol{b}_1 \oplus \cdots \oplus \boldsymbol{b}_{n-1}$$

respectively. By hypothesis these are equal and without loss of generality we may suppose that A, B are one-one equivalent in the old sense. By a result of Myhill ([4], § 5) there is a recursive permutation ρ mapping A onto B. For $i < m$, $j < n$ define

$$A_{i,j} = \{x \mid \rho(mx + i) = j \bmod n \quad \text{and} \quad x \in A_i\}\ .$$

Letting $\boldsymbol{a}_{i,j}$ be the one-one degree of $A_{i,j}$ the lemma follows by inspection.

THEOREM 3. *If a one-one degree has a canonical decomposition*

then its canonical decomposition is unique to within a permutation.

Proof. Let $\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_{m-1}, \boldsymbol{b}_0 \oplus \boldsymbol{b}_1 \oplus \cdots \boldsymbol{b}_{n-1}$ be two canonical decompositions of the same one-one degree. Applying Lemma 1 and using the notation above, since $\boldsymbol{a}_i$ is indecomposable, there exists $\alpha(i)$ such that $\boldsymbol{a}_i = \boldsymbol{a}_{i,\alpha(i)}$. Similarly there exists $\beta(i)$ such that $\boldsymbol{b}_i = \boldsymbol{a}_{\beta(i),i}$. As i runs through the numbers $<m$, $\alpha(i)$ must run through all numbers $<n$; for suppose $\alpha(i)$ is never zero then

$$\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_{m-1} \leqq \boldsymbol{b}_1 \oplus \boldsymbol{b}_2 \oplus \cdots \oplus \boldsymbol{b}_{n-1}$$

which contradicts the definition of canonical decomposition. Similarly $\beta(i)$ runs through all numbers $<m$. Hence $m = n$ and α, β are both permutations of the numbers $<m$. We claim that $\boldsymbol{a}_i = \boldsymbol{b}_{\alpha(i)}$ and $\boldsymbol{b}_i = \boldsymbol{a}_{\beta(i)}$ for each $i < m$. It is clear that $\boldsymbol{a}_i \leqq \boldsymbol{b}_{\alpha(i)}$ and $\boldsymbol{b}_i \leqq \boldsymbol{a}_{\beta(i)}$ for each $i < m$. For reductio ad absurdum suppose that our claim is false then the set

$$S = \{\boldsymbol{a}_i \mid \boldsymbol{a}_i < \boldsymbol{b}_{\alpha(i)} \quad \text{and} \quad i < m\} \cup \{\boldsymbol{b}_i \mid \boldsymbol{b}_i < \boldsymbol{a}_{\beta(i)} \quad \text{and} \quad i < m\}$$

is nonempty, and without loss of generality we may suppose that $\boldsymbol{a}_k < \boldsymbol{b}_{\alpha(k)}$ and that $\boldsymbol{a}_k$ is a maximal member of S. By choice of k we have

$$\boldsymbol{a}_k < \boldsymbol{b}_{\alpha(k)} = \boldsymbol{a}_{\beta\alpha(k)} = \boldsymbol{b}_{\alpha\beta\alpha(k)} = \boldsymbol{a}_{\beta\alpha\beta\alpha(k)} = \cdots .$$

But $\beta\alpha$ is a permutation and so one of the suffixes of $\boldsymbol{a}$ in the chain of equalities is equal to k. This contradiction completes the proof of the theorem.

Let $\boldsymbol{a}, \boldsymbol{b}$ be one-one degrees then $\boldsymbol{b}/\boldsymbol{a}$ is defined to be $\sup\{n \mid n\boldsymbol{a} \leqq \boldsymbol{b}\}$; $\boldsymbol{b}/\boldsymbol{a}$ may take any of the values $0, 1, 2, \cdots, \infty$. Notice that for any indecomposable $\boldsymbol{a}$, $(\boldsymbol{b} \oplus \boldsymbol{c})/\boldsymbol{a} = (\boldsymbol{b}/\boldsymbol{a}) + (\boldsymbol{c}/\boldsymbol{a})$ where it is understood that $x + \infty = \infty + x = \infty$. If $\boldsymbol{a}$ is not a cylinder then $\boldsymbol{a}, 2\boldsymbol{a}, 3\boldsymbol{a}, \cdots$ is a strictly increasing sequence of one-one degrees whence $\boldsymbol{a}/\boldsymbol{a} = 1$. If $\boldsymbol{a}$ is a cylinder then $\boldsymbol{b}/\boldsymbol{a}$ is zero or ∞, and if $\boldsymbol{a} \leqq \boldsymbol{b}$ and $\boldsymbol{a}$ is a cylinder then $\boldsymbol{b} \oplus \boldsymbol{a} = \boldsymbol{b}$.

Notice that if a one-one degree $\boldsymbol{a}$ has only finitely many predecessors then some sequence of the degrees $\leqq \boldsymbol{a}$ must constitute a canonical decomposition of $\boldsymbol{a}$; for example if $\boldsymbol{a}$ is indecomposable then it is its own canonical decomposition.

LEMMA 2. *Let $\boldsymbol{a}, \boldsymbol{b}$ be one-one degrees such that $\boldsymbol{a}$ has only finitely many prodecessors. Let $\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_m$ be the canonical decomposition of $\boldsymbol{a}$. Then $\boldsymbol{a} \leqq \boldsymbol{b}$ if and only if $\boldsymbol{a}/\boldsymbol{a}_i$ for all $i \leqq m$.*

Proof. The "only if" part is immediate. For the "if" part we

proceed by induction on m. Since the result is immediate for $m = 0$ we may suppose that $m > 0$. Let $\boldsymbol{a}^* = \boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \boldsymbol{a}_{m-1}$ and suppose that $\boldsymbol{a}/\boldsymbol{a}_i \leqq \boldsymbol{b}/\boldsymbol{a}_i$ for all $i \leqq m$. It follows from our induction hypothesis that $\boldsymbol{a}^* \leqq \boldsymbol{b}$ since it is easy to see that $\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_{m-1}$ is the canonical decomposition of $\boldsymbol{a}^*$. Without loss of generality we assume that $\boldsymbol{a}_m$ is minimal among $\boldsymbol{a}_0, \boldsymbol{a}_1, \cdots, \boldsymbol{a}_m$.

Let B be a representative of $\boldsymbol{b}$. It is convenient to introduce a map Ψ from the r.e. sets onto the one-one degrees $\leqq \boldsymbol{b}$. If W is finite let $\Psi(W) = \boldsymbol{0}$, if W is an infinite r.e. set let ω be any one-one recursive function whose range is W and let $\Psi(W)$ be the one-one degree of $\{x \mid \omega(x) \in B\}$. It is easy to see that $\Psi(W)$ does not depend upon the choice of w, that Ψ is order preserving, and that

$$\Psi(W) = \boldsymbol{w}_0 \oplus \boldsymbol{w}_1 \oplus \cdots \oplus \boldsymbol{w}_{e-1}$$

if and only if W is the union of disjoint r.e. sets $W_0, W_1, \cdots, W_{e-1}$ such that $\Psi(W_i) = \boldsymbol{w}_i$ for each $i < e$. Since $\boldsymbol{a}^* \leqq \boldsymbol{b}$ there exist disjoint r.e. sets $A_0, A_1, \cdots, A_{m-1}$ such that $\Psi(A_i) = \boldsymbol{a}_i$ for each $i < m$. There are two cases to consider:

Case 1. $\boldsymbol{a}/\boldsymbol{a}_m$ is finite. Then $\boldsymbol{a}_m$ is not a cylinder because of the remarks that precede this lemma. Let $\boldsymbol{a}^*/\boldsymbol{a}_m = k$ so that $\boldsymbol{b}/\boldsymbol{a}_m \geqq \boldsymbol{a}/\boldsymbol{a}_m \geqq k + 1$. It easily follows that there are disjoint r.e. sets $C_0, C_1, \cdots, C_k$ such that $\Psi(C_j) = \boldsymbol{a}_m$ for each $j \leqq k$. Effectively enumerate the r.e. sets $A_0, A_1, \cdots, A_{m-1}, C_0, C_1, \cdots, C_k$ simultaneously stage by stage letting A_i^s and C_i^s be the finite sets enumerated up to stage s for each $i < m$ and $j \leqq k$. We then define

$$F_i = \{x \mid (\exists s)[x \in A_i^s \quad \text{and} \quad (\forall j)_{j \leqq k}(\forall r)_{r<s}[x \notin C_j^r]]\}\,, \qquad i < m\,;$$
$$G_j = \{x \mid (\exists s)[x \in C_j^s \quad \text{and} \quad (\forall i)_{i<m}(\forall r)_{r \leqq s}[x \notin A_j^r]]\}\,, \qquad j \leqq k\,.$$

Also, for each $i<m$ let F_i^* be the r.e. set $A_i - F_i$, and for each $j \leqq k$ let G_j^* be the r.e. set $C_j - G_j$. Observe that $F_i^* \subseteqq G_0 \cup G_1 \cup \cdots \cup G_k$ for all $i < m$ and $G_j^* \subseteqq F_0 \cup F_1 \cup \cdots \cup F_{m-1}$ for all $j \leqq k$. Also, it can be seen that $F_0, F_1, \cdots, F_{m-1}, G_0, G_1, \cdots, G_k$ constitute a decomposition of $A_0 \cup A_1 \cup \cdots \cup A_{m-1} \cup C_0 \cup C_1 \cup \cdots \cup C_k$ into disjoint r.e. sets.

For each $i \leqq m$, $\boldsymbol{a}_i$ is indecomposable. This implies that either $\Psi(F_i) = \boldsymbol{a}_i$ or $\Psi(F_i^*) = \boldsymbol{a}_i$ for each $i < m$ and that either $\Psi(G_j) = \boldsymbol{a}_m$ or $\Psi(G_j^*) = \boldsymbol{a}_m$ for each $j \leqq k$. Define P, Q by

$$P = \{i \mid i < m \quad \text{and} \quad \Psi(F_i) = \boldsymbol{a}_i\}\,,$$
$$Q = \{i \mid i < m \quad \text{and} \quad i \notin P\}\,.$$

Now if $i \in Q$ then $\Psi(F_i^*) = \boldsymbol{a}_i$. But F_i^* is the union of the disjoint r.e. sets $F_i^* \cap G_0, \cdots, F_i^* \cap G_k$ so that for each $i \in Q$ there exists $\alpha(i) \leqq k$

such that $\Psi(F_i^* \cap G_{\alpha(i)}) = \boldsymbol{a}_i$.

Consider the set

$$D = \alpha(Q) \cup \{j \mid j < k \text{ and } \Psi(G_j^*) = \boldsymbol{a}_m\} ,$$

and let β be a map from $\alpha(Q)$ into Q such that $\alpha\beta(j) = j$ for all $j \in \alpha(Q)$. For each $j \in D$ we define

$$D_j = \begin{cases} G_j^* \text{ if } \Psi(G_j^*) = \boldsymbol{a}_m , \\ F_{\beta(j)}^* \cap G_j \text{ otherwise} . \end{cases}$$

This is a sound definition because if $\Psi(G_j^*) \neq \boldsymbol{a}_m$ then $j \in \alpha(Q)$ and so $\beta(j)$ is defined.

We first claim that $\Psi(D_j) = \boldsymbol{a}_m$ for all $j \in D$. For, if $\Psi(G_j^*) \neq \boldsymbol{a}_m$ then $\beta(j)$ is defined and $\Psi(G_j) = \boldsymbol{a}_m$. Since Ψ is order preserving it follows that

$$\boldsymbol{a}_{\beta(j)} = \Psi(F_{\beta(j)}^* \cap G_j) \leqq \Psi(G_j) = \boldsymbol{a}_m ,$$

and because $\boldsymbol{a}_m$ is minimal we deduce that $\boldsymbol{a}_{\beta(j)} = \boldsymbol{a}_m$. It follows that $\Psi(F_{\beta(j)}^* \cap G_j) = \Psi(D_j) = \boldsymbol{a}_m$. This proves our first claim.

Next we use this to deduce that $j \notin D$ for some $j \leqq k$. For, otherwise $(k+1)\boldsymbol{a}_m = \Psi(D_0 \cup \cdots \cup D_k) \leqq \Psi(A_0 \cup \cdots \cup A_{m-1}) = \boldsymbol{a}^*$, because $D_0, D_1, \cdots, D_k$ are disjoint r.e. subset of $A_0 \cup \cdots \cup A_{m-1}$. But $\boldsymbol{a}^*/\boldsymbol{a}_m = k$ and so this is impossible.

Now let j be a fixed number $\leqq k$ which does not belong to D and for each $i \leqq m$ define

$$E_i = \begin{cases} F_i \text{ if } i \in P \\ F_i^* \cap G_{\alpha(i)} \text{ if } i \in Q , \\ G_j \text{ if } i = m . \end{cases}$$

Then $E_0, E_1, \cdots, E_m$ are disjoint r.e. sets and $\Psi(E_i) = \boldsymbol{a}_i$ for all $i \leqq m$. Since $\Psi(E_0 \cup \cdots \cup E_m) = \boldsymbol{a}$ it follows that $\boldsymbol{a} \leqq \boldsymbol{b}$ which completes this case.

Case 2. $\boldsymbol{a}/\boldsymbol{a}_m = \infty$. Then $\boldsymbol{a}_m$ is cylinder, otherwise $\boldsymbol{a}$ would have an infinite number of predecessors. Suppose $\boldsymbol{a}_m \leqq \boldsymbol{a}^*$, then by the remarks preceding the lemma we have $\boldsymbol{a} = \boldsymbol{a}^* \oplus \boldsymbol{a}_m = \boldsymbol{a}^*$ which contradicts $\boldsymbol{a}_0 \oplus \boldsymbol{a}_1 \oplus \cdots \oplus \boldsymbol{a}_m$ being a canonical decomposition of $\boldsymbol{a}$. Thus $\boldsymbol{a}^*/\boldsymbol{a}_m = \boldsymbol{0}$. Now, taking $k = 0$, proceed just as in the last case.

Before turning to the next theorem, we note that any finite upper semilattice is in fact a lattice as long as it has a least element. For, if least upper bounds exist then greatest lower bounds exist by the definition:

$$\boldsymbol{b} \cap \boldsymbol{c} = \sup \{\boldsymbol{x} \mid \boldsymbol{x} \leqq \boldsymbol{b} \quad \text{and} \quad \boldsymbol{x} \leqq \boldsymbol{c}\} .$$

THEOREM 4. *Let $\boldsymbol{a}$ be a one-one degree which has only a finite number of predecessors then the predecessors of $\boldsymbol{a}$ form a lattice.*

Proof. In view of the observation that precedes the theorem, we have only to show that the predecessors of $\boldsymbol{a}$ form an upper semilattice. Let $\boldsymbol{b}, \boldsymbol{c}$ be any one-one degrees $\leqq \boldsymbol{a}$ and suppose that $\boldsymbol{a}$ has only a finite number of predecessors. Let $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_p$ be an enumeration of all the indecomposable one-one degrees $\leqq \boldsymbol{a}$ arranged so that for any i, j

$$p \geqq i > j \geqq 1 \text{ implies } \boldsymbol{a}_i \not\geqq \boldsymbol{a}_j .$$

To show that $\boldsymbol{b} \cup \boldsymbol{c}$ exists we shall define an increasing sequence $\boldsymbol{e}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_p$ of one-one degrees $\leqq \boldsymbol{a}$ and simultaneously we shall show that for all i, $1 \leqq i \leqq p$, and all $\boldsymbol{d} \leqq \boldsymbol{a}$

(1) $$\boldsymbol{b} \leqq \boldsymbol{d} \quad \text{and} \quad \boldsymbol{c} \leqq \boldsymbol{d} . \rightarrow \boldsymbol{e}_i \leqq \boldsymbol{d}$$

and

(2) $$\boldsymbol{e}_i/\boldsymbol{a}_i \geqq \max (\boldsymbol{b}/\boldsymbol{a}_i, \boldsymbol{c}/\boldsymbol{a}_i) .$$

Define

$$\boldsymbol{e}_1 = \begin{cases} \boldsymbol{a}_1 \text{ if one of } \boldsymbol{b}/\boldsymbol{a}_1, \boldsymbol{c}/\boldsymbol{a}_1 \text{ is } \infty \\ \max (\boldsymbol{b}/\boldsymbol{a}_1, \boldsymbol{c}/\boldsymbol{a}_1)\boldsymbol{a}_1 \text{ otherwise} \end{cases}$$

then $\boldsymbol{e}_1 \leqq \boldsymbol{a}$ and (1), (2) are certainly satisfied for $i=1$. Suppose that $\boldsymbol{e}_i$ has been defined for $1 \leqq i \leqq k$ where $k < p$ so that (1), (2) are satisfied for $1 \leqq i \leqq k$. If

$$\boldsymbol{e}_k/\boldsymbol{a}_{k+1} \geqq \max (\boldsymbol{b}/\boldsymbol{a}_{k+1}, \boldsymbol{c}/\boldsymbol{a}_{k+1}) ,$$

then set $\boldsymbol{e}_{k+1} = \boldsymbol{e}_k$, in which case (1), (2) are clearly satisfied for $i = k+1$. Otherwise let r, s be the numbers on the left, right respectively of the last inequality with the proviso that if $\boldsymbol{a}_{k+1}$ is a cylinder then $s = 1$. Set $\boldsymbol{e}_{k+1} = \boldsymbol{e}_k \oplus (s - r)\boldsymbol{a}_{k+1}$. Any degree appearing in the canonical decomposition of $\boldsymbol{e}_{k+1}$ is clearly one of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{k+1}$. Hence by Lemma 2, $\boldsymbol{e}_{k+1} \leqq \boldsymbol{a}$. Suppose that $\boldsymbol{b} \leqq \boldsymbol{d}$ and $\boldsymbol{c} \leqq \boldsymbol{d}$ then from (1) for $i = k$ we know that $\boldsymbol{e}_k \leqq \boldsymbol{d}$ which implies that $\boldsymbol{d}/\boldsymbol{a}_i \geqq \boldsymbol{e}_k/\boldsymbol{a}_i$ for $1 \leqq i \leqq k$. But since we are assuming $\boldsymbol{e}_{k+1} \neq \boldsymbol{e}_k$ we can deduce that $\boldsymbol{a}_{k+1}$ is different from all of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_k$ and does not exceed any of them. Thus

$$\boldsymbol{d}/\boldsymbol{a}_i \geqq \boldsymbol{e}_k/\boldsymbol{a}_i = \boldsymbol{e}_{k+1}/\boldsymbol{a}_i , \qquad 1 \leqq i \leqq k .$$

Also

$$\boldsymbol{d}/\boldsymbol{a}_{k+1} \geqq s = \boldsymbol{e}_{k+1}/\boldsymbol{a}_{k+1} .$$

Any degree which appears in the canonical decomposition of $\boldsymbol{e}_{k+1}$ is one of $\boldsymbol{a}_1, \boldsymbol{a}_2, \cdots, \boldsymbol{a}_{k+1}$. Hence $\boldsymbol{e}_{k+1} \leqq \boldsymbol{d}$ by Lemma 2. It is immediate that (2) is satisfied for $i = k+1$. This completes the induction step. It is clear that $\boldsymbol{e}_p$ is the least upper bound of $\boldsymbol{b}, \boldsymbol{c}$. For $\boldsymbol{b} \leqq \boldsymbol{e}_p$ and $\boldsymbol{c} \leqq \boldsymbol{e}_p$ by (2) and Lemma 2, and $\boldsymbol{e}_p$ is the least among the common upper bounds of $\boldsymbol{b}, \boldsymbol{c}$ from (1). This completes the proof of the theorem.

The next theorem gives a necessary condition for a finite lattice to be an initial segment of one-one degrees.

THEOREM 5. *Let $\boldsymbol{a}$ be a one-one degree with only a finite number of predecessors then there exist finite separated distributive lattices L_1, L_2 and an order preserving map Ψ of L_2 onto the one-one degrees $\leqq \boldsymbol{a}$ satisfying the following four conditions in which $\subseteq$, $\varnothing$ denote the partial order and least element of L_2 respectively.*

(i) *L_1 is an initial segment of L_2 containing at least $\varnothing$ and $\Psi(L_1)$ is a distributive lattice,*

(ii) *if $x \cap y = \varnothing$ then $\Psi(x \cup y) = \Psi(x)$ if and only if $y \in L_1$ and $\Psi(y) \leqq \Psi(x)$,*

(iii) *if $x_1 \cap x_2 = y_1 \cap y_2 = \varnothing$ and $\Psi(x_i) = \Psi(y_i)$ for $i = 1, 2$ then $\Psi(x_1 \cup x_2) = \Psi(y_1 \cup y_2)$,*

(iv) *for any $x \in L_2$ and $\boldsymbol{y} \leqq \Psi(x)$ there exists $y \subseteq x$ such that $\Psi(y) = \boldsymbol{y}$.*

Proof. We proceed by induction on the number of predecessors of $\boldsymbol{a}$. When $\boldsymbol{a} = \boldsymbol{0}$ the theorem is obvious. Suppose that $\boldsymbol{b} > \boldsymbol{0}$ has only a finite number of predecessors and assume the result for $\boldsymbol{a} < \boldsymbol{b}$. Assume in addition that for any $\boldsymbol{a} < \boldsymbol{b}$ we can choose L_1, L_2, Ψ such that $\Psi(L_1)$ consists of the cylinders $\leqq \boldsymbol{a}$, and if $x_1 \cap x_2 = \varnothing$ then $\Psi(x_1 \cup x_2) = \Psi(x_1) \oplus \Psi(x_2)$.

Case 1. Suppose that $\boldsymbol{b}$ has just one maximal predecessor $\boldsymbol{b}_1$. Let $L_1^{(1)}, L_1^{(1)}, \Psi^{(1)}$ be the finite separated distributive lattices and map whose existence is predicted by the theorem, and satisfying the additional conditions, when $\boldsymbol{a} = \boldsymbol{b}_1$. Let $L_2 = L_2^{(1)} \cup \{b\}$ where $b \notin L_2^{(1)}$, and let

$$L_1 = \begin{cases} L_1^{(1)} \cup \{b\} \text{ if } \boldsymbol{b} \text{ is a cylinder} \\ L_1^{(1)} \text{ otherwise.} \end{cases}$$

Let $x \subseteq y$ in L_2 if and only if $y = b$ or x, y are both in $L_2^{(1)}$ and $x \subseteq y$ in $L_2^{(1)}$. Let $\Psi(b) = \boldsymbol{b}$ and otherwise let Ψ agree with $\Psi^{(1)}$. It is easy

to check (i) – (iv) and the additional conditions for $\boldsymbol{a} = \boldsymbol{b}$.

Case 2. Suppose that $\boldsymbol{b}$ has at least two maximal predecessors $\boldsymbol{b}_1, \boldsymbol{b}_2$. Using the technique of Lemma 2 it is easy to show that there exist $\boldsymbol{c}_1 \leqq \boldsymbol{b}_1$ and $\boldsymbol{c}_2 \leqq \boldsymbol{b}_2$ such that $\boldsymbol{b} = \boldsymbol{c}_1 \oplus \boldsymbol{c}_2$. For $i = 1, 2$ let $L_1^{(i)}$, $L_2^{(i)}$, $\Psi^{(i)}$ be the finite separated distributive lattices and the map predicted by the theorem, and satisfying the additional conditions, when $\boldsymbol{a} = \boldsymbol{c}_i$. For $j = 1, 2$ let L_j be the direct union of $L_j^{(1)}$, $L_j^{(2)}$. Then for $j = 1, 2$, L_j is a finite separated distributive lattice, and L_1 is an initial segment of L_2. For any pair (x, y) in L_2 define

$$\Psi(x, y) = \Psi^{(1)}(x) \oplus \Psi^{(2)}(y) .$$

It is again easy to check (i)–(iv) and the additional conditions for $\boldsymbol{a} = \boldsymbol{b}$. This completes the proof of the theorem.

We conjecture that the converse of Theorem 5 is true in the following sense. Let P be any partially ordered set such that there exist finite separated distributive lattice L_1, L_2 and an order preserving map Ψ of L_2 onto P such that (i)–(iv) are satisfied. We conjecture that there exists a one-one degree $\boldsymbol{p}$ such that the lattice of one-one degree $\leqq \boldsymbol{p}$ is isomorphic to P, and that under the isomorphism $\Psi(L_1)$ maps into the set of cylinders $\leqq \boldsymbol{p}$, and that for any x_1, x_2 disjoint in L_2 the image of $\Psi(x_1 \cup x_2)$ under the isomorphism is the disjoint union of the images of $\Psi(x_1)$, $\Psi(x_2)$. In the next section we shall prove a particular case of this conjecture. Although it is by no means obvious this conjecture is consistent with the earlier results of this section, e.g., it follows from the conditions imposed upon P that P is a lattice. This can be shown by a suitable modification of the proof of the earlier results.

There are a number of observations which are almost immediate from what has been proved above. First notice that if $\boldsymbol{a}_1, \boldsymbol{a}_2$ both have only a finite number of predecessors then $\boldsymbol{a}_1 \oplus \boldsymbol{a}_2$ has only a finite number of predecessors and so from Theorem 4, $\boldsymbol{a}_1 \cup \boldsymbol{a}_2$ and $\boldsymbol{a}_1 \cap \boldsymbol{a}_2$ exist in the partial ordering of one-one degrees. Next, if $\boldsymbol{a}_1$, $\boldsymbol{a}_2, \cdots, \boldsymbol{a}_n$ are distinct one-one degrees minimal $>\boldsymbol{0}$, then

$$\boldsymbol{a}_1 \cup \boldsymbol{a}_2 \cup \cdots \cup \boldsymbol{a}_n = \boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \oplus \cdots \oplus \boldsymbol{a}_n$$

and the one-one degrees $\leqq \boldsymbol{a}_1 \oplus \boldsymbol{a}_2 \oplus \cdots \oplus \boldsymbol{a}_n$ form a Boolean algebra of order 2^n. This shows that neither of the two nondistributive lattices of order five can be isomorphic to initial segments of the one-one degrees. One can also show rather easily that many other nondistributive lattices of small order cannot be isomorphic to initial segments of the one-one degrees, e.g., the lattices depicted in the usual way:

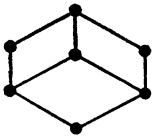


Fig. 1

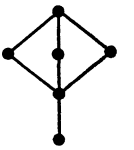


Fig. 2



Fig. 3

As an example take the second of these and suppose for reductio and absurdum that the initial segment whose greatest member is $\boldsymbol{a}$ is isomorphic to this lattice. Let L_1, L_2, Ψ be the separated distributive lattices and the map given by Theorem 5 and let the additional conditions of the proof of Theorem 5 be satisfied. Let $\boldsymbol{a}_1$, $\boldsymbol{a}_2$, $\boldsymbol{a}_3$ be maximal below $\boldsymbol{a}$, and let $\boldsymbol{b}$ be minimal above $\mathbf{0}$. Let a_1, a_2 be elements of L_2 such that $\Psi(a_i) = \boldsymbol{a}_i$ for $i = 1, 2$. Since L_2 is separated there exist b_1, b_2 such that $b_i \subseteqq a_i$ for $i = 1, 2$ and such that $b_1 \cup b_2 = a_1 \cup a_2$ while $b_1 \cap b_2 = \varnothing$. Clearly, $\Psi(b_i) \leqq \boldsymbol{a}_i$ for $i = 1, 2$. Also,

$$\Psi(a_1 \cup a_2) = \Psi(b_1 \cup b_2)$$

is $\geqq$ both $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ since Ψ is order preserving. Thus $\Psi(b_1 \cup b_2) = \boldsymbol{a}$. From (iv) there exists $a_3 \subseteqq b_1 \cup b_2$ such that $\Psi(a_3) = \boldsymbol{a}_3$. Now for $i = 1, 2$ $\Psi(a_3 \cap b_i)$ is $\leqq$ both $\boldsymbol{a}_3$ and $\Psi(b_i)$ and hence is either $\mathbf{0}$ or $\boldsymbol{b}$. But $\Psi(a_3) = \Psi(a_3 \cap b_1) \oplus \Psi(a_3 \cap b_2)$, whence $\boldsymbol{a}_3 = \boldsymbol{b} \oplus \boldsymbol{b}$. Similarly, $\boldsymbol{a}_1 = \boldsymbol{b} \oplus \boldsymbol{b}$, contradiction.

3. Not every initial segment is distributive. In this section we verify one particular consequence of our conjecture and in so doing we construct an initial segment of one-one degrees which is isomorphic to the lattice:

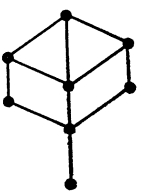


Fig. 4

Let M be a maximal r.e. set. Let $\mathscr{K}$, $\mathscr{K}(M)$, $\mathscr{L}(M)$ denote the classes of all r.e. sets, all r.e. subsets of M, all recursive subsets of M respectively. For our construction we shall require the following:

LEMMA 3. *There exists a class $\mathscr{M}$ of* r.e. *sets, $\mathscr{L}(M) \subseteqq \mathscr{M} \subseteqq \mathscr{K}(M)$, such that if W is a finite union of members of $\mathscr{M}$ then $W \in \mathscr{M}$ and $M - W$ is infinite, and such that if $W \in \mathscr{K}(M) - \mathscr{M}$ then some member of $\mathscr{M}$ complements W relative to M.*

Since this is easy we omit its proof; notice that the property we require of $\mathscr{M}$ is simply that $\mathscr{K}(M) - \mathscr{M}$ should be the intersection with $\mathscr{K}(M)$ of a nonprincipal ultrafilter over M which contains no member of $\mathscr{L}(M)$. It is clear that there exists an increasing sequence $M_0, M_1, \cdots$ of members of $\mathscr{M}$ such that every member of $\mathscr{M}$ is $\subseteqq M_i$ for some i. Choose a strictly increasing sequence $m_0, m_1, \cdots$ of members of M such that for all i, $m_i \in M - M_i$. Since $M - M_i$ is infinite for each i, the sequence $m_0, m_1, \cdots$ can be found. Let $M^* = \{m_i \mid i \geqq 0\}$, then $M^* \cap M_i$ is finite for each i. For $j = 0, 1$ let $R_j = \{2x + j \mid x \geqq 0\}$ and $S_j = \{2x + j \mid x \in M\}$. Let P_0, P_1 be infinite disjoint sets both $\subseteqq M'$. Let $\boldsymbol{a}$ be the one-one degree of the set

$$A = \{2x + j \mid x \in M^* \text{ and } j \leqq 1\} \cup \{2x + j \mid x \in P_j \text{ and } j \leqq 1\} .$$

We shall show that the degrees $\leqq \boldsymbol{a}$ with their natural ordering do form a lattice of order eight isomorphic to the one depicted above. We shall again use the map Ψ introduced in Lemma 2. If W is finite then $\Psi(W) = \mathbf{0}$, while if W is an infinite r.e. set we let ω be any one-one recursive function whose range is W and $\Psi(W)$ is defined to be the one-one degree of $\{x \mid \omega(x) \in A\}$. We shall use the properties of Ψ which were stated above without proof. Let $\boldsymbol{a}_j$ be $\Psi(R_j)$ for $j = 0, 1$ and let $\boldsymbol{m} = \Psi(S_0) = \Psi(S_1)$; it is apparent that $\Psi(S_0) = \Psi(S_1)$ when one considers the map: $2x$ goes to $2x+1$, restricted to S_0. It is clear that $\boldsymbol{a} = \boldsymbol{a}_0 \oplus \boldsymbol{a}_1$, that $\boldsymbol{m} \leqq \boldsymbol{a}_j$ for $j = 0, 1$. Thus all of $\mathbf{0}$, $\boldsymbol{m}$, $\boldsymbol{m} \oplus \boldsymbol{m}$, $\boldsymbol{a}_0$, $\boldsymbol{a}_1$, $\boldsymbol{a}_0 \oplus \boldsymbol{m}$, $\boldsymbol{a}_1 \oplus \boldsymbol{m}$, $\boldsymbol{a}$ are $\leqq \boldsymbol{a}$. It now suffices to show that these are the only one-one degrees $\leqq \boldsymbol{a}$ and that they are all different. Let $\boldsymbol{b}$ be any one-one degree $\leqq \boldsymbol{a}$ then there exists some r.e. set B such that $\Psi(B) = \boldsymbol{b}$. Now

$$\Psi(B) = \Psi(B \cap R_0) \oplus \Psi(B \cap R_1) ,$$

thus to prove that the degrees listed are the only ones $\leqq \boldsymbol{a}$ it is sufficient for $j = 0, 1$ to show that $\Psi(B \cap R_j)$ is one of $\boldsymbol{a}_j$, $\boldsymbol{m}$, $\mathbf{0}$. Now Ψ does not distinguish between r.e. sets whose symmetric difference is finite, hence because M is maximal we may suppose that $B \cap R_j \subseteqq S_j$ or $(B \cap R_j) \cup S_j = R_j$. We now consider three cases which exhaust all the possibilities. Let $\mathscr{M}_j = \{\{2x + j \mid x \in X\} \mid X \in \mathscr{M}\}$.

Case 1. $B \cap R_j \in \mathscr{M}_j$. Then $B \cap R_j \cap A$ is finite, and so

$$\Psi(B \cap R_j) = \mathbf{0} .$$

Case 2. $B \cap R_j \in \mathscr{K}(S_j) - \mathscr{M}_j$. Then there exists W in $\mathscr{M}_j$ such that $(B \cap R_j) \cup W = S_j$. Since $\mathscr{K}$ is separated there exist disjoint r.e. sets T, U included in $B \cap R_j$, W respectively such that $T \cup U = S_j$. Now $\Psi(S_j) = \Psi(T) \oplus \Psi(U)$. But $\Psi(W) = \mathbf{0}$, whence $\Psi(U) = \mathbf{0}$. Thus

$\Psi(T)=\Psi(S_j)=\boldsymbol{m}$. Since $T \subseteq B \cap R_j \subseteq S_j$, we also have $\Psi(B \cap R_j)=\boldsymbol{m}$.

Case 3. $(B \cap R_j) \cup S_j = R_j$. Then since $\mathscr{K}$ is separated there exist recursive sets T, U included in $B \cap R_j$, S_j respectively such that $T \cup U = R_j$. Since $\mathscr{L}(M) \subseteq \mathscr{M}$, we have $U \in \mathscr{M}_j$ whence $\Psi(U) = \mathbf{0}$. Thus $\Psi(T) = \Psi(R_j) = \boldsymbol{a}_j$. Therefore $\Psi(B \cap R_j) = \boldsymbol{a}_j$.

This completes the proof that each one-one degree $\leqq \boldsymbol{a}$ is one of the eight listed. We now turn to the task of showing that all eight are distinct. It is easy to show that no cylinder $>\mathbf{0}$ can be represented by an immune or coimmune set. But $A \cap S_j$ is immune since otherwise $A \cap S_j$ would have an infinite recursive subset which is impossible. Hence $\boldsymbol{m} = \Psi(S_j)$ is represented by an immune set, whence $\boldsymbol{m} \oplus \boldsymbol{m} \neq \boldsymbol{m}$. Suppose for reductio ad absurdum that $\boldsymbol{a}_0 \leqq \boldsymbol{a}_1$ then there exists a one-one partial recursive function ρ such that $\operatorname{dom} \rho \subseteq R_0$, $\operatorname{rng} \rho \subseteq R_1$, such that $\operatorname{dom} \rho$ and $R_0 \cap (A \cup \operatorname{dom} \rho)$ are both recursive, and such that for all x in $\operatorname{dom} \rho$

$$x \in A \Leftrightarrow \rho(x) \in A \,. \tag{3}$$

It is easy to see that $R_0 \cap (\operatorname{dom} \rho)' \cap A$ is finite since A is immune. Therefore $(\operatorname{dom} \rho)'$ cannot contain all but a finite number of members of $R_0 - S_0$, therefore $\operatorname{dom} \rho$ does contain all but a finite number of members of $R_0 - S_0$. Now $\rho(R_0 - S_0)$ is clearly cohesive and so either $S_1 \cap \rho(R_0 - S_0)$ is finite or $(R_1 - S_1) \cap \rho(R_0 - S_0)$ is finite.

Case 1. $S_1 \cap \rho(R_0 - S_0)$ is finite. Define the one-one partial recursive function σ by

$$\sigma(x) = y \Leftrightarrow \rho(2x) = 2y + 1 \,.$$

Then $\sigma(M') \cap M$ is finite. Write $x \sim y$ if either $x = y$, or x can be obtained by iterating σ on y, or y can be obtained by iterating σ on x. Since σ is one-one, $\sim$ is an equivalence relation on N. Consider the equivalence classes intersecting M'; none of them can have infinite intersection with M' because this would mean that M' had an infinite r.e. subset. Thus there are an infinite number of equivalence classes intersecting M'. Since $\sigma(M') \cap M$ is finite and M is maximal, $M' - \sigma(M')$ is also finite. Hence only a finite number of equivalence classes intersect both M and M'. Since P_0, P_1 are disjoint it follows from (3) that $\sigma(x) \neq x$ for infinitely many x in M'. Hence there exist infinitely many equivalence classes $\subseteq M'$ of cardinality >1. We now have a contradiction, because we can construct two disjoint r.e. sets each of which intersects all the finite equivalence classes of cardinality >1, and hence each of which has infinite intersection with M'.

Case 2. $\rho(R_0 - S_0) \cap (R_1 - S_1)$ is finite. By Theorem 3 of [2] it follows that $R_1 - \rho(R_0 - S_0)$ is r.e. and hence there exists a recursive set $T \supseteq \rho(R_0 - S_0)$ such that $T - S_1$ is finite. Now $T \cap S_1$ being recursive is in $\mathscr{M}_1$ and so has only a finite intersection with A. Hence $\rho(R_0 - S_0)$ has only a finite intersection with A. This contradicts (3), because ρ is one-one and $R_0 - S_0$ has infinite intersection with A.

We have now shown that $\boldsymbol{a}_0 \not\leqq \boldsymbol{a}_1$; similarly $\boldsymbol{a}_1 \not\leqq \boldsymbol{a}_0$. It only remains to show that $\boldsymbol{a}_j/\boldsymbol{m} = 1$ for $j = 0, 1$. Suppose for reductio ad absurdum that $\boldsymbol{a}_j/\boldsymbol{m} \neq 1$, then $\boldsymbol{a}_j = \boldsymbol{m} \oplus \boldsymbol{m}$, because we have shown above that the only one-one degrees $< \boldsymbol{a}_j$ are $\boldsymbol{0}$ and $\boldsymbol{m}$. Since $\boldsymbol{a}_j = \Psi(R_j) = \boldsymbol{m} \oplus \boldsymbol{m}$ there exist disjoint recursive sets T, U whose union is R_j such that $\Psi(T) = \Psi(U) = \boldsymbol{m}$. One of T, U has only finite intersection with $R_j - S_j$. Thus we may suppose $T \subseteq S_j$, whence $T \in \mathscr{M}_j$, which means that $T \cap A$ is finite. Therefore $\Psi(T) = \boldsymbol{0}$, again a contradiction. Using the results of § 2 it now follows easily that the initial segment of one-one degrees $\leqq \boldsymbol{a}$ is isomorphic to the nondistributive lattice depicted above.

The use of Theorem 3 of [2] may be eliminated if we construct $A \cap (R_0 - S_0)$, $A \cap (R_1 - S_1)$ with the immediate intention that $\boldsymbol{a}_0, \boldsymbol{a}_1$ should be incomparable and both $> \boldsymbol{m}$.

4. Recursively enumerable initial segments.

In this section we prove a theorem about r.e. one-one degrees and deduce that all r.e. finite initial segments are distributive. The theorem we shall prove is of the same genre as Theorem 3 of [2] quoted above.

THEOREM 6. *Let* $\boldsymbol{a}, \boldsymbol{b}$ *be indecomposable* r.e. *one-one degrees such that* $\boldsymbol{b} < \boldsymbol{a}$, *then* $\boldsymbol{a} = \boldsymbol{a} \oplus \boldsymbol{b}$.

Proof. Let A be a representative of $\boldsymbol{a}$. Let Ψ be the map of the r.e. sets onto the one-one degrees $\leqq \boldsymbol{a}$ defined in the last section. It is clear that $\Psi(W) = \Psi(W \cup A)$ for every r.e. set W. Let $\{\varphi_i\}$ be a recursive enumeration of all p.r. functions of one argument. Let $\{\varphi_{x,y}\}$ be an s.r.e. double sequence of finite functions such that for each x, $\varphi_{x,0} \subseteq \varphi_{x,1} \subseteq \cdots$ and $\lim_y \varphi_{x,y} = \varphi_x$. Let B be an infinite r.e. set such that $B \supseteq A$ and $\Psi(B) = \boldsymbol{b}$. Let β be a one-one recursive function enumerating B. We shall effectively enumerate in steps $0, 1, 2, \cdots$ disjoint r.e. sets B^0, B^1 whose union is B. For $j = 0, 1$ let B^j_s be the finite set of numbers enumerated in B_j before step s. While enumerating B^0, B^1 we simultaneously enumerate A. Let A_s be the finite set of numbers enumerated in A before step s. Let P be the set of all pairs (e, j) of natural numbers with $j \leqq 1$. We will order P according to

$$(e, j) < (f, k) \Leftrightarrow [e < f \cdot \bigvee \cdot e = f \quad \text{and} \quad j < k] .$$

In order to describe step s we need two auxiliary functions:

$$\gamma_s^j(e) = \sup \{x \mid x = e \bigvee (y)_{y<x}[y \in B_s^j \rightarrow : \varphi_{e,s}(y)$$

is defined $\cdot\&\cdot\ \varphi_{e,s}(y) \in B_s^{1-j} \cdot\&\cdot y \in A_s \Leftrightarrow \varphi_{e,s}(y) \in A_s]\}$,

$$\delta_s^j(e) = \sup [\gamma_x^k(y) \mid x \leqq s \quad \& \quad (y, k) \leqq (e, j) \quad \& \quad (y, k) \in P\}$$

each defined only for (e, j) in P. The construction can now be stated very easily. In step s let (e, j) be the least pair in P such that $\beta(s) < \delta_s^j(e)$. Enumerate $\beta(s)$ in B^j.

To prove the theorem notice that

$$\boldsymbol{b} = \Psi(B) = \Psi(B^0) \oplus \Psi(B^1) .$$

Since $\boldsymbol{b}$ is indecomposable, for either $j = 0$ or $j = 1$ we have $\Psi(B^j) \leqq \Psi(B^{1-j})$. Thus there exists a partial recursive one-one function φ_e such that $B^j - \operatorname{dom} \varphi_e$, $B^j - (A \cup \operatorname{dom} \varphi_e)$ are both r.e., such that $\operatorname{dom} \varphi_e \subseteqq B^j$, $\operatorname{rng} \varphi_e \subseteqq B^{1-j}$, and

$$y \in A \Leftrightarrow \varphi_e(y) \in A \tag{4}$$

for all y in $\operatorname{dom} \varphi_e$. Now $A \cap B^{1-j}$ is obviously infinite and r.e., thus we may suppose that B^j-$\operatorname{dom} \varphi_e \subseteqq A'$. Further, if $B^{1-j} - A$ has an infinite recursive subset, we may suppose that $B^j = \operatorname{dom} \varphi_e$. However, if $B^{1-j} - A$ is immune, we may suppose that B was chosen so that $B - A$ is since $\Psi(B^{1-j}) = \boldsymbol{b}$. Thus in any case we may suppose that $B^j - \operatorname{dom} \varphi_e$ is finite. By a suitable adjustment of φ_e we can make $\operatorname{dom} \varphi_e = B^j$ unless $\Psi(B^j) = \Psi(B^{1-j})$. However, if $\Psi(B^j) = \Psi(B^{1-j})$, $\boldsymbol{b}$ is a cylinder and the theorem is immediate. Thus we shall suppose that $\operatorname{dom} \varphi_e = B^j$. Clearly, $\lim_x \gamma_x(e, j) = \infty$ whence $\lim_x \delta_x(e, j) = \infty$. Let (e, j) now be the least pair in P such that $\lim_x \delta_x(e, j) = \infty$, then $\lim_x \gamma_x(e, j) = \infty$ and $\Psi(B^j) \leqq \Psi(B^{1-j}) = \boldsymbol{b}$. For each pair (f, k) in P which is $< (e, j)$, $\lim_x \delta_x(f, k)$ is finite because $\delta_s(f, k)$ is increasing with s. Let

$$m = \sup \{\delta_x(f, k) \mid x \geqq 0 \quad \& \quad (f, k) \in P \quad \& \quad (f, k) < (e, j)\} .$$

At any step s where $\beta(s) \geqq m$, we can have $\beta(s)$ enumerated in B^{1-j} only if $\delta_s(e, j) \leqq \beta(s)$. But $\delta_s(e, j)$ increases with s and $\lim_x \delta_x(e, j) = \infty$, therefore B^{1-j} is recursive. Now

$$\boldsymbol{a} = \Psi(N) = \Psi(B^{1-j}) \oplus \psi((B^{1-j})') .$$

But $\boldsymbol{a}$ is indecomposable and $\Psi(B^{1-j}) = \boldsymbol{b} < \boldsymbol{a}$, whence $\Psi((B^{1-j})') = \boldsymbol{a}$. The last equation is now $\boldsymbol{a} = \boldsymbol{b} \oplus \boldsymbol{a}$ which proves the theorem.

THEOREM 7. *Let $\boldsymbol{a}$ be an* r.e. *one-one degree which has only a finite number of predecessors then the one-one degrees $\leqq \boldsymbol{a}$ form a distributive lattice.*

Proof. Let $\boldsymbol{b}, \boldsymbol{c}, \boldsymbol{a}^*$ be one-one degrees $\leqq \boldsymbol{a}$ such that one of $\boldsymbol{b}/\boldsymbol{a}^*$, $\boldsymbol{c}/\boldsymbol{a}^*$ is finite. Then $\min(\boldsymbol{b}/\boldsymbol{a}^*, \boldsymbol{c}/\boldsymbol{a}^*)\boldsymbol{a}^*$ is $\leqq$ both $\boldsymbol{b}$ and $\boldsymbol{c}$. Hence

$$(5) \qquad \boldsymbol{b} \cap \boldsymbol{c}/\boldsymbol{a}^* = \min(\boldsymbol{b}/\boldsymbol{a}^*, \boldsymbol{c}/\boldsymbol{a}^*) .$$

It is clear that we can relax the condition that one of $\boldsymbol{b}/\boldsymbol{a}^*, \boldsymbol{c}/\boldsymbol{a}^*$ be finite. Now let $\boldsymbol{a}^*$ be a maximal indecomposable degree $\leqq \boldsymbol{a}^*$. Then in the proof of Theorem 4 we may choose $\boldsymbol{a}_1 = \boldsymbol{a}^*$ and so from the construction of $\boldsymbol{b} \cup \boldsymbol{c}$ in that proof

$$(6) \qquad \boldsymbol{b} \cup \boldsymbol{c}/\boldsymbol{a}^* = \max(\boldsymbol{b}/\boldsymbol{a}^*, \boldsymbol{c}/\boldsymbol{a}^*) .$$

It is clear that (6) also holds when $\boldsymbol{a}^*$ is a cylinder.

Now let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ be any r.e. one-one degrees $\leqq \boldsymbol{a}$, to prove the theorem it is sufficient to prove

$$(\boldsymbol{x} \cup \boldsymbol{y}) \cap \boldsymbol{z} \leqq (\boldsymbol{x} \cap \boldsymbol{z}) \cup (\boldsymbol{y} \cap \boldsymbol{z}) .$$

From Lemma 2 it is therefore sufficient to show that

$$\{(\boldsymbol{x} \cup \boldsymbol{y}) \cap \boldsymbol{z}\}/\boldsymbol{a}^* \leqq \{(\boldsymbol{x} \cap \boldsymbol{z}) \cup (\boldsymbol{y} \cap \boldsymbol{z}\}/\boldsymbol{a}^*$$

for every indecomposable degree $\boldsymbol{a}^* \leqq \boldsymbol{a}$. From Theorem 6 any such $\boldsymbol{a}^*$ is either a maximal indecomposable degree $\leqq \boldsymbol{a}$ or is a cylinder, thus (5) and (6) are applicable. Using (5) and (6) this reduces to showing that

$$\min(\max(\boldsymbol{x}/\boldsymbol{a}^*, \boldsymbol{y}/\boldsymbol{a}^*), \boldsymbol{z}/\boldsymbol{a}^*) \leqq \max(\min(\boldsymbol{x}/\boldsymbol{a}^*, \boldsymbol{z}/\boldsymbol{a}^*) ,$$

$\min(\boldsymbol{y}/\boldsymbol{a}^*, \boldsymbol{z}/\boldsymbol{a}^*))$.

However, it is easy to verify that equality obtains in the last line using only the definitions of max and min. This completes the proof of the theorem.

5. **Conclusion.** Above we have hardly touched upon the general question of characterizing arbitrary initial segments of the one-one degrees. A certain amount of information can be obtained by studying the map Ψ introducted in the proof of Lemma 2. However, there is a serious obstacle to a complete solution of the problem. Suppose that we are considering the one-one degrees $\leqq \boldsymbol{a}$ where $\boldsymbol{a}$ is a cylinder. If there is a noncylinder $\boldsymbol{b} \leqq \boldsymbol{a}$ then from the work of Young in [7] there exists a collection of one-one degrees $\leqq \boldsymbol{a}$, and in the same many-one degree as $\boldsymbol{b}$, which has the order type of the rationals. Thus it

seems that the approach to initial segments via finite substructures which has been successful for the many-one degrees in [3], is doomed to failure in the case of one-one degrees.

Our final comment concerns the relation between the notion of one-one degree used in this paper and the usual notion. Let $\boldsymbol{a}$ be the one-one degree of a simple set such that the order types of the initial segments of one-one degrees $\leqq \boldsymbol{a}$, under our definition and the usual definition, are θ and θ^* respectively. From Theorem 1 of [6] it is easy to see that

$$\theta^* = \omega \oplus \{(\theta - \{1\}) \circ (\omega^* \oplus \omega)\} \oplus \omega^* ,$$

where $\theta - \{1\}$ is the order type obtained from θ by deleting its greatest member, where $\oplus$ and $\circ$ denote ordinal sum, and product respectively, where ω is the order type of the natural numbers, and where ω^* is the converse of ω.

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RANK k GRASSMANN PRODUCTS

M. J. S. LIM

The general question concerning the structure of subspaces of a symmetry class of tensors in which every nonzero element has an irreducible representation as a sum of decomposable (or pure) elements of a given length is as yet largely unanswered. This problem relates to the problem of characterizing the linear transformations on such a symmetry class which map the set of tensors of "irreducible length" k into itself; i.e., preserves the *rank* k of the tensors. Another related problem is: "Is it possible to obtain algebraic relations involving the components of a tensor which imply it has rank ("Irreducible length") k, for any positive integer k"?

This paper is concerned mostly with the third question for the $\binom{n}{r}$-dimensional Grassmann Product Space $\wedge^r U$, where U is an n-dimensional vector space over a field F. It includes some discussion of the first question for F *algebraically closed* and $r = 2$.

A vector in $\wedge^r U$ is said to have *rank* k if it can be expressed as the sum of k, and *not less than* k, nonzero pure r-vectors in $\wedge^r U$. We denote the set of such vectors by $C_k^r(U)$. The nonzero pure products in $\wedge^r U$ have rank one.

The results obtained in this paper are as follows: (i) the rank of a vector in $\wedge^r U$ is unchanged if we extend U, (ii) in the Grassmann Algebra $\wedge^0 U + \wedge^1 U + \cdots + \wedge^r U + \cdots$, multiplication of a Grassmann product by a nonzero vector in U either annihilates it or preserves its rank, (iii) we can associate with each vector z in $C_k^r(U)$ a unique subspace $U(z)$ in U, (iv) if $z \in C_k^r(U)$ and dim $U(z)$ is rk, then z has rank k, $(v) x_1 \wedge y_1 + \cdots + x_s \wedge y_s \in C_s^2(U)$ if and only if $\{x_1, y_1, \cdots, x_s, y_s\}$ is independent. Finally, we discuss the rank two subspaces in $\wedge^2 U$ when dim $U = 4$. If F is *algebraically closed*, these subspaces are of dimension one. Otherwise, they can be different, as the examples show.

In this paper, $Q(k, t, n)$ will denote the totality of strictly increasing sequences of k integers chosen from $t, t+1, \cdots, n$; $S(k, t, n)$ the totality of sequences of k integers chosen from $t, t+1, \cdots, n$.

Let $x_1, \cdots, x_n$ be a basis of U. For $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$, we denote the product $x_{i_1} \wedge \cdots \wedge x_{i_r}$ by x_ω.

Let p be an r-linear alternating function from $\pi_{i=1}^r E \to F$, $E = \{1, \cdots, n\}$.

We will need the following known result.

THEOREM 1. (See [2], p. 289–312.) *Let*

$$z = \sum p(\omega)\boldsymbol{x}_\omega, \ (\omega \in Q(r, 1, n)) \ .$$

Then z is a pure vector if and only if

$$(1) \qquad \sum_{\mu=0}^{r} (-1)^\mu p(\alpha, j_\mu)p(j_0, \cdots, j_{\mu-1}, j_{\mu+1}, \cdots, j_r) = 0$$

for all $\alpha \in S(r-1, 1, n)$ and all $(j_0, \cdots, j_r) \in S(r+1, 1, n)$.

Furthermore, there are $(n-r)$ independent equations in the system of equations (1).

The following lemma will be useful.

LEMMA 2. *Let $z = \sum p(\omega)\boldsymbol{x}_\omega$, $(\omega \in Q(r, 1, n); z \in C_k^r(U))$. Let s, m be integers, $0 \leqq s \leqq r$, $0 \leqq m \leqq n$, and let*

$$z' = \sum p(1, \cdots, s, \alpha)x_1 \wedge \cdots \wedge x_s \wedge \boldsymbol{x}_\alpha, \qquad (\alpha \in Q(m-s, s+1, m)) \ .$$

Then $z' \in C_l^r(U)$, for some l, $0 \leqq l \leqq k$.

Proof. We prove first the case $k = 1$.
Let $\omega = (i_1, \cdots, i_r) \in Q(r, 1, n)$. We set

$$p'(i_1, \cdots, i_r) = p(i_1, \cdots, i_r)$$

if $i_1 = 1, \cdots, i_s = s$, and $s+1 \leqq i_{s+1} < \cdots < i_r \leqq m$. Otherwise, $p'(i_1, \cdots, i_r) = 0$. Then $z' = \sum p'(\omega)\boldsymbol{x}_\omega; (\omega \in Q(r, 1, n))$. It is easy to show that the system of equations (1) holds for the p' s; (there are 3 cases to check; viz., $i_t > m$ or $j_t > m$ for some t; not all of the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ or not all of the integers $1, \cdots, s$ are present in $j_0, \cdots, j_r$; and, thirdly, all the integers $1, \cdots, s$ are present in $i_1, \cdots, i_{r-1}$ and in $j_0, \cdots, j_r$ with $i_t \leqq m$ $(t = 1, \cdots, r-1)$ and $j_l \leqq m$ $(l = 0, \cdots, r)$). Thus, by Theorem 1, $z' \in C_1^r(U)$ or is zero.

For $z = z_1 + \cdots + z_k \in C_k^r(U)$, $z_i \in C_1^r(U)$ $(i = 1, \cdots, k)$, we apply the above result to each term z_i, noting that

$$z' = (z_1 \cdots + z_k)' = z_1' + \cdots + z_k' \ .$$

THEOREM 3. *Let $U' \subseteq U$ be a subspace.*
Then $C_k^r(U') \subseteq C_k^r(U)$.

Proof. Let $x_1, \cdots, x_s$ be a basis of U', and let $x_1, \cdots, x_n$ be an extension of this basis to a basis of U. Let

$$y_1 + \cdots + y_k \in C_k^r(U'), \ y_i \in C_1^r(U') \ .$$

Suppose $y_1 + \cdots + y_k = z_1 + \cdots + z_l \in C_l^r(U), z_i \in C_1^r(U)$. Clearly

$l \leqq k$.

To show $l \geqq k$, let

$$z_j = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega,\ \omega \in Q(r, 1, n), \qquad 1 \leqq j \leqq l\ .$$

Since $y_i \in C_1^r(U'), 1 \leqq i \leqq k$, then

$$\sum_{j=1}^{l} p^{(j)}(\omega) = 0$$

whenever $\omega = (i_1, \cdots, i_r)$ and $\{i_1, \cdots, i_r\} \not\subseteq \{1, \cdots, s\}$. Hence

$$z'_j = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega,\ \omega \in Q(r, 1, s)\ ,$$

is in $C_1^r(U')$ by Lemma 2, and since $z'_1 + \cdots + z'_l = z_1 + \cdots + z_l = y_1, \cdots, y_k$, the $l \geqq k$.

DEFINITION. For $z \in C_k^r(U)$, we define $R_r(z) = k$; i.e., R_r: $\wedge^r U \to J$ such that $R_r(z) = k$ if and only if $z \in C_k^r(U)$.

We will drop the index r when no confusion arises.

If $x \in U, z \in \wedge^r U$ such that $z = \sum p(\omega)\boldsymbol{x}_\omega, \omega \in Q(r, 1, n)$, where $x_1, \cdots, x_n$ is a basis of U, then we write $x \wedge z$ for the vector

$$\sum p(\omega)x \wedge \boldsymbol{x}_\omega, \omega \in Q(r, 1, n)\ .$$

If $z = x_1 \wedge \cdots \wedge x_r$ is a nonzero pure vector in $\wedge^r U$, then we shall denote the r-dimensional space $\langle x_1, \cdots, x_n\rangle$ by $U(z)$.

THEOREM 4. *Let* $y = y_1 + \cdots + y_k \in C_k^r(U), y_i \in C_1^r(U), 1 \leqq i \leqq k$.

(i) *Suppose* $x \wedge (y_1 + \cdots + y_k) = 0, x \in U$. *Then* $x \in U(y_i)$, $i = 1, \cdots, k$.

(ii) *Suppose* $x \in U, x \notin U(y_1) + \cdots + U(y_k)$. *Then* $x \wedge y \in C_k^{r+1}(U)$.

Proof. (i) Suppose on the contrary that $x \notin U(y_1)$. Then

$$x \wedge y_1 = x \wedge \left(-\sum_{i=2}^{k} y_i\right) \neq 0\ .$$

Thus, we can choose a basis $x_1, \cdots, x_n$ of U such that

$$x = x_1, y_1 = x_2 \wedge \cdots \wedge x_{r+1}\ .$$

Then

$$\left(-\sum_{i=2}^{k} y_i\right) = x_2 \wedge \cdots \wedge x_{r+1} + \sum p(1, \alpha)x_1 \wedge \boldsymbol{x}_\alpha, (\alpha \in Q(r-1, 2, n))\ .$$

Hence $(-\sum_{i=2}^{k} y_i) = y_1 + x \wedge v$, where $v = \sum p(1, \alpha)\boldsymbol{x}_\alpha \in \wedge^{r-1} U$. Taking

$s = 1$, $m = n$ in Lemma 2, it is easy to see that since $R(-\sum_{i=2}^{k}) = k - 1$, then $R(x \wedge v) \leqq k - 1$. But $x \wedge v = -(y_1 + \cdots + y_k)$ which implies $R(x \wedge v) = k$. We have a contradiction. Therefore $x \in U(y_1)$. Similarly, $x \in U(y_i)$, $i = 2, \cdots, k$.

(ii) Suppose that

$$x \wedge y = z_1 \cdots + z_l \in C_l^{r+1}(U), z_i \in C_1^{r+1}(U) , \qquad 1 \leqq i \leqq l .$$

Clearly $l \leqq k$.

To show $l \geqq k$, we choose a basis $x_1, \cdots, x_n$ of U such that $x = x_1$ and $x_2, \cdots, x_s$ is a basis of $U(y_1) + \cdots + U(y_k)$. Then

$$y = \sum p(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, n)) .$$

Using (i) and the fact that $x \wedge (x \wedge y) = x_1 \wedge (z_1 + \cdots + z_l) = 0$, we can express each $z_j = x_1 \wedge (\sum p^{(j)}(\omega)\boldsymbol{x}_\omega); \omega \in Q(r, 2, n))$, $1 \leqq j \leqq l$.

Now $\sum_{j=1}^{l} p^{(j)}(\omega) = 0, (\omega = (i_1, \cdots, i_r))$, *unless*

$$\{i_1, \cdots, i_r\} \subseteq \{2, \cdots, s\} .$$

In the latter case, $\sum_{j=1}^{l} p^{(j)}(\omega) = p(\omega)$. Therefore, $z_1 + \cdots + z_l = z_1' + \cdots + z_l' = x \wedge y$ where

$$z_j' = \sum p^{(j)}(\omega)x_1 \wedge \boldsymbol{x}_\omega, (\omega \in Q(r, 2, s)) .$$

Hence $y = z_1'' + \cdots + z_l''$, where $z_j'' = \sum p^{(j)}(\omega)\boldsymbol{x}_\omega, (\omega \in Q(r, 2, s))$, which implies $R(y) \leqq l$, i.e., $k \leqq l$.

THEOREM 5. *Let* $y_i \in C_1^r(U), z_i \in C_1^r(U)$, $(i = 1, \cdots, k)$ *such that* $y_1 + \cdots + y_k = z_1 + \cdots + z_k$.

Then $U(y_1) + \cdots + U(y_k) = U(z_1) + \cdots + U(z_k)$.

Proof. Suppose on the contrary that there exists a vector $x \in U(y_1)$ such that $x \notin U(z_1) + \cdots + U(z_k)$. Since $x \wedge (y_1 + \cdots + y_k) = x \wedge (z_1 + \cdots + z_k)$, then

$$R(x \wedge (y_1 + \cdots + y_k)) = R(x \wedge (z_1 + \cdots + z_k)) \leqq k - 1 .$$

But, by Theorem 4 (ii), $R(x \wedge (z_1 + \cdots + z_k)) = k$, which is a contradiction.

DEFINITION. Let

$$z = z_1 + \cdots + z_k \in C_k^r(U), z_i \in C_1^r(U), \qquad i = 1, \cdots, k .$$

Then we define $U(z)$ to be the subspace $U(z_1) + \cdots + U(z_k)$.

THEOREM 6. *Let* $z_i \in C_1^r(U)$, $i = 1, \cdots, k$, *and let*

$$\dim [U(z_1) + \cdots + U(z_k)] = rk .$$

Then $R(z_1 + \cdots + z_k) = k$.

Proof. Suppose the Theorem is false. Let k be the smallest integer for which it fails. Clearly $k \geqq 2$. Let

$$z_1 + \cdots + z_k = y_1 + \cdots + y_l \in C_l^r(U), y_i \in C_1^r(U) .$$

Let $x \in U(z_1)$. Then $x \notin U(z_2) + \cdots + U(z_k)$. By the choice of

$$k, z_2 + \cdots + z_k \in C_{k-1}^r(U) .$$

Hence, by Theorem 4 (ii),

$$x \wedge (z_2 + \cdots + z_k) = x \wedge (z_1 + \cdots + z_k) = x \wedge (y_1 + \cdots + y_l) ,$$

and $l \geqq k - 1$. But we assumed $l < k$. Therefore $l = k - 1$.

By Theorem 5,

$$U(x \wedge z_2) + \cdots + U(x \wedge z_k) = U(x \wedge y_1) + \cdots + U(x \wedge y_{k-1}) .$$

Hence $\langle x \rangle + U(z_2) + \cdots + U(z_k) = \langle x \rangle + U(y_1) + \cdots + U(y_{k-1})$.

Now let $x' \in U(z_1)$, independent of x. Then again

$$\langle x' \rangle + U(z_2) + \cdots + U(z_k) = \langle x' \rangle + U(y_1) + \cdots + U(y_{k-1}) .$$

Taking intersections, we obtain

$$U(z_2) + \cdots + U(z_k) = U(y_1) + \cdots + U(y_{k-1}) .$$

By a similar argument,

$$\begin{aligned} V_i &= U(z_1) + \cdots + U(z_{i-1}) + U(z_{i+1}) + \cdots + U(z_k) \\ &= U(y_1) + \cdots + U(y_{k-1}) . \end{aligned}$$

Hence $U(y_1) + \cdots + U(y_{k-1}) = \bigcap_{i=1}^k V_i = \{0\}$, which is impossible. The result follows.

THEOREM 7. $\sum_{i=1}^s x_i \wedge y_1 \in C_s^2(U)$ *if and only if* $(\{x_1, y_1, \cdots x_s, y_s\}$ *is independent.*

Proof. If $\{x_1, y_1, \cdots, x_s, y_s\}$ is dependent, it is easy to show that $R(\sum_{i=1}^s x_i \wedge y_i) \leqq s - 1$. It follows that the condition is necessary.

The converse follows easily from Theorem 6.

COROLLARY 8. *Let* $f = \sum_{i=1}^s x_i \wedge y_i$, *and* $\dim \langle x_1, y_1, \cdots, x_s, y_s \rangle < 2k, k \leqq s$. *Then* $R(f) \leqq k - 1$.

We shall now direct our attention to the rank 2 subspaces of $\wedge^2 U$.

DEFINITION. A *rank* 2 *subspace* H in $\wedge^2 U$ is a subspace whose nonzero members are in $C_2^2(U)$.

In this paper, we shall restrict our considerations to the case dim $U = 4$. It is clear from Theorem 7 that $C_2^2(U)$ is empty when dim $U < 4$.

LEMMA 9. *Let* $f \in C_2^2(U)$ *and let* $\{y_1, \cdots, y_4\}$ *be any basis of* $U(f)$. *Then* f *has a representation* $f = y_1 \wedge u + v \wedge w$, *where* $\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle$.

Proof. Since $f \in \wedge^2\langle y_1, \cdots, y_4\rangle$, then

$$\begin{aligned} f &= \sum p(\omega)\boldsymbol{y}_\omega, (\omega \in Q(2, 1, 4)), p(\omega) \in F , \\ &= y_1 \wedge (\textstyle\sum_{j=2}^4 p(1, j)y_j) + \sum p(\alpha)\boldsymbol{y}_\alpha ; \qquad (\alpha \in Q(2, 2, 4)) , \end{aligned}$$

which is of the form $y_1 \wedge u + v \wedge w$. It follows from Theorem 7 and its corollary, and the fact that $R(f) = 2$ that

$$\langle u, v, w\rangle = \langle y_2, y_3, y_4\rangle .$$

THEOREM 10. *Let* dim $U = 4$ *and let* H *be a rank* 2 *subspace in* $\wedge^2 U$. *Then* dim $H = 1$, *provided* F *is algebraically closed.*

Proof. Let f be a nonzero member of H. Then f has a representation $f = x_1 \wedge x_2 + x_3 \wedge x_4$ in $C_2^2(U)$. By Theorem 7,

$$U = U(f) = \langle x_1, \cdots, x_4\rangle .$$

If f' is any other nonzero member of H, then $U(f') = \langle x_1, \cdots, x_4\rangle$. By Lemma 9, $f' = x_1 \wedge u + v \wedge w, \langle u, v, w\rangle = \langle x_2, x_3, x_4\rangle$. Hence dim $\langle v, w\rangle \bigcap \langle x_3, x_4\rangle \leqq 1$. Without loss of generality, we shall assume $x_3 \in \langle v, w\rangle \bigcap \langle x_3, x_4\rangle$. Hence

$$f' = x_1 \wedge u + x_3 \wedge w', \langle u, w'\rangle \subset \langle x_2, x_3, x_4\rangle .$$

Let $u = \sum_{i=2}^4 b_i x_i$; $w' = \sum_{i=2,4} d_i x_i$; $b_i, d_i \in F$. Then for

$$\begin{aligned} \lambda \in F, z = \lambda f + f' = x_1 \wedge (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4) \\ + x_3 \wedge (\lambda x_4 + d_2 x_2 + d_4 x_4) . \end{aligned}$$

The condition that the vectors

$$x_1, (\lambda x_2 + b_2 x_2 + b_3 x_3 + b_4 x_4), x_3, (\lambda x_4 + d_2 x_2 + d_4 x_4)$$

be independent; i.e., $R(z) = 2$, is equivalent to the condition that the determinant

$$\Gamma(\lambda, f_1, f_2) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda + b_2 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ 0 & d_2 & 0 & \lambda + d_4 \end{vmatrix} \text{ be}$$

nonzero. Now

$$\Gamma(\lambda, f_1, f_2) = \lambda^2 + \lambda(d_4 + b_2) + (b_2 d_4 - d_2 b_4) = g(\lambda) .$$

Since u, w' are independent, $(b_2 d_4 - d_2 b_4) \neq 0$. Hence $g(\lambda)$ is a non-trivial polynomial in λ, and hence, for some nonzero λ in F, $g(\lambda) = 0$; i.e., $\Gamma(\lambda, f_1, f_2) = 0$. For such a λ, $R(z) \leqq 1$. It follows that dim $H = 1$.

The above theorem is false when F is *nonalgebraically closed.* For example, the vectors

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (x_3 + x_4) + (x_3 - x_2) \wedge x_4$$

in $C_2^2(U)$, where $U = \langle x_1, \cdots, x_4 \rangle$, dim $U = 4$, $F \equiv$ Reals, generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

It is interesting to note that if F (nonalgebraically closed) has an *irreducible quadratic* polynomial $h(\lambda)$, and dim $U = 4$, then we can construct 2 independent vectors f_1, f_2 in $C_2^2(U)$, which will generate a 2-dimensional rank 2 subspace in $\wedge^2 U$, and such that $\Gamma(\lambda, f_1, f_2) = h(\lambda)$ (see Theorem 10). The construction is as follows:

Let dim $U = 4$, $U = \langle x_1, \cdots, x_4 \rangle$. Let $h(\lambda) = \lambda^2 + a_1\lambda + a_0$ be irreducible in F. The companion matrix of $h(\lambda)$ is

$$B = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}; \qquad \lambda I - B = \begin{bmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{bmatrix}.$$

Now

$$\det(\lambda I - B) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & a_0 & 0 & \lambda + a_1 \end{vmatrix} = h(\lambda) \neq 0 .$$

Taking this determinant to be $\Gamma(\lambda, f_1, f_2)$ corresponding to $z = \lambda f_1 + f_2$, where $f_1, f_2 \in C_2^2(U), \lambda \in F$, we have

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$
$$f_2 = x_1 \wedge (-x_4) + x_3 \wedge (a_0 x_2 + a_1 x_4) .$$

The construction is complete. Thus, for example, if $F \equiv$ Rationals and $h(\lambda) = \lambda^2 - 2$, then

$$f_1 = x_1 \wedge x_2 + x_3 \wedge x_4$$

and

$$f_2 = x_1 \wedge (-x_4) + (-2)x_3 \wedge x_2 ,$$

and f_1, f_2 generate a 2-dimensional rank 2 subspace in $\wedge^2 U$.

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University of British Columbia

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MULTIPLIER ALGEBRAS OF BIORTHOGONAL SYSTEMS

RAYMOND J. MCGIVNEY AND WILLIAM RUCKLE

Let $\{e_i, E_i\}$ be a total biorthogonal system in a linear topological space X. The multiplier algebra of X with respect to $\{e_i, E_i\}$ written $M(X)$ is the set of all scalar sequences $(t^{(i)})$ such that for each $x \in X$ there is $y \in X$ with

$$E_i(y) = t^{(i)}E_i(x) .$$

The form of $M(X)$ is determined when $\{e_i, E_i\}$ is a norming complete biorthogonal system in a Banach space or a basis in a complete barreled space. It is shown that a sequence space is the multiplier algebra for a basis in a Banach space if and only if it is a γ-perfect BK-algebra.

A *biorthogonal system* is a double sequence $\{e_i, E_i\}$ with each e_i in a locally convex space X and each E_i a continuous linear functional on X (i.e., in X^*) which satisfies the relationship

$$E_i(e_j) = \delta_{ij} \text{ (Kronecker } \delta) \quad i, j = 1, 2, \cdots .$$

The biorthogonal system is *total* if $\{E_i\}$ is total on X; that is, $E_i(x) = 0$ for each i implies $x = 0$. If $\{e_i, E_i\}$ is a total biorthogonal system then the space X can be identified with the space of all sequences $(E_i(x))$ by means of the natural correspondence x to $(E_i(x))$. Under this correspondence e_i becomes the ith coordinate vector, the sequence which has a one in the ith coordinate and 0's elsewhere and E_i becomes the ith coordinate functional, the functional whose value on the sequence $(x^{(1)}, x^{(2)}, \cdots)$ is $x^{(i)}$. This identification will be assumed whenever a total biorthogonal system is considered.

DEFINITION 1.1. Let $\{e_i, E_i\}$ be a total biorthogonal system in a locally convex space X. A scalar sequence

$$t = (t^{(1)}, t^{(2)}, \cdots)$$

is a *multiplier of X with respect to $\{e_i, E_i\}$* if for each $x \in X$ there is $y \in X$ for which

$$E_i(y) = t^{(i)}E_i(x) \qquad i = 1, 2, \cdots .$$

The set of all such t is written $M(X; e_i, E_i)$ or simply $M(X)$ and called the *multiplier algebra of X (with respect to $\{e_i, E_i\}$)*.

In other words $M(X)$ is the set of all t such that

$$tx \in X \text{ whenever } x \in X$$

where X is now considered a sequence space and multiplication of sequences is defined coordinatewise. It is now obvious that $M(X)$ forms a linear algebra of operators from X into X; namely the operators which are diagonal with respect to $\{e_i, E_i\}$. Multiplication in this algebra is defined coordinatewise. The purpose of this paper is to study the properties of the space $M(X)$ and the possible forms which it can assume with varying hypotheses on X or on $\{e_i, E_i\}$. Results of this type were obtained by Yamazaki in [11] and [12] for $\{e_i\}$ a basis in a Banach space. The concept of imultiplier space is implicitly treated in [4].

Throughout this paper it is immaterial whether the scalar field considered consists of the real or complex numbers.

2. Sequence spaces: notation and basic facts. A set of scalar sequences which is closed under coordinatewise addition and scalar multiplication is a *sequence space*; if it is closed under coordinatewise multiplication as well it will be called a *sequence algebra*. The ith coordinate vector is written e_i; the ith coordinate functional, E_i. If each E_i is continuous on a locally convex sequence space (algebra) X and $e_i \in X$ for each i then X is called a *K-space* (algebra). If in addition X is an F-space (complete metric linear space) X will be called an FK-space or FK-algebra as the case may be. If X is a Banach space (algebra), X will be called a BK-space (algebra). Note that in an FK-algebra X the functions $x \to tx$ and $x \to xt$ are continuous in x for fixed t by the continuity of the coordinate functionals and the closed graph theorem. This is enough to conclude that a BK-algebra is a Banach algebra without identity. See p. 860 and 861 of [3].

The following are well known sequence spaces. For additional discussion see Chapter IV of [2], [5] or p. 289 of [10]:

ω sometimes called s is the set of all scalar sequences. Endowed with the topology of coordinatewise convergence it is an FK-algebra.

φ is the linear span of $\{e_i\}$ in ω, i.e., the space of all finitely nonzero sequences.

l^1 is the set of all sequences t such that

$$\| t \| = \sum_{i=1}^{\infty} | t^{(i)} | < \infty$$

which is a BK-space with this norm.

m is the set of all sequences t such that

$$\| t \| = \sup_i | t^{(i)} | < \infty$$

which is a BK algebra with this norm.

bs is the set of all sequences t such that

$$\| t \| = \sup_n | \sum_{i=1}^{n} t^{(i)} | < \infty$$

which is a BK space with this norm.

cs is the closed linear span of $\{e_i\}$ in bs; it consists of all sequences t such that $\sum_{i=1}^{\infty} t^{(i)}$ converges.

bv is the set of all sequences t such that

$$\| t \| = \lim_n | t_n | + \sum_{i=1}^{\infty} | t_i - t_{i+1} | < \infty$$

which is a BK-algebra with this norm. See §3 of [4] or p. 3 of [11]. Yamazaki denoted bv by w.

NOTATION 2.1. (a) Let $t, u \in \omega$ be such that $ut \in cs$; the sum $\sum_{i=1}^{\infty} u^{(i)} t^{(i)}$ is denoted by (u, t).

(b) For $A \subseteqq \omega$

A^{α}, the α-dual of A is $\{t: ut \in l^1, u \in A\}$

A^{β}, the β-dual of A is $\{t: ut \in cs, u \in A\}$

A^{γ}, the γ-dual of A is $\{t: ut \in bs, u \in A\}$.

(c) For A and $B \subseteqq \omega$

$AB = \{uv: u \in A, v \in B\}$.

(d) For $t \in \omega$ and $A \subseteqq \omega$

$t^{-1}A = \{u \in \omega: tu \in A\}$.

(e) For $A \subseteqq \omega$

$A^{\varphi} = \{t \in \varphi: | (t, u) | \leqq 1, u \in A\}$.

(f) For $A \subseteqq \varphi$

$A^{\omega} = \{t \in \omega: | (t, u) | \leqq 1, u \in A\}$.

(g) For X a K-space, X^{δ} is the space of all sequences $(f(e_i))$ as f ranges over X^*. Note that for $t \in X^{\delta}$ with $t^{(i)} = f(e_i)$, $E_i(t) = t^{(i)} = f(e_i)$.

Gamma-perfect BK-spaces can be constructed by means of sequential norms. A sequential norm (s.n.) is a function P from ω into R^* which is an extended norm and in addition satisfies the condition

$$P(x) = \sup_n P\left(\sum_{i=1}^{n} x^{(i)} e_i\right) x \in \omega \ .$$

If

$$0 < \inf_n P(e_n) \leqq \sup_n P(e_n) < \infty$$

P is a proper sequential norm (p.s.n.). For P a s.n, S_P is the set of all $x \in \omega$ for which $P(x) < \infty$ endowed with the topology determined by $P\varepsilon$. The closed linear span of $\{e_1, e_2, \cdots\}$ in S_P is denoted by S_P^0. The

following proposition contains information about sequential norms which was derived in [7] and [8] and which we shall use in §6.

PROPOSITION 2.2. (a) *If P is a* s.n. *S_P is a γ-perfect BK-space. If X is a γ-perfect BK-space there is an* s.n. *P such that $X = S_P$.*

(b) *If P is a* s.n. *the function P' given by*

$$P'(x) = \sup\left\{\sup_n |\sum_{i=1}^{n} x^{(i)}y^{(i)}| : P(y) \leqq 1\right\}$$

is a s.n. *and $P'' = P$. If P is a* p.s.n. *so is P'.*

(c) $(S_P^0)^\delta = S_{P'}$ *and* $(S_{P'}^0)^\delta = S_P$.

(d) *An* s.n. *P is a* p.s.n. *if and only if $l^1 \subseteqq S_P \subseteqq m$.*

3. Preliminary results.

PROPOSITION 3.1. *If $\{e_i, E_i\}$ is a total biorthogonal system in X*

$$M(X) = \cap\{y^{-1}X : y \in S\} \tag{3-1}$$

where S is any absorbing subset of X.

Proof. Let R denote the set on the right of (3-1). If $t \in R$ and $x \in X$ there is $a > 0$ such that $ax \in S$. Thus $atx \in X$ so that $tx \in X$ which implies $t \in M(X)$. If $t \in M(X)$ then $tx \in X$ for every $x \in X$ so in particular for every $x \in S$.

A complete biorthogonal system is a total biorthogonal system $\{e_i, E_i\}$ on X such that sp $\{e_i\}$ $(= \varphi)$, the linear span of $\{e_i\}$ is dense in X.

PROPOSITION 3.2. *Let $\{e_i, E_i\}$ be a complete biorthogonal system in X.*

(a) *For each $t \in M(X)$, the mapping $x \to tx$ is a closed linear operator from X into X.*

(b) *The set $M_c(X)$ of all $t \in M(X)$ for which $x \to tx$ is continuous is a closed sub-algebra of $\mathscr{L}(X)$ where $\mathscr{L}(X)$ has any topology containing the topology of simple convergence. Here $\mathscr{L}(X)$ denotes the space of continuous operators from X into X.*

Proof. (a) Obvious.

(b) Define $E_i \otimes e_j$ on $\mathscr{L}(X)$ by

$$E_i \otimes e_j(T) = E_i(Te_j) .$$

Then $E_i \otimes e_j$ is a continuous linear functional on $\mathscr{L}(X)$ given the topology of simple convergence. Therefore

$$S = \cap\{[E_i \otimes e_j]^{-1}(0) : i \neq j\}$$

is closed in this topology and every topology containing it.

The following statement generalizes Theorem 1 of [11].

COROLLARY 3.3. *If $\{e_i, E_i\}$ is a complete biorthogonal system in a Banach space X then there is a topology on $M(X)$ which makes it a BK algebra.*

Proof. In this case $M_c(X) = M(X)$ since the mapping $x \to tx$ is closed. Thus by 3.2 $M(X)$ is a Banach algebra with the norm

$$\| t \| = \sup\{\| tx \| : \| x \| \leqq 1\} .$$

Each E_i is continuous on $M(X)$ since

$$E_i(T) = E_i \otimes e_i(T_t)$$

where $T_t(x) = tx$; and $E_i \otimes e_i$ is continuous on $\mathscr{L}(X)$.

PROPOSITION 3.4. *If $\{e_i, E_i\}$ is a total biorthogonal system in a linear topological space X then*

$$M_c(X; e_i, E_i) \subseteqq M_c(X_0; e_i, E_i)$$

where X_0 is the closed linear span of $\{e_i\}$ in X.

Proof. If $t \in M_c(X)$ then $tx \in \varphi$ for $x \in \varphi$. Since φ is dense in X_0 and t is continuous $tx \in X_0$ for $x \in X_0$ so that $t \in M_c(X_0)$.

If $\{e_i, E_i\}$ is a complete biorthogonal system on X, X^* is isomorphic to X^δ under the correspondence of f in X^* to $(f(e_i))$ in X^δ and $\{e_i, E_i\}$ is a total biorthogonal system on X^δ.

PROPOSITION 3.5. *If $\{e_i, E_i\}$ is a complete biorthogonal system in a locally convex space X then*

$$M_c(X; e_i, E_i) \subseteqq M(X^\delta; e_i, E_i) .$$

Proof. If $t \in M_c$ for $f \in X^*$ let $f_t(x) = f(tx)$, $x \in X$. Then $f_t \in X^*$ and $f_t(e_i) = t^{(i)} f(e_i)$ for each i so that $ty \in X^\delta$ for $y \in X^\delta$.

4. Multiplier algebras of a norming biorthogonal system in a Banach space. A biorthogonal system $\{e_i, E_i\}$ in a normed space X is called *norming* if the topology of X is determined by a norm of the type

$$\| x \| = \sup\{| f(x) | : f \in A\}$$

where A is a subset of the linear span of $\{E_i\}$ in X^*. An equivalent condition is that the above norm be given by

$$\|x\| = \sup\{|(x, t)| : t \in A\} \tag{4-1}$$

where A is a subset of φ.

If $\{e_i, E_i\}$ is a complete biorthogonal system which is norming on X and the norm of X is given by (4-1) it may be assumed that A consists of all sequences t in φ for which

$$|(t, x)| \leqq \|x\| \qquad x \in X\,. \tag{4-2}$$

Denote by $\hat{X}$ the space of all $x \in \omega$ for which

$$\|x\| = \sup\{|(x, t)| : t \in A\} < \infty\,. \tag{4-3}$$

The function defined in (4-3) is a norm since $a_n^{-1} e^n \in A$ for $n = 1, 2, \cdots$ where a_n is the norm of E_n as a member of X^*. With this norm $\hat{X}$ is a BK-space in which X is the closed linear span of $\{e_i\}$.

PROPOSITION 4.1. *The space X^δ consists of all $y \in \omega$ for which*

$$\|y\|' = \sup\{|(y, x)| : x \in A^\varphi\} < \infty\,.$$

The correspondence

$$f \text{ to } (f(e_i)) \qquad f \in X^* \tag{4-4}$$

is an isometry from X^* onto $(X^\delta, \|\|')$. The correspondence

$$g \text{ to } (g(e_i)) \qquad g \in (X_0^\delta)^* \tag{4-5}$$

is an isometry from $(X_0^\delta)^*$ onto $\hat{X}$, where X_0^δ denotes the closed linear span of $\{e_i\}$ in X^δ.

Proof. The correspondence in (4-4) is clearly well defined and linear.

If $f \in X^*$ and $x \in A^\varphi$ then $x \in X$ and $\|x\| \leqq 1$. Thus

$$\begin{aligned} |(f(e_i), x)| &= |\sum_i x^{(i)} f(e_i)| \\ &= |f(x)| \leqq \|f\| \end{aligned}$$

so that

$$\|(f(e_i))\|' \leqq \|f\|\,.$$

If $y \in X^\delta$ define f on $\{\varphi, \|\|\}$ by

$$f(x) = (y, x)\,.$$

Then f is a bounded linear functional on $\{\varphi, ||\,||\}$ for which

$$|f(x)| \leqq ||y||' x \in \varphi, ||x|| \leqq 1$$

because

$$A^{\varphi} = \{x \in \varphi: ||x|| \leqq 1\} .$$

Since φ is dense in X, f can be continuously extended to X with

$$||f|| \leqq ||y||' .$$

For this extended f

$$(f(e_i)) = (y, e_i) = y_i \qquad i = 1, 2, \cdots .$$

Therefore, the correspondence in (4–4) is an isometry.

That (4–5) is an isometry from $(X_0^{\delta})^*$ onto $\hat{X}$ will follow from an analogous argument if it is shown that

$$A^{\varphi\varphi} = A .$$

When A has the form given by (4–2).
That $A^{\varphi\varphi} \supseteq A$ is clear, if $z \in A^{\varphi\varphi}$ then

$$|(z, x)| \leqq 1 \qquad x \in A^{\varphi}$$

but A^{φ} is dense in the unit ball of X so that

$$|(z, x)| \leqq 1 \qquad x \in X, ||x|| \leqq 1 .$$

Thus if

$$f(x) = (z, x) \qquad x \in X$$

we have $||f|| \leqq 1$ and

$$f(e_i) = z_i \qquad i = 1, 2, \cdots$$

so that $z \in A$. It is here that the assumption that $\{e_i, E_i\}$ is norming was used.

THEOREM 4.2. *If $\{e_i, E_i\}$ is a norming complete biorthogonal system in the Banach space X then $M(X)$ is of the form*

$$\bigcup_{n=1}^{\infty} n(AA^{\omega})^{\omega} \tag{4–6}$$

where A is a coordinatewise bounded subset of φ which contains a multiple of e_i for each i.

Proof. Let A be given by (4–2) and let Z denote the sequence space (4–6).

By 3.4, 3.5 and the fact that $M_c(Y) = M(Y)$ for Y a Banach space we have

$$M(X) \subseteq M(X^\beta) \subseteq M(X_0^\beta) \subseteq M(\hat{X}) \subseteq M(X)$$

so that $M(X)$ and $M(\hat{X})$ are equal. It will be shown that $M(\hat{X}) = Z$. Suppose $t \in M(\hat{X})$, then there is k such that

$$\| tx \| \leqq k \| x \| \qquad x \in \hat{X} .$$

If $s \in A$ and $x \in A^\omega$

$$| (sx, t) | = | (s, tx) | \leqq k$$

so that

$$t \in k(AA^\omega)^\omega \subseteq Z .$$

If $z \in Z$ and $x \in \hat{X}$, $x/\| x \| \in A^\omega$ so if n is such that $z \in n(AA^\omega)^\omega$

$$| (t, zx) | = | (tx, z) | \leqq n \| x \|$$

for $t \in A$. Therefore $zx \in \hat{X}$.

Question. If $\{e_i, E_i\}$ is a complete biorthogonal system in the Banach space X and $M(X)$ has the form (4-6) is $\{e_i, E_i\}$ norming?

5. **Multiplier algebras of bases.** A biorthogonal system $\{e_i, E_i\}$ in a linear topological space X is a (Schauder) *basis* for X if

$$x = \sum_{i=1}^{\infty} E_i(x) e_i \qquad x \in X . \tag{5-1}$$

It is an *unconditional basis* if the convergence in (5-1) is unconditional. If X is a l.c.s. $\{e_i, E_i\}$ is an *absolute basis* if the convergence in (5-1) is absolute, i.e., if

$$\sum_{i=1}^{\infty} | E_i(x) | p(e_i) < \infty$$

for each $x \in X$ and each continuous seminorm p on X. It is clear that an absolute basis is unconditional.

The proofs of 5.1, 5.2, and 5.3 are omitted since these statements are essentially known. See p. 205 of [10] and Propositions 4 and 5 of [1].

LEMMA 5.1. *For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled* l.c.s. *X it is always true that $X^\gamma \subseteq X^\beta$.*

LEMMA 5.2. *For $\{e_i, E_i\}$ a basis in a locally convex space X, $X^\beta \subseteq X^\beta$.*

PROPOSITION 5.3. *For $\{e_i, E_i\}$ a complete biorthogonal system in a barreled locally convex space X the following are equivalent.*

(a) *$\{e_i, E_i\}$ is a basis for X.*

(b) $X^\delta = X^\beta$.

(c) $X^\delta = X^\gamma$.

THEOREM 5.4. *If $\{e_i, E_i\}$ is a basis of a complete barreled space X then*

$$M_c(X) = M(X) = (XX^\beta)^\beta = (XX^\gamma)^\gamma .$$

Proof. Suppose $t \in M(X)$, $x \in X$ and $y \in X^\beta$. Then $tx \in X$ so that $txy \in cs$ by 5.1 which implies $t \in (XX^\beta)^\beta$.

Let P denote the family of all continuous seminorms on X. Let $\hat{X}$ be the linear space of all $x \in \omega$ such that

$$p'(x) = \sup_n p\left(\sum_{i=1}^{n} x^{(i)} e_i\right) < \infty, \quad p \in P .$$

Since X is barreled, p' restricted to X is continuous and since φ is dense in X, $p'(x) \geqq p(x)$ for $x \in X$. Thus X is the closed linear span of $\{e_i\}$ in the space $\hat{X}$ with the topology determined by the seminorms $\{p' : p \in P\}$.

For $t \in (XX^\gamma)^\gamma$ define

$$p_t(x) = \sup_n p\left(\sum_{i=1}^{n} t^{(i)} x^{(i)} e_i\right) .$$

Since $t \in (XX^\gamma)^\gamma$, $\{\sum_{i=1}^{n} t^{(i)} x^{(i)} e_i : n = 1, 2, \cdots\}$ is a weakly bounded, thus a strongly bounded subset of X so that

$$P_t(x) < \infty \qquad x \in X ,$$

and p_t is a continuous seminorm on X. If $x \in \hat{X}$, $tx \in \hat{X}$ and

$$p'(tx) = p'_t(x)$$

so that $t \in M_c(\hat{X})$. By 3.4, $(XX^\gamma)^\gamma \subseteqq M_c(X)$. Thus

$$M_c(X) \subseteqq M(X) \subseteqq (XX^\gamma)^\beta \subseteqq (XX^\gamma)^\gamma \subseteqq M_c(X)$$

which establishes the result.

COROLLARY 5.5. *If $\{e_i, E_i\}$ is an unconditional basis of a complete barreled space X then*

$$M(X) = (XX^\alpha)^\alpha .$$

Proof. Since $\{e_i, E_i\}$ is an unconditional basis $X^\alpha = X^\delta$. If $t \in (XX^\gamma)^\gamma$

and $u \in XX^\gamma$ let $u = xy$ with $x \in X, y \in X^\gamma = X^\delta = X^\alpha$. Let $v^{(i)} = \operatorname{sgn} t^{(i)}u^{(i)}$ then $vy \in X^\alpha$ so that $vu \in XX^\gamma$. Hence, $vut \in bs$ so that $ut \in l^1$. Therefore, $(XX^\gamma)^\gamma \subseteqq (XX^\alpha)^\alpha$ from which the conclusion follows.

THEOREM 5.6. *Let $\{e_i, E_i\}$ be an absolute basis of a sequentially complete locally convex space X. If P is any family of continuous seminorms which determines the topology of X then*

$$M(X) = (AA^\alpha)^\alpha$$

where

$$A = \{(p(e_i)) : p \in P\} .$$

Proof. The hypotheses of this theorem imply that $X = A^\alpha$. Now $t \in M(X)$ if and only if $tx \in X$ whenever $x \in X$ which happens if and only if $txy \in l^1$ whenever $x \in A^\alpha = X$ and $y \in A$. This will hold if and only if $t \in (AA^\alpha)^\alpha$.

EXAMPLES. $M(\omega) = M(\varphi) = \omega$; $M(cs) = M(bv_0) = bv$; $M(c_0) = M(l^1) = m$.

Let X be the space of all real sequences x for which

$$p_k(x) = \sum_n |x_n| k^n < \infty \qquad k = 1, 2, \cdots .$$

Then X with the seminorms $p_1, p_2, \cdots$ is a nuclear F-space which is equivalent to the space of all infinitely differentiable real functions of period 2π. See §5 of [6]. If $A = (k^n) : k = 1, 2, \cdots\}$ it is clear that $X = A^\alpha$. If $x \in A^\alpha$, $x = xe \in AA^\alpha$ and if $x \in AA^\alpha$, $x = y(k^n)$ for some k and $y \in A^\alpha$. But

$$\sum_{n=1}^{\infty} |t_n| k^n h^n = \sum_{n=1}^{\infty} |t_n| (kh)^n$$

so $x \in A^\alpha$. Thus $M(X) = A^{\alpha\alpha} = X^\delta$.

The following theorem is a version of Theorem 3 and Corollary 2 of [4]. For the definition of B_r-complete see p. 162 of [9].

THEOREM 5.7. *A complete biorthogonal system $\{e_i, E_i\}$ in a space X which is barreled and B_r complete is a basis for X if and only if $M(X) \supseteqq bv$. It is an unconditional basis for X if and only if $M(X) \supseteqq m$.*

6. **Gamma-perfect BK-algebras.** A proper sequential norm P which satisfies the inequality

$$P(xy) \leqq P(x)P(y) \tag{6-1}$$

will be called an algebraic p.s.n. (a.p.s.n.).

THEOREM 6.1. *The following statements are equivalent for a sequence space* M:

(a) M *is a multiplier algebra for a basis in a Banach space*;

(b) M *is a* γ*-perfect BK-algebra containing* e;

(c) $M = S_P$ *for* P *an* a.p.s.n. *with* $P(e) < \infty$.

Proof. (a) $\Rightarrow$ (b). If M is a multiplier algebra for a basis in a Banach space X it is a BK-algebra containing e by 3.3 and γ-perfect since it is the γ-dual of XX^{γ} by 5.4.

(b) $\Rightarrow$ (c). Suppose M is a γ-perfect BK-algebra containing e. By 2.2 (a) there is a sequential norm Q such that $M = S_Q$. It is routine to verify that P given by

$$P(x) = \sup \{Q(xy): Q(y) \leqq 1\}$$

is a s.n. equivalent to Q (i.e., $S_P = S_Q$) which satisfies (6-1). It remains to show

$$0 < \inf_n P(e_n) \leqq \sup_n P(e_n) < \infty. \tag{6-2}$$

Since $P(e_n) = P(e_n e_n) \leqq P(e_n)^2$, the left inequality of (6-2) is valid. Since $e \in S_P$, $bs \supseteqq S_P^{\gamma}$ so that $bv \subseteqq S_P$. That S_P is γ-perfect follows form 2.2 (a). The identity map from bv into S_P is continuous and $\{e_n: n = 1, 2, \cdots\}$ is bounded in S_P. Hence the right inequality in (6-2) is true. Therefore, P is an a.p.s.n.

(c) $\Rightarrow$ (a). If P is an a.p.s.n. with $P(e) < \infty$, S_P is a BK-algebra with identity so $M(S_P) = S_P$. But since $S_P = (S_{P'}^0)^{\delta}$ and $S_{P'} = (S_P^0)^{\delta}$, $M(S_P^0) = M$ and $\{e_i, E_i\}$ is a basis for S_P^0 (2.2).

The following theorem gives a means of constructing γ-perfect BK-algebras with identity which are distinct from bv and m. Let N denote the sequence of positive integers and N_k a subsequence of the form

$$N_k = \{k(1) < k(2) < \cdots\}\,. \tag{6-3}$$

THEOREM 6.2. (a) *Let* $N_1, N_2, \cdots, N_r$ *be a partition of* N *with each* N_k *given by* (6-3). *For each* k *let* P_k *be an* a.p.s.n. *for which* $P_k(e) < \infty$. *Define* P *by*

$$P(x) = \max \{P_k(x^{k(1)}, x^{k(2)}, \cdots): k = 1, 2, \cdots r\}\,.$$

Then P *is an* a.p.s.n. *and* S_P *is a* γ*-perfect BK-algebra containing* e.

(b) *Let* $N_1, N_2, \cdots$ *be an infinite partition of* N *with each* N_k *given by* (6-3). *For each* k *let* P_k *be an* a.p.s.n. *for which*

(6-4) $$\sup_k P_k(e) < \infty \ .$$

Define P by

$$P(x) = \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \cdots)\} \ .$$

Then P is an a.p.s.n. *and S_P is a γ-perfect BK-algebra containing e.*

Proof. The proof of (a) is omitted since it is similar to but less difficult than that of (b).

(b) It is straightforward to verify that P is a norm. That P is a.s.n. follows from the equalities:

$$\begin{aligned}
&\sup_n P\Big(\sum_{i=1}^{n} x^{(i)} e_i\Big) \\
&\quad = \sup_n \sup_k \Big\{P_k\Big(\sum_{k(i)\leqq n} x^{k(i)} e_i\Big)\Big\} \\
&\quad = \sup_k \Big\{\sup_n P_k\Big(\sum_{k(i)\leqq n} x^{k(i)} e_i\Big)\Big\} \\
&\quad = \sup_k \{P_k(x^{k(1)}, x^{k(2)}, \cdots)\}
\end{aligned}$$

since each P_k is an s.n. That S_P is an algebra follows since

$$\begin{aligned}
P(xy) &= \sup_k \{P_k(x^{k(i)} y^{k(i)})\} \\
&\leqq \sup_k \{P_k(x^{k(i)}) P_k(y^{k(i)})\} \\
&\leqq P(x)P(y) \ .
\end{aligned}$$

Therefore, S_P is a γ-perfect BK-algebra; it contains e because of (6-4).

EXAMPLE. Let M be the set of all sequences x such that

$$P(x) = \sup_k \Big\{\sum_{i=1}^{\infty} |\, x^{k(i)} - x^{k(i+1)} \,| + \lim_{k(i)} |\, x^{k(i)} \,|\Big\} < \infty$$

for $N_1, N_2, \cdots$ a partition of the integers with each N_k given by (6-3). Then M is a γ-perfect BK-algebra containing e but is neither m nor bv. The sequence y with

$$y^{(i)} = \begin{cases} 1 \text{ for } i = k(1) \text{ for each } k \\ 0 \text{ otherwise} \end{cases}$$

is in M but not in bv. The sequence z with

$$z^{(i)} = \begin{cases} 1 \text{ for } i = k(2j-1) \text{ for each } j < k \text{ and all } k \\ 0 \text{ otherwise} \end{cases}$$

is in m but not M.

Question. Are there γ-perfect BK-algebras other than those in

the smallest class of BK-algebras which contain bv_∞ and m and are closed under the operations described in Theorem 6.2.?

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LAFAYETTE COLLEGE
LEHIGH UNIVERSITY

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ON THE RATE OF DECAY OF SOLUTIONS OF PARABOLIC DIFFERENTIAL EQUATIONS

J. K. ODDSON

In this paper upper and lower bounds are determined for the rate of growth or decay of solutions of parabolic equations for indefinitely increasing time. These bounds are obtained from comparison functions which are constructed systematically with the aid of elliptic extremal operators and in many cases are the best possible for the class of problems which we consider.

The behavior of solutions of parabolic equations for indefinitely increasing time has been investigated by several authors in recent years. We refer in particular to the papers [2], [3] and [9] and the survey article [7]. Most of these articles have been concerned with establishing the approach of solutions to a steady state condition under various assumptions on the coefficients. In the present paper we determine upper and lower bounds for the rate at which solutions decay (or possibly grow) in a half cylinder as $t \to \infty$. These results, presented in Theorems 1 and 2, are obtained for a class of parabolic equations, assuming only uniform parabolicity and boundedness of the coefficients. The main tool is the maximum principle and the use of comparison functions constructed with the aid of elliptic extremal operators [15, 16]. By this device the estimates which we obtain are often the best possible for the class of operators which we consider.

2. Notations and basic hypotheses. Let Ω be a bounded, open, connected subset of $E^n (n \geqq 1)$ with boundary Γ and closure $\bar{\Omega} = \Omega \cup \Gamma$. Let D denote the $(n+1)$-dimensional half cylinder $D = \Omega \times (0, \infty)$.

In this paper we shall be concerned with the linear differential operator

$$L - k\frac{\partial}{\partial t} \equiv \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial}{\partial x_i} + c(x,t) - k(x,t)\frac{\partial}{\partial t} \tag{2.1}$$

with real valued coefficients defined for all $(x,t) \equiv (x_1, \cdots, x_n, t) \in D$ and satisfying there the following hypothesis I:

(1) $\sum_{i=1}^{n} a_{ii}(x,t) \equiv 1$ for all $(x,t) \in D$;

(2) There exists a constant α, $0 < \alpha \leqq 1/n$, such that $\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqq \alpha \sum_{i=1}^{n} \xi_i^2$ for all $(x,t) \in D$ and all real n-vectors $(\xi_1, \cdots, \xi_n)$.

(3) $b_1(x,t), \cdots, b_n(x,t), c(x,t)$ are bounded in D;

(4) There exist constants k_0, k_1 such that $0 < k_0 \leqq k(x, t) \leqq k_1$ in D.

We remark that any uniformly parabolic operator of the form $\sum_{i,j=1}^{n} A_{ij}\,(\partial^2/\partial x_i \partial x_j) + \sum_{i=1}^{n} B_i\,(\partial/\partial x_i) + c - (\partial/\partial t)$ with bounded coefficients can be normalized to the form (2.1) by dividing by $\sum_{i=1}^{n} A_{ii}$.

Note that the conditions $I(1)$ and $I(2)$ imply that $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leqq [1-(n-1)\alpha]\sum_{i=1}^{n}\xi_i^2$ in D for all real n-vectors $(\xi_1, \cdots, \xi_n)$ and that in the case $\alpha = 1/n$ the principal part of the operator L reduces to $(1/n)\Delta$, where Δ denotes the n-dimensional Laplacian.

It is well known [4-7, 10, 14, 17-18] that under the hypotheses I the operator (2.1) enjoys the maximum and boundary point principles. Since we wish to permit possible discontinuities in our solutions $u(x, t)$ on the initial surface $\bar{\Omega} \times \{0\}$ we state these principles in the following form: Suppose that $u(x, t)$ is a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on $\bar{\Omega} \times (0, \infty)$, and satisfying $Lu - k(\partial u/\partial t) \geqq 0$ in D. Suppose that $u \leqq 0$ on $\Gamma \times (0, \infty)$ and that $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq 0$. Then $u(x, t) \leqq 0$ in $\bar{\Omega} \times (0, \infty)$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in D$ then $u(x, t) \equiv 0$ for all $x \in \bar{\Omega}$, $0 < t \leqq t_0$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in \Gamma \times (0, \infty)$ and if Γ has the inner sphere property[1] at x^0 then either $u(x, t) \equiv 0$ for all $x \in \bar{\Omega}$, $0 < t \leqq t_0$ or else there exists a *positive* constant m such that $u(x, t_0) \leqq -m \,||\, x - x^0 \,||$ for all x within the prescribed sphere along a fixed line segment from x^0 and sufficiently close to x^0. Here we define

$$||\, x \,|| = \Big(\sum_{i=1}^{n} x_i^2\Big)^{1/2}.$$

With the aid of the maximum principle we establish the following

LEMMA 1. *Suppose that hypotheses I hold. Let $u(x, t)$ be a real valued function, differentiable with respect to t and twice differentiable with respect to x in D, continuous on $\bar{\Omega} \times (0, \infty)$, and satisfying $Lu - k(\partial u/\partial t) \geqq 0$ in D. Suppose that $u \leqq 0$ on $\Gamma \times (0, \infty)$ and that $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq M$. If $u(x^0, t_0) = 0$ for some point $(x^0, t_0) \in \Gamma \times (0, \infty)$ and if Γ has the exterior sphere property[1] at x^0 then there exists a positive constant C such that $u(x, t_0) \leqq C \,||\, x - x^0 \,||$ for all x in $\bar{\Omega}$ and sufficiently close to x^0.*

Proof. Assume first that $c(x, t) \leqq 0$. Without loss of generality we may suppose that the axes have been translated so that the exterior sphere prescribed at x^0 is centered at the origin and has radius $r_0 = ||\, x^0 \,||$. Define $r = ||\, x \,||$ and $\rho^2 = ||\, x \,||^2 + (t - t_0)^2$. Consider now the $(n + 1)$-dimensional sphere $B = \{(x, t): \rho^2 < r_0^2\}$, which is exterior to

[1] Γ has the inner (exterior) sphere property at x^0 if there exists an open ball B contained in Ω (the complement of $\bar{\Omega}$) such that $\bar{B} \cap \Gamma = \{x^0\}$.

D at the point (x^0, t_0), and the concentric sphere $B' = \{(x, t): \rho^2 < (r_0 + \varepsilon)^2\}$ where $\varepsilon > 0$ is chosen to be suitably small.

Define Ω' to be the open subset of D enclosed by B' for $0 < t < t_0$, Γ' to be the closure of $B' \cap \{\Gamma \times (0, t_0)\}$, and S' to be the subset of the surface of B' contained in D for $0 < t \leqq t_0$.

In the closure of Ω' consider the function $h(x, t) = e^{-\delta r_0^2} - e^{-\delta \rho^2}$, where δ is a constant to be chosen subsequently. Note that $h > 0$ in the closure of Ω' with the exception of the point (x^0, t_0) where we have $h(x^0, t_0) = 0$.

In Ω' we have

$$Lh - k\frac{\partial h}{\partial t} = ch + 2\delta e^{-\delta\rho^2}\{1 - 2\delta \sum_{i,j=1}^{n} a_{ij}x_i x_j + \sum_{i=1}^{n} b_i x_i - k(t - t_0)\}$$

$$\leqq 2\delta e^{-\delta\rho^2}\{A - 2\delta\alpha \,||\, x^0 \,||^2\}$$

$$< 0 \text{ if } \delta \text{ is chosen to be sufficiently large.}$$

It is an easy consequence of the weak form of the maximum principle mentioned earlier that $u(x, t) \leqq \max\{0, M\}$ in D and hence there exists a positive constant C such that $u \leqq Ch$ for $(x, t) \in S'$.

If we now define $w = u - Ch$ we have $Lw - k(\partial w/\partial t) > 0$ in Ω' while $w \leqq 0$ in $S' \cup \Gamma'$. It follows from the weak maximum principle that $w \leqq 0$ in the closure of Ω'. In particular we have $u(x, t_0) \leqq C\{e^{-\delta r_0^2} - e^{\delta r^2}\}$ for $x \in \bar{\Omega}$, $r \leqq r_0 + \varepsilon$, which immediately implies the result of the lemma. Note that the constant C is independent of t_0 in this case.

For the more general case when $c(x, t) \leqq \gamma$, $\gamma > 0$, we apply the above result to the function $v(x, t) = e^{-\gamma t/k_0} u(x, t)$ which satisfies the inequality $Lv - k(\partial v/\partial t) \geqq 0$ in D for an operator of type (2.1) with $c(x, t) \leqq 0$. We obtain $v(x, t_0) \leqq C\{e^{-\delta r_0^2} - e^{-\delta r^2}\}$ and thus $u(x, t_0) \leqq Ce^{\gamma t_0/k_0}\{e^{-\delta r_{20}} - e^{-\delta r^2}\}$.

3. Extremal operators and the class $\mathscr{L}_\alpha$. Let us denote by $\mathscr{L}_\alpha$ the class of uniformly elliptic operators of the form

$$L' = \sum_{i,j=1}^{n} a_{ij}(x, t)\frac{\partial^2}{\partial x_i \partial x_j}$$

with real valued coefficients defined in D and satisfying there the hypotheses $I(1)$ and $I(2)$. In recent papers [15, 16] Pucci has introduced the *maximizing* and *minimizing* operators, relative to the class $\mathscr{L}_\alpha$, defined respectively by

$$M_\alpha[u](x, t) = \sup_{L' \in \mathscr{L}_\alpha} \{L'u(x, t)\}$$

$$m_\alpha[u](x, t) = \inf_{L' \in \mathscr{L}_\alpha} \{L'u(x, t)\} \tag{3.1}$$

for any given function $u(x, t)$ twice differentiable with respect to x in D.

He has shown that these operators have the nonlinear representations

$$\begin{aligned} M_\alpha[u] &= \alpha\Delta u + (1 - n\alpha)C_n(u) \\ m_\alpha[u] &= \alpha\Delta u + (1 - n\alpha)C_1(u) \end{aligned} \tag{3.2}$$

where Δ denotes the n-dimensional Laplacian and $C_1(u) \leqq C_2(u) \leqq \cdots \leqq C_n(u)$ denote the ordered eigenvalues of the $n \times n$ Hessian matrix $(u_{ij}(x, t))$.

For functions $u(x, t) = R(r, t)$ which depend only upon t and the distance $r \equiv \| x \|$ from a fixed point (taken to be the origin) the representations (3.2) reduce to the simpler forms

$$\begin{aligned} M_\alpha[u] &= h_1 \frac{\partial^2 R}{\partial r^2} + \frac{(1 - h_1)}{r}\frac{\partial R}{\partial r} \\ m_\alpha[u] &= h_2 \frac{\partial^2 R}{\partial r^2} + \frac{(1 - h_2)}{r}\frac{\partial R}{\partial r} \end{aligned} \tag{3.3}$$

where $h_1 = \alpha$ and $h_2 = 1 - (n - 1)\alpha$ if $(\partial^2 R/\partial r^2) < (1/r)(\partial R/\partial r)$ while $h_1 = 1 - (n - 1)\alpha$ and $h_2 = \alpha$ if $(\partial^2 R/\partial r^2) \geqq (1/r)(\partial R/\partial r)$.

From the definitions it is clear that for any function $u(x, t)$, twice differentiable with respect to x in D, and any $L' \in \mathscr{L}_\alpha$ we have the inequality $m_\alpha[u](x, t) \leqq L'u(x, t) \leqq M_\alpha[u](x, t)$ for each $(x, t) \in D$. However, it may also be shown that there exist operators $L_1', L_2' \in L_\alpha$ (with coefficients determined by u) such that $L_1'u(x, t) = m_\alpha[u](x, t)$ and $L_2'u(x, t) = M_\alpha[u](x, t)$ for all $(x, t) \in D$, thereby justifying the terminology.

We refer to the papers [8, 11–13, 16] for a more detailed development of the theory of these extremal operators and their applications. Their significance for our purposes in this paper rests on the fact that they provide a systematic method for determining comparison functions for our parabolic operator (2.1). The construction of these functions is based upon the following lemma. Here J_μ denotes the Bessel function of the first kind of order μ in the notation of [1], and $r = \| x \| = (\sum_{i=1}^n x_i^2)^{1/2}$.

LEMMA 2. *Let $\alpha, \beta, \gamma, \kappa$ be constants with $0 < \alpha \leqq 1/n$, and $\beta > -1$. If we set*

$$\mu = \frac{\beta + 1}{2[1 - (n - 1)\alpha]} - 1 \quad \text{and} \quad \lambda = \frac{\xi_0^2[1 - (n - 1)\alpha] - \gamma r_0^2}{\kappa r_0^2},$$

where ξ_0 is the first positive zero of $J_\mu(\xi)$, then the function $r^{-\mu}J_\mu\{(\xi_0 r/r_0)\}e^{-\lambda t}$ is a solution of the equation

$$M_\alpha[u] + \frac{\beta}{r}\frac{\partial u}{\partial r} + \gamma u - \kappa\frac{\partial u}{\partial t} = 0$$

in the $(n+1)$*-dimensional cylinder* $Q=\{(x, t): r < r_0, t > 0\}$ *which is positive in* Q*, decreasing with respect to* r*, and zero on the cylindrical surface* $r = r_0$*. If, instead, we set*

$$\mu = \frac{(\beta+1)}{2\alpha} - 1 \quad \text{and} \quad \gamma = \frac{\alpha\xi_0^2 - \gamma r_0^2}{\kappa r_0^2}$$

then this function is a solution of the equation

$$m_\alpha[u] + \frac{\beta}{r}\frac{\partial u}{\partial r} + \gamma u - \kappa \frac{\partial u}{\partial t} = 0$$

in Q *which is positive in* Q*, decreasing with respect to* r*, and zero on the surface* $r = r_0$*.*

4. The decay estimates. With the aid of the maximum principle and the previous lemmas we can now establish our main results on the decay rate of solutions associated with our parabolic operator (2.1).

To obtain an upper bound on the decay rate we suppose that the set Ω has diameter $2r_0$ and is contained in the sphere of radius r_0 centered at the point $x^0 \in E^n$. The set D is then enclosed in the $(n+1)$-dimensional half cylinder $Q = \{(x, t): \| x - x^0 \| < r_0, t > 0\}$.

THEOREM 1. *Suppose that hypotheses I hold and that*

$$\inf_D \sum_{i=1}^n b_i(x_i - x_i^0) \equiv \beta > -1 \quad \text{and} \quad \sup_D c(x, t) = \gamma .$$

Define

$$\mu = \frac{(\beta+1)}{2[1-(n-1)\alpha]} - 1 \quad \text{and} \quad \lambda = \frac{\xi_0^2[1-(n-1)\alpha] - \gamma r_0^2}{\kappa r_0^2}$$

where ξ_0 *is the first positive zero of* $J_\mu(\xi)$ *and* $\kappa = k_1$ *if* $\xi_0^2[1-(n-1)\alpha] \geqq \gamma r_0^2$*, while* $\kappa = k_0$ *if* $\xi_0^2[1-(n-1)\alpha] < \gamma r_0^2$*.*

Suppose that $u(x, t)$ *is a real valued function, differentiable with respect to* t *and twice differentiable with respect to* x *in* D*, continuous on* $\bar{\Omega} \times (0, \infty)$*, and satisfying* $Lu - k(\partial u/\partial t) \geqq 0$ *in* D*. If* $u(x, t) \leqq 0$ *on* $\Gamma \times (0, \infty)$ *and* $\overline{\lim}_{(x,t)\to\bar{\Omega}\times\{0\}} u(x, t) \leqq M$ *then there exists a positive constant* C *such that* $u(x, t) \leqq C\rho e^{-\lambda t}$ *in* $\bar{\Omega} \times (0, \infty)$ *where* ρ *denotes the distance from* x *to the lateral surface of* Q*.*

Proof. We need only consider the case $M > 0$ for otherwise the result follows trivially from the weak maximum principle. Moreover without loss of generality we may suppose that the axes have been translated so that the axis of the cylinder Q is the t-axis.

Using the definition of the maximizing operator, the assumptions

of the theorem, and Lemma 2 we note that the function $V(r, t) = r^{-\mu}J_\mu(\xi_0 r/r_0)e^{-\lambda t}$ satisfies the differential inequality

$$LV - k\frac{\partial V}{\partial t} \leqq M_\alpha[V] + \frac{(\sum_{i=1}^{n} b_i x_i)}{r}\frac{\partial V}{\partial r} + cV - k\frac{\partial V}{\partial t}$$

$$\leqq M_\alpha[V] + \frac{\beta}{r}\frac{\partial V}{\partial r} + \gamma V - \kappa\frac{\partial V}{\partial t} = 0$$

in D, where κ is prescribed in the hypothesis of the theorem.

Let $0 < \varepsilon < 1$ be an arbitrary constant and denote by D_ε the half cylinder $D_\varepsilon = \Omega \times (\varepsilon, \infty)$. Note that Γ has the exterior sphere property at any point $x^0 \in \Gamma$ such that (x^0, ε) is a common boundary point of both D_ε and Q. The result of Lemma 1 then implies that $\overline{\lim}_{x\to\Gamma}\{u(x, \varepsilon)/V(r, \varepsilon)\}$ is bounded above by a constant which may be chosen independent of ε. Moreover from the weak maximum principle we have $u(x, \varepsilon)$ bounded above, by M if $\gamma \leqq 0$ and by $Me^{\gamma\varepsilon/k_0}$ if $\gamma > 0$, for all $x \in \Omega$. It follows that there exists a positive constant a, independent of ε, such that $u(x, \varepsilon) - aV(r, \varepsilon) \leqq 0$ for all $x \in \Omega$.

Combining these results we conclude that the function $w(x, t) = u(x, t) - aV(r, t)$ satisfies the differential inequality $Lw - k(\partial w/\partial t) \geqq 0$ in D_ε and the inequality $w \leqq 0$ on $\Gamma \times [\varepsilon, \infty]$ and on $\Omega \times \{\varepsilon\}$. By the weak maximum principle we have $w \leqq 0$ in $\bar{\Omega} \times [\varepsilon, \infty)$, i.e., there exists a positive constant, a, such that

$$u(x, t) \leqq ar^{-\mu}J_\mu\left(\frac{\xi_0 r}{r_0}\right)e^{-\lambda t} \quad \text{in} \quad \bar{\Omega} \times [\varepsilon, \infty)$$

and since a is independent of ε the same result holds in $\bar{\Omega} \times (0, \infty)$. This clearly implies the result of the theorem.

To obtain a lower bound for the decay rate we define r_1 to be the maximum of the radii of all open spheres contained in Ω and suppose one such largest sphere is B, centred at the point $x^1 \in \Omega$. The set D then contains the $(n+1)$-dimensional half cylinder $Q_1 = B \times (0, \infty) = \{(x, t): \| x - x^1 \| < r_1, t > 0\}$.

THEOREM 2. *Suppose that hypotheses I hold and that* $\sup_{Q_1}\sum_{i=1}^{n} b_i(x_i - x_i^1) = \beta$, $\inf_{Q_1} c(x, t) = \gamma$. *Define* $\mu = \{(\beta + 1)/2\alpha\} - 1$ *and* $\lambda = \dfrac{\alpha\xi_0^2 - \gamma r_1^2}{Kr_1^2}$, *where* ξ_0 *is the first positive zero of* $J_\mu(\xi)$ *and* $K = k_0$ *if* $\alpha\xi_0^2 \geqq \gamma r_1^2$ *while* $K = k_1$ *if* $\alpha\xi_0^2 < \gamma r_1^2$.

Suppose that $u(x, t)$ *is a real valued function, differentiable with respect to* t *and twice differentiable with respect to* x *in* D, *continuous on* $\bar{\Omega} \times (0, \infty)$, *and satisfying* $Lu - k(\partial u/\partial t) \leqq 0$ *in* D. *If* $u \geqq 0$ *on* $\Gamma \times (0, \infty)$, $\underline{\lim}_{(x,t)\to\Gamma\times\{0\}} u(x, t) \geqq 0$, *and* $\underline{\lim}_{(x,t)\to\Omega\times\{0\}} u(x, t) > 0$ *then there exists a*

positive constant m such that $u(x, t) \geqq m\rho e^{-\lambda t}$ in $B \times [\varepsilon, \infty)$ where ρ denotes the distance from x to the boundary of B and $\varepsilon > 0$ is arbitrarily small.

Proof. Again we may assume that the axes are translated so that the sphere B is centred at the origin. Using the definition of the minimizing operator, the assumptions of the theorem, and Lemma 2 we note that the function $v(r, t) = r^{-\mu}J_{\mu}\{(\xi_0 r/r_1)\}e^{-\lambda t}$ satisfies the differential inequality

$$Lv - k\frac{\partial v}{\partial t} \geqq m_{\alpha}[v] + \frac{(\sum_{i=1}^{n} b_i x_i)}{r}\frac{\partial v}{\partial r} + cv - k\frac{\partial v}{\partial t}$$

$$\geqq m_{\alpha}[v] + \frac{\beta}{r}\frac{\partial v}{\partial r} + \gamma v - K\frac{\partial v}{\partial t} = 0$$

in Q_1, where K is prescribed in the hypotheses of the theorem.

Let $0 < \varepsilon < 1$ be an arbitrary constant. Note that $u > 0$ on $B \times \{\varepsilon\}$ by the strong minimum principle and the condition $\underline{\lim}_{(x,t)\to\Omega\times\{0\}} u(x,t) > 0$. Moreover the boundary point principle implies that $\underline{\lim}_{x\to\partial B}\{u(x, \varepsilon)/v(r, \varepsilon)\}$ is bounded below by a positive constant, since Γ has the inner sphere property at any point common to Γ and ∂B. It follows that there exists a positive constant δ such that $u(x, \varepsilon) - \delta v(r, \varepsilon) \geqq 0$ for all $x \in B$.

Combining these results we conclude that the function $w(x, t) = u(x, t) - \delta v(r, t)$ satisfies the differential inequality $Lw - k(\partial w/\partial t) \leqq 0$ in $B \times (\varepsilon, \infty)$ and the inequality $w \geqq 0$ on $\partial B \times [\varepsilon, \infty)$ and on $B \times [\varepsilon]$. By the weak minimum principle we have $w \geqq 0$ in $B \times [\varepsilon, \infty)$, i.e., there exists a positive constant δ such that $u(x, t) \geqq \delta r^{-\mu}J_{\mu}\{\xi_0 r/r_1)\}e^{-\lambda t}$ in $B \times [\varepsilon, \infty)$. The result of the theorem now follows directly.

5. **Concluding remarks.** We emphasize that for the class of operators of the form (2.1) with coefficients $b_i \equiv 0$, $i \equiv 1, \cdots, n$ and the case that Ω is a sphere our Theorems 1 and 2 cannot be improved. In fact from the discussion of the properties of the extremal operators it follows that there exists an operator in this class for which our comparison function $V(r, t)$ $(v(r, t))$ is a solution in $\Omega \times (0, \infty)$.

Finally let us note that Theorems 1 and 2 may be used to obtain comparison theorems for more general nonhomogeneous parabolic equations by the device of subtracting off a suitable approximate solution of the equations.

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UNIVERSITY OF CALIFORNIA, RIVERSIDE

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GEOMETRIES ON SURFACES

HELMUT SALZMANN

Among the topological geometries, two classes have so far attracted special attention, namely the locally compact, 2-dimensional projective and affine planes. Such a plane has a pointset which is homeomorphic to the pointset of the real arguesian plane, hence is a 2-dimensional manifold. In this paper, all the 2-manifolds that can carry topological geometries with connected lines will be determined.

THEOREM. *Let M be a surface carrying a topological geometry such that any two distinct points are on a unique line. Then either*

(1) *M is an open disk, and each line is open, i.e., homeomorphic to the real line R, or*

(2) *M is a compact surface of characteristic* 1, *and each line is closed, i.e., homeomorphic to a circle, or*

(3) *M is a Moebius strip, and through each point there pass closed lines and at least one open line.*

The proof follows immediately by combining statements 1.10 and 2.3, 6, 9, 10, 13 below. From 2.9 and 2.11 we get

COROLLARY. *M is orientable if and only if the space $\mathfrak{L}$ of lines is a Moebius strip, and M is nonorientable if and only if $\mathfrak{L}$ is compact.*

1. Let $\mathfrak{L}$ be a family of subsets of a nontrivial topological space M, and assume that each two distinct "points" $p, q \in M$ are joined by a unique "line" $L = p \cup q \in \mathfrak{L}$. The system $\boldsymbol{E} = (M, \mathfrak{L})$ is called [3] a "plane" whenever $\mathfrak{L}$ can be provided with a topology (necessarily unique) for which $\boldsymbol{E}$ becomes a topological geometry in the following sense:

(a) $p \cup q$ depends continuously on (p, q), and

(b) the set $\mathfrak{D}$ of pairs of intersecting lines is open in $\mathfrak{L} \times \mathfrak{L}$, and intersection is a continuous map from $\mathfrak{D}$ onto M.

Condition (b) is equivalent to

(b′) If $U \subseteq M$ is open, then $\{(K, L); K \cap L \in U\}$ is open in $\mathfrak{L} \times \mathfrak{L}$.

We shall be concerned with "flat" planes only, i.e., with those planes in which the underlying space M is a 2-dimensional manifold or "surface". The line space $\mathfrak{L}$ of a flat plane is also a surface, and the incident point-line pairs or "flags" form a 3-dimensional closed submanifold F of $M \times \mathfrak{L}$; each line is closed in M and is locally homeomorphic to the real line [3, 2.3].

If B is a connected open subset of M, then the system

$$E_B = (B, \mathfrak{L}_B = \{L \cap B \neq \emptyset; L \in \mathfrak{L}\})$$

is again a flat plane. This shows that, in general, the lines of a geometry will not be connected sets; it is therefore reasonable to determine only those flat planes that have connected lines, i.e. planes satisfying

(c) every line is homeomorphic to the real line or to the circle.

Each line can then be provided with two opposite orientations, and the space $\mathfrak{L}^+$ of oriented lines becomes a two-fold covering of $\mathfrak{L}$. The lines homeomorphic to R will be called "open", and the others compact or "closed".

LEMMA 1.1. *A compact disk $D \subseteq M$ cannot contain any line.*

Proof. A line $K \subseteq D$ would have to be a closed line. Hence K would separate some points $a \in D$ and $b \in M$, and then $a \cup b$ would intersect K twice.

DEFINITION 1.2. A set $C \subseteq M$ is convex if $L \cap C$ is connected for every line L. If the boundary bd C consists of finitely many, resp. of 4 or 3 line segments, C will be called a polygon resp. a quadrangle or a triangle.

LEMMA 1.3. *Each point $p \in M$ has a neighborhood basis consisting of quadrangles (triangles).*

Proof. Let W be a neighborhood of p. Choose points a_1, a_2 such that $p \notin a_1 \cup a_2$, and let J_k be a sufficiently small connected neighborhood of p on $a_k \cup p$. Then

$$U = \{(x_1 \cup a_2) \cap (x_2 \cap a_1); x_k \in J_k\} \subseteq W$$

is a neighborhood of p homeomorphic to $J_1 \times J_2$, which is easily seen to be a convex quadrangle.

COROLLARY 1.4. *The space $\mathfrak{L}_p^+$ of oriented lines through p is homeomorphic to a circle.*

Convergence in M and in $\mathfrak{L}$ are related by

LEMMA 1.5. $L_n \xrightarrow[n]{} L$ *if and only if* $L = \{x; \vee x_n \in L_n x_n \xrightarrow[n]{} x\}$.

This follows immediately from the continuity of join and intersection, (a) and (b).

A set is termed "bounded" if it has compact closure.

LEMMA 1.6. *A set $\mathfrak{S}$ of lines is bounded in $\mathfrak{L}$ if and only if there is a bounded set $C \subseteq M$ intersecting every line $S \in \mathfrak{S}$.*

Proof. Let $S_n \in \mathfrak{S}$ and $x_n \in S_n \cap C$. We may assume $x_n \xrightarrow[n]{} x$. Choose a triangle with boundary T containing x in its interior; then for each sufficiently large n there exists a point $t_n \in S_n \cap T$. The t_n accumulate at some $t \in T$, and a subsequence of the S_n converges to $t \cup x$. For the converse note that M is a union of a countable family of compact subsets.

LEMMA 1.7. *Let K be a closed line, and $B \subseteq M$ a bounded open set containing K. Then $\mathfrak{B} = \{L \in \mathsf{L};\ L \subseteq B\}$ is an open neighborhood of K in $\mathfrak{L}$ consisting of closed lines only.*

Proof. Since bounded lines are closed, $\mathfrak{B}$ cannot contain any open line. If $\mathfrak{B}$ were not a neighborhood of K, we would find a sequence of lines $L_n \xrightarrow[n]{} K$ with $L_n \cap B \neq \emptyset$ and $L_n \notin \mathfrak{B}$. By (c) each line L_n would contain a point x_n of the compact boundary bd B, in contradiction to 1.5.

COROLLARY 1.8. *The space $\mathfrak{R}$ of closed lines is open in $\mathfrak{L}$.*

LEMMA 1.9. *A closed line K intersects every line $L \in \mathfrak{L}$.*

Proof. For $a \notin K$ the map $\alpha = (x \mapsto a \cup x)$ maps K into the set $\mathfrak{L}_a$ of lines through a. Since $\mathfrak{L}_a$ is homeomorphic to a circle by 1.4, and α is one-to-one, α is even a bijection.

COROLLARY 1.10. *If M is a compact surface, then $(M, \mathfrak{L})$ is a projective plane, and M is homeomorphic to the real projective plane.*

Proof. Any two different lines intersect by 1.9, and this is the first assertion. The second statement follows now from [2].

PROPOSITION 1.11. If a geometry contains a closed line K, then each point lies on a closed line.

Proof. Assume that each line through s is homeomorphic to R, and let $s \cup z^+$ denote the line $s \cup x$ oriented from s to x. Then $(x \mapsto s \cup x^+): K \to \mathfrak{L}_s^+$ would be a homeomorphism from K onto a set of oriented lines through s containing exactly one of any two lines of opposite orientation. This contradicts 1.4.

2. From now on we shall assume for the sake of simplicity that

M is a surface of finite connectivity, i.e., that M can be embedded in a compact surface $\bar{M}$ by adding a finite number of "endpoints" $e_1, \cdots, e_r$.

LEMMA 2.1. *Let $g: R \to M$ represent an oriented open line $L = g(R)$. Then L has unique endpoints*

$$e^{\pm}(L) = \lim_{t \to \pm\infty} g(t)$$

in $\bar{M}$.

Proof. Choose small disjoint open neighborhoods U_k of e_k in $\bar{M}$. Since each line is closed in M, there exists an $s \in R$ such that the connected set $\{g(t);\ t \geqq s\}$ is contained in $\bigcup_{k=1}^{r} U_k$ and hence in one of the U_k.

LEMMA 2.2. *The oriented lines L having a given endpoint $e^+(L) = e$ form a closed subset of $\mathfrak{L}^+$.*

Proof. Let $L_n \xrightarrow[n]{} L$ with $e^+(L_n) = e$, and choose a small neighborhood U of e in $\bar{M}$. Then all the L_n intersect the compact boundary $\operatorname{bd} U \subseteqq M$. Hence $L \cap \operatorname{bd} U \neq \emptyset$ by 1.5, and $e^+(L) = e$.

LEMMA 2.3. *For each $a \in M$ and each endpoint e there is an oriented line $L \in \mathsf{L}_a^+$ with $e^+(L) = e$.*

Proof. Let $b_n \xrightarrow[n]{} e$ in $\bar{M}$, and $a \cup b_n \xrightarrow[n]{} L$.

COROLLARY 2.4. *Let $(M, \mathfrak{L})$ be a flat plane with connected lines. Then M is compact if and only if each line is closed.*

PROPOSITION 2.5. A surface M carrying a plane geometry with connected lines has at most one endpoint.

Proof. $\mathfrak{L}_{ik} = \{L \in \mathfrak{L}^+;\ e^-(L) = e_i,\ e^+(L) = e_k\}$ is closed in $\mathfrak{L}^+$ by 2.2. Assume $r > 1$, choose small disjoint neighborhoods U_k of e_k in $\bar{M}$, and let $b_{kn} \xrightarrow[n]{} e_k$ in U_k. For $i \neq k$ the lines $b_{in} \cup b_{kn}$ meet the compact complement C of $\bigcup_{k=1}^{r} U_k$, and hence accumulate at a line in $\mathfrak{L}_{ik}$. Since $\mathfrak{L}^+$ is connected, the boundary $\operatorname{bd} \mathfrak{L}_{ik} \neq \emptyset$, and there is a sequence of closed lines K_n converging to a line $L \in \mathfrak{L}_{ik}$. For large n a line K_n will contain two arcs joining U_i and U_k; thus a line H intersecting L at a point of C would meet some line K_n twice, a contradiction.

PROPOSITION 2.6. A plane geometry on an orientable surface M

has no compact lines.

Proof. A compact line K on an orientable surface M would have a bounded neighborhood $B \subseteqq M$ which is homeomorphic to a cylinder. By 1.7 the set $\mathfrak{B}$ of all lines $L \subseteqq B$ is an open neighborhood of K in $\mathfrak{L}$, and each $L \in \mathfrak{B}$ is closed. Moreover, it follows from 1.1 and 1.9 that a line $L \subseteqq B$ decomposes the cylinder and intersects K. But then K and L would have to intersect twice.

PROPOSITION 2.7. The complement of an open line L is homeomorphic to an open subset of R^2.

Proof. For $a, b \in L$ consider

$$(x \mapsto (a \cup x, b \cup x)) : M - L \to (\mathfrak{L}_a - \{L\}) \times (\mathfrak{L}_b - \{L\}) .$$

PROPOSITION 2.8. If M is orientable, then any line L decomposes M into two open half-planes, each of which is homeomorphic to an open disk. Consequently $\bar{M}$ is homeomorphic to a 2-sphere.

Proof. Let $L = a \cup b$, and choose homeomorphisms

$$\alpha : \mathfrak{L}_a - \{L\} \to (0, 1) , \qquad \beta : \mathfrak{L}_b - \{L\} \to (0, 1)$$

corresponding to the same orientation of M. For $\xi = A^\alpha$ let

$$\delta(\xi) = \inf \{B^\beta ; A \cap B = \varnothing\} .$$

Then δ is a monotonically increasing, left-continuous mapping. According to the orientation of M one of the half-planes defined by L will be mapped by

$$x \mapsto ((a \cup x)^\alpha, (b \cup x)^\beta)$$

onto the Jordan domain

$$D = \{(\xi, \eta) \in R^2 ; 0 < \xi < 1, 0 < \eta < \delta(\xi)\} .$$

COROLLARY 2.9. *Let $(M, \mathfrak{L})$ be a flat plane with connected lines. If M is orientable, then M is homeomorphic to R^2, and $\mathfrak{L}$ is a Moebius strip.*

For the last assertion see [1, § 3(6)] or [3, 2.14].

An argument completely analogous to the proof of 2.8 shows

COROLLARY 2.10. *Each point of a plane geometry on a non-*

orientable surface is incident with a compact line.

THEOREM 2.11. *The line space of a plane geometry* $(M, \mathfrak{L})$ *on a nonorientable surface is a compact surface of characteristic* 1; *the space of oriented lines is homeomorphic to a 2-sphere.*

Proof. If H, K are closed lines through a, then $\mathfrak{L} - \mathfrak{L}_a$ is homeomorphic to $(H - \{a\}) \times (K - \{a\})$.

LEMMA 2.12. *In a plane geometry* $(M, \mathfrak{L})$ *on a nonorientable surface the set of closed lines through a point a is connected.*

Proof. Because of 2.5 we may assume that M has exactly one endpoint e. Choose an open line L and two closed lines H and K through a. We have to show that no open line through a is separated from L by H, K. By 2.7 the Jordan curve theorem holds in $M - L$. Hence the union $H \cup K$ decomposes $M - L$ into two connected subsets, only one of which contains points of a small neighborhood U of e. Therefore, a line G which is separated from L by H, K must be disjoint from U and hence is compact.

THEOREM 2.13. *An open nonorientable surface M carrying a plane geometry with connected lines is a Moebius strip.*

Proof. The dual $(\mathfrak{L}, M)$ of the geometry defined by the system $\mathfrak{K}$ of closed lines on M is a flat plane by 2.10, 1.8, 1.9, and 2.12. The theorem follows now from 2.9 and 2.11.

Note that in general the lines of a topological geometry on a Moebius strip need not be connected.

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ARTINIAN, ALMOST ABELIAN GROUPS AND THEIR GROUPS OF AUTOMORPHISMS

ANNEMARIE SCHLETTE

The principal results of this paper are several characterizations of the class of artinian almost abelian groups. Several corollaries of the main characterization are given, one of which states that a torsion group G which has a finite group of automorphisms is finite, and an abelian torsion group which has a countable automorphism group is finite. In a concluding section, the class of artinian almost abelian groups and the class of finite groups are characterized by formal group-theoretical properties.

A group is called artinian, if the minimum condition is satisfied by its subgroups. The structure of an abelian, artinian group is completely determined. Every artinian group that is known so far is an extension of an abelian, artinian group by a finite group. An extension of an abelian group by a finite group is called almost abelian. It is not known whether there exist further artinian groups. For this reason it seems to be worth while to consider the structure of artinian and almost abelian groups and we will give some characterizations of them. We will prove the following

MAIN THEOREM. *The following properties of the group G are equivalent*:

(1) *G is artinian and almost abelian.*

(2) (a) *Every abelian subnormal subgroup of G is artinian;*
(b) *Every infinite epimorphic image of G possesses an abelian subnormal subgroup, not* 1;
(c) *G is a torsion group.*

(3) (a) *Every abelian subgroup of G is artinian;*
(b) *Every epimorphic image, not* 1, *of G possesses an almost abelian normal subgroup, not* 1.

(4) (a) *G is a torsion group;*
(b) *Every torsion group of automorphisms of G is artinian and almost abelian.*

(5) (a) *G is a torsion group;*
(b) *The central quotient group of G is artinian and almost abelian;*
(c) *Primary elementary abelian groups of automorphisms of G are countable.*

Sections 1 to 4 deal with preliminary lemmas, some of which may

be of independent interest. We show, for instance, that the central quotient group of G is artinian and almost abelian if, and only if, G is almost abelian with artinian commutator subgroup G' (Proposition 3.2). Section 5 is dedicated to the proof of the Main Theorem. A simple application of the Main Theorem yields a characterization of the artinian groups with finite central quotient group (Corollary 6.1) and of the finite groups (Corollary 6.2), both by their automorphism groups. The results of the Main Theorem and of its corollaries finally are used to characterize the class of all artinian and almost abelian groups on the one hand and the class of all finite groups on the other hand by merely formal group theoretical properties. This is done in § 7 (Theorem 7.1 and Theorem 7.2).

NOTATIONS.

$\mathfrak{z}G$ = center of the group G,
$\mathfrak{c}_G U$ = centralizer of U in G,
$\mathfrak{n}_G U$ = normalizer of U in G,
$[G:U]$ = index of U in G,
$\mathfrak{f}G$ = intersection of all subgroups U of G with finite $[G:U]$,
$\{\cdots\}$ = subgroup generated by the enclosed elements,
$x \circ y = x^{-1}x^y = x^{-1}y^{-1}xy$ for elements x, y in G,
$X \circ Y = \{x \circ y$ with x in X, y in $Y\}$,
$G \circ G = G' = G^1$ = commutator subgroup of G,
$G^{(n+1)} = G^{(n)} \circ G^{(n)}$,
G_p = set of all p-elements in G for p a prime,
G^p = set of all g^p with g in G [if G is multiplicatively written],
pA = set of all pa with a in A [if A is additively written].

DEFINITIONS. A *factor* of G is an epimorphic image of a subgroup of G.

The *socle* of an abelian p-group is the set of its elements of order p or 1.

The group G is called

artinian, if its subgroups satisfy the minimum condition;

almost abelian, if it possesses an abelian [characteristic] subgroup of finite index;

soluble, if $G^{(n)} = 1$ for almost all n;

locally finite, if every finitely generated subgroup of G is finite;

hyper (almost) abelian, if every epimorphic image, not 1, of G possesses an (almost) abelian normal subgroup, not 1.

$$\left.\begin{matrix}\mathfrak{h}\mathfrak{a}G\\ \mathfrak{a}\mathfrak{a}G\\ \mathfrak{l}\mathfrak{f}G\end{matrix}\right\} = \text{product of all } \left\{\begin{matrix}\text{hyperabelian}\\ \text{almost abelian}\\ \text{locally finite}\end{matrix}\right\} \text{ normal subgroups of } G.$$

An abelian group A is called *divisible* (*radicable*), if $pA = A$ ($A^p = A$) for all primes p, and *reduced*, if it possesses no divisible (radicable) subgroup $\neq 0$ ($\neq 1$).

1. **Lemmas on abelian groups.** In this chapter we consider (additively written) abelian groups A. Any homomorphism σ of A maps the element x in A upon $x\sigma$.

LEMMA 1.1. *A reduced abelian p-group A with finite A/pA is finite.*

Proof. Since A/pA is finite, there exists a finitely generated subgroup F of A with $A = F + pA$. Since finitely generated, abelian p-groups are finite, F is finite. Therefore there exists a positive integer n with $p^nF = 0$. Hence

$$p^nA = p^nF + p^{n+1}A = p(p^nA)$$

is divisible. But A is reduced, so that $p^nA = 0$. From $A = F + pA$ we deduce

$$\begin{aligned} A &= F + [pF + p^2A] = F + p^2A = \cdots = \\ &= F + p^nA = F , \end{aligned}$$

so that A is finite.

LEMMA 1.2. *If A is an abelian, reduced, infinite p-group and if B is an abelian p-group* $\neq 0$, *then* Hom (A, B) *contains an uncountably infinite, elementary abelian p-subgroup.*

Proof. Since A is an abelian, reduced, infinite p-group, A/pA is infinite, by Lemma 1.1. Since $B \neq 0$, it contains a subgroup Z of order p. Now

$$\text{Hom}\,(A, Z) \subseteq \text{Hom}\,(A, B)$$

and

$$\text{Hom}\,(A, Z) \cong \text{Hom}\,(A/pA, Z) ,$$

which is an uncountably infinite, elementary abelian p-group, (cf. [5], p. 206; E, and p. 208; Lemma 54.2).

LEMMA 1.3. (a) *If A and B are abelian, artinian groups, then the torsion subgroup of* Hom (A, B) *is finite.*

(b) *Torsion groups of automorphisms of an abelian, artinian group are finite.*

Proof. The abelian artinian group A is a direct sum

$$A = F \oplus D$$

of a finite group F and a divisible group D, (cf. [5], p. 65; Th. 19.2). Hence there exists a positive integer n with $nF = 0$. Let X be the subgroup of all x in B with $nx = 0$. Since B is abelian and artinian, it is a consequence of ([5], p. 65; Th. 19.2) that X is finite.

If σ is a torsion element of order i in $\mathrm{Hom}(A, B)$, then

$$D\sigma = (iD)\sigma = D(i\sigma) = D0 = 0$$

and

$$n(A\sigma) = (nF \oplus nD)\sigma = n(D\sigma) = 0 ,$$

so that $A\sigma \subseteq X$. Therefore the torsion subgroup of $\mathrm{Hom}(A, B)$ is essentially the same as a group of homomorphisms of the finite group $A/D \cong F$ into the finite group X, which implies its finiteness. This proves (a), and (b) is a consequence of (a) if we let $A = B$.

LEMMA 1.4. *If the abelian group A is the direct sum of infinitely many subgroups $A_\nu \neq 0$, then A possesses an uncountably infinite, elementary abelian 2-group of automorphisms.*

The simple proof of this lemma may be left to the reader.

2. Stabilizing automorphism groups.

PROPOSITION 2.1. *Suppose B is a normal subgroup of the group G and A is a subgroup of G with $A \subseteq B \cap \mathfrak{z}G$. Let $\Sigma = \Sigma_G(A, B)$ be the group of all automorphisms σ of G which fix every element in G/A and in B. Then*

$$\Sigma_G(A, B) \cong \mathrm{Hom}(G/B, A) \cong \mathrm{Hom}(G/BG', A) .$$

Proof. If σ is an automorphism in Σ, then $\sigma - 1$ is a single valued mapping of G into A. If x and y are elements in G, then

$$(xy)^{\sigma-1} = x^\sigma y^\sigma y^{-1} x^{-1} = x^\sigma y^{\sigma-1} x^{-1} = x^{\sigma-1} y^{\sigma-1} ,$$

since $y^{\sigma-1}$ belongs to $A \subseteq \mathfrak{z}G$. Therefore $\sigma - 1$ is a homomorphism of G into A, whose kernel contains B. Thus

(a) If σ is an element in Σ, then $\sigma - 1$ induces a homomorphism σ^* of G/B into A with

$$(Bg)^{\sigma^*} = g^{\sigma-1} .$$

Let α, β be two automorphisms in Σ and g be any element in G. Then

$$g^{\alpha\beta^{-1}} = g^{\alpha\beta}(g^{-1})^{\beta}g^{\beta}g^{-1} = (g^{\alpha^{-1}})^{\beta}g^{\beta^{-1}} = g^{\alpha^{-1}}g^{\beta^{-1}} ,$$

since $g^{\alpha^{-1}}$ belongs to $A \subseteqq B$ and every element in B is a fixed element of β. Therefore

$$(\alpha\beta)^* = \alpha^* + \beta^* \qquad \text{for } \alpha, \beta \text{ in } \Sigma ,$$

so that $\sigma \rightarrow \sigma^*$ is a homomorphism of Σ into Hom $(G/B, A)$. If $\sigma^* = 0$ and g is an element in G, then

$$1 = (Bg)^{\sigma^*} = g^{\sigma^{-1}} .$$

This implies $g = g^{\sigma}$ for every g in G, so that $\sigma = 1$. Hence

(b) The mapping $\sigma \rightarrow \sigma^*$ is a monomorphism of Σ into Hom $(G/B, A)$.

For every γ in Hom $(G/B, A)$ define the mapping δ of G into G by the rule

$$g^{\delta} = (Bg)^{\gamma}g \qquad \text{for every } g \text{ in } G.$$

This is clearly a single valued mapping of G into itself.

If x, y are two elements in G, then

$$(xy)^{\delta} = (Bxy)^{\gamma}xy = (Bx)^{\gamma}(By)^{\gamma}xy = (Bx)^{\gamma}x(By)^{\gamma}y = x^{\delta}y^{\delta}$$

since $(By)^{\gamma}$ belongs to $A \subseteqq {}_{\delta}G$. Thus δ is an endomorphism of G. If $x^{\delta} = 1$, then $(Bx)^{\gamma}x = 1$, so that x belongs to B (since $(Bx)^{\gamma}$ is in $A \subseteqq B$). This implies $(Bx)^{\gamma} = B^{\gamma} = 1$ so that $x = 1$. Hence δ is a monomorphism of G into G. Because of $B^{\gamma} = 1$ and $(Bg)^{\gamma} \in A$ it is clear that δ induces the identity in both B and G/A. Now let g be any element in G and let

$$x = (Bg^{-1})^{\gamma}g .$$

Then

$$x^{\delta} = (Bx)^{\gamma}x = [B(Bg^{-1})^{\gamma}g]^{\gamma}(Bg^{-1})^{\gamma}g = (Bg)^{\gamma}[(Bg)^{\gamma}]^{-1}\, g = g$$

since $(Bg^{-1})^{\gamma}$ belongs to $A \subseteqq B$. This shows that δ is an automorphism of G which belongs to Σ. Finally

$$g^{\delta^{-1}} = (Bg)^{\gamma} \qquad \text{for every } g \text{ in } G ,$$

shows $\delta^* = \gamma$ and

(c) $\sigma \rightarrow \sigma^*$ is an isomorphism of Σ onto Hom $(G/B, A)$.

The isomorphy of Hom $(G/B, A)$ and Hom $(G/BG', A)$ is an immediate consequence of the commutativity of A.

LEMMA 2.2. *If the group G possesses a characteristic subgroup A such that the torsion groups of automorphisms of A and of G/A are finite, then every torsion group Γ of automorphisms of G is almost abelian: The subgroup Δ of Γ which fixes every element in A and in G/A is abelian and has finite index in Γ.*

Proof. Let Γ be a torsion group of automorphisms of G. Denote by Γ_A resp. $\Gamma_{G|A}$ the set of all those automorphisms in Γ, which induce the identity in A and in G/A respectively. Since A is characteristic in G, the sets Γ_A and $\Gamma_{G|A}$ are both normal subgroups of Γ. Now Γ/Γ_A resp. $\Gamma/\Gamma_{G|A}$ is essentially the same as the group of automorphisms of A resp. G/A, induced by Γ. Therefore it is (essentially) a torsion group of automorphisms of A resp. G/A, and hence, by hypothesis, finite. If $\Delta = \Gamma_A \cap \Gamma_{G|A}$, then Δ is a normal subgroup of Γ with finite Γ/Δ. By its definition, Δ is a subgroup of the stabilizer of the characteristic subgroup A of G. As such it is known to be abelian, see ([7], p. 88; Satz 19). This proves Lemma 2.2.

3. Central quotient group and commutator subgroup. In this chapter we aim at characterizing the groups with almost abelian, artinian central quotient group (Proposition 3.2).

LEMMA 3.1. *If G is a group with abelian, radicable, not torsionfree central quotient group $G/\mathfrak{z}G$, then G is abelian.*

Proof. Suppose that G is not abelian. Then $G/\mathfrak{z}G \neq 1$ and since it is not torsionfree there exists a subgroup $W/\mathfrak{z}G$ in $G/\mathfrak{z}G$ of order a prime p. Now $G' \subseteq \mathfrak{z}G \subset W$, so that W is normal in G. Since $G/\mathfrak{z}G$ is abelian, it follows that $W \circ G \subseteq \mathfrak{z}G$ and that

$$w(w \circ g) = (w \circ g)w \quad \text{for every} \quad w \in W, g \in G .$$

Since w^p also belongs to $\mathfrak{z}G$, it is centralized by every element in G. Therefore it follows from $w^g = w(w \circ g)$ that

$$w^p = (w^p)^g = (w^g)^p = (w(w \circ g))^p = w^p(w \circ g)^p .$$

Hence

$$(w \circ g)^p = 1 \quad \text{for every} \quad w \in W, g \in G .$$

By complete induction we get

$$w^{g^i} = w(w \circ g)^i \quad \text{for every positive integer } i$$

and in particular

$$w^{g^p} = w(w \circ g)^p = w ,$$

which shows

$$G'G^p \subseteqq G^p \mathfrak{z}G \subseteqq \mathfrak{c}_G W .$$

Therefore the epimorphic image $G/\mathfrak{c}_G W$ of $G/\mathfrak{z}G$ is an elementary abelian p-group. Since elementary abelian p-groups $\neq 1$ possess finite non-trivial epimorphic images, $G/\mathfrak{c}_G W$ is equal to 1, because $G/\mathfrak{z}G$ is radicable. Hence $\mathfrak{c}_G W = G$, implying $W \subseteqq \mathfrak{z}G \subset W$, a contradiction. Therefore G is abelian.

DEFINITION. $\mathfrak{f}G$ = intersection of all subgroups U of G with finite index $[G: U]$. This is a well determined characteristic subgroup of G.

PROPOSITION 3.2. *The following properties of the group G are equivalent*:

(i) *$G/\mathfrak{z}G$ is artinian and almost abelian.*

(ii) *If F is the uniquely determined characteristic subgroup of G with $\mathfrak{z}G \subseteqq F$ and $F/\mathfrak{z}G = \mathfrak{f}(G/\mathfrak{z}G)$, then F is abelian, G/F is finite, and $G \circ F$ is artinian.*

(iii) *There exists an abelian normal subgroup A of G with finite G/A and artinian $G \circ A$.*

(iv) *G is almost abelian and G' is artinian.*

Proof. (i) $\rightarrow$ (ii). Since $G/\mathfrak{z}G$ is artinian and almost abelian, application of ([1], pp. 3, 4, 22(B)) on $G/\mathfrak{z}G$ shows

(a) $G/F = (G/\mathfrak{z}G)/\mathfrak{f}(G/\mathfrak{z}G)$ is finite and $\mathfrak{f}(G/\mathfrak{z}G) = F/\mathfrak{z}G$ does not possess finite epimorphic images. $F/\mathfrak{z}G$ is abelian because $G/\mathfrak{z}G$ is almost abelian. Hence $F' \subseteqq \mathfrak{z}G \subseteqq \mathfrak{z}F \subseteqq F$ and therefore

(b) $F/\mathfrak{z}F$ is an abelian radicable torsion group. Apply Lemma 3.1 on F to see

(c) F is abelian.

Let Γ be the group of automorphisms which is induced in F by G. Then $\Gamma \cong G/\mathfrak{c}_G F$ is finite because $F \subseteqq \mathfrak{c}_G F$ by (c), and G/F is finite, by (a). Every automorphism γ in Γ fixes every element in $\mathfrak{z}G$. Hence $\gamma - 1$ is an endomorphism of the abelian group F whose kernel contains $\mathfrak{z}G$. Consequently $F^{\gamma-1}$ is an epimorphic image of the artinian group $F/\mathfrak{z}G$ and as such $F^{\gamma-1}$ is artinian. Since Γ is finite and F is abelian,

$$F^{\Gamma-1} = \prod_{\gamma \in \Gamma} F^{\gamma-1}$$

is likewise artinian. From the definition of Γ,

$$G \circ F = F^{\Gamma-1}$$

so that

(d) $G \circ F$ is artinian.

(ii) $\rightarrow$ (iii). Let $A = F$.

(iii) $\rightarrow$ (iv). It is an immediate consequence of (iii) that G is almost abelian. Since A is normal in G, so is $G \circ A$. Denote by σ the canonical epimorphism of G upon $H = G/(G \circ A)$. Then $B = A^\sigma$ is an abelian normal subgroup of H with finite H/B, as G/A is finite. From $H \circ B = (G \circ A)^\sigma = 1$ we get $B \subseteqq \mathfrak{z}H$, so that $H/\mathfrak{z}H$ is finite. This implies the finiteness of H', see ([6], p. 443; 15.1.13). But $H' = G'^\sigma \cong G'/(G \circ A)$, so that G' is artinian as an extension of the artinian group $G \circ A$ by the finite group H'.

(iv) $\rightarrow$ (iii). Obvious.

(iii) $\rightarrow$ (i). The group Γ of automorphisms of A which are induced in A by G is isomorphic to $G/\mathfrak{c}_G A$ and it is finite since $A \subseteqq \mathfrak{c}_G A$ and G/A is finite. If γ is an automorphism in Γ, then $\gamma - 1$ is an endomorphism of the abelian group A with $A^{\gamma-1} \subseteqq G \circ A$. It follows that $A^{\gamma-1}$ is artinian and $A^{\gamma-1} \cong A/K(\gamma)$, where $K(\gamma)$ denotes the kernel of γ, i.e., $K(\gamma) = \mathfrak{c}_A g$ if g induces γ. Now

$$A \cap \mathfrak{z}G = \bigcap_{\gamma \in \Gamma} K(\gamma) .$$

Therefore $A/(A \cap \mathfrak{z}G)$ is isomorphic to a subgroup of the direct product of the finitely many artinian groups $A/K(\gamma)$ and as such $A/(A \cap \mathfrak{z}G) \cong A\mathfrak{z}G/\mathfrak{z}G$ is itself artinian. $G/A\mathfrak{z}G$ is finite as an epimorphic image of G/A. Finally $G/\mathfrak{z}G$ is an extension of the abelian artinian group $A\mathfrak{z}G/\mathfrak{z}G$ by the finite group $G/A\mathfrak{z}G$. Hence $G/\mathfrak{z}G$ is artinian and almost abelian.

4. **Products of normal subgroups.** In this section we shall establish a few lemmas on products of normal subgroups, which are needed later. In this context see also Corollary 6.4.

DEFINITION. $\mathfrak{h}\mathfrak{a}G$ = product of all hyperabelian normal subgroups of G.

This is a well determined characteristic subgroup of G.

LEMMA 4.1. *If every abelian subnormal subgroup of the torsion group G is artinian, then*

(a) $\mathfrak{h}\mathfrak{a}G$ *is artinian and soluble;*

(b) $\mathfrak{h}\mathfrak{a}G$ *contains every hyperabelian subnormal subgroup of* G;

(c) 1 *is the only hyperabelian subnormal subgroup of* $G/\mathfrak{h}\mathfrak{a}G$;

(d) $\mathfrak{h}\mathfrak{a}(G/\mathfrak{h}\mathfrak{a}G) = 1$.

Proof. If S is a subnormal subgroup of G, then every abelian subnormal subgroup of $\mathfrak{h}\mathfrak{a}S$ is a subnormal subgroup of G and hence, by hypothesis, artinian. Nontrivial epimorphic images of $\mathfrak{h}\mathfrak{a}S$ possess

by definition hyperabelian normal subgroups, not 1, and consequently abelian subgroups, not 1; and $\mathfrak{h}\mathfrak{a}S$ is a torsion group, as is G. Apply ([4], p. 345, Satz 6.1, equivalence of (1) and (2)) to $\mathfrak{h}\mathfrak{a}S$ to show that

(e) $\mathfrak{h}\mathfrak{a}S$ is artinian and soluble for every subnormal subgroup S of G. This implies (a) in the particular case $S = G$.

Since S is subnormal in G, there exists a finite chain of subgroups S_i of G with

$$S = S_0,\ S_i \text{ normal in } S_{i+1},\ S_n = G\ .$$

If S is hyperabelian, then $S = \mathfrak{h}\mathfrak{a}S_0$. By (e), $\mathfrak{h}\mathfrak{a}S_i$ is a soluble, hence hyperabelian characteristic subgroup of the normal subgroup S_i of S_{i+1}. Hence $\mathfrak{h}\mathfrak{a}S_i \subseteqq \mathfrak{h}\mathfrak{a}S_{i+1}$, showing

$$S = \mathfrak{h}\mathfrak{a}S_0 \subseteqq \cdots \subseteqq \mathfrak{h}\mathfrak{a}S_n = \mathfrak{h}\mathfrak{a}G\ ,$$

so that $\mathfrak{h}\mathfrak{a}G$ contains every hyperabelian subnormal subgroup of G, proving (b). Now let $T = S/\mathfrak{h}\mathfrak{a}G$ be a hyperabelian subnormal subgroup of $G/\mathfrak{h}\mathfrak{a}G$. Then S is as an extension of the, by (a), soluble group $\mathfrak{h}\mathfrak{a}G$ by the hyperabelian group T hyperabelian. Since S is subnormal in G, it follows from (b) that $S \subseteqq \mathfrak{h}\mathfrak{a}G$. Hence $T = 1$, proving (c).

(d) is an immediate consequence of (c).

LEMMA 4.2. *If N is a product of minimal normal subgroups of the group G, then N is a direct product of minimal normal subgroups of G.*

This is proven by the customary lattice theoretical arguments; see for instance ([8], p. 208; Basis Theorem of Lattice Theory).

LEMMA 4.3. *If a minimal normal subgroup M of G is contained in a product N of finite normal subgroups of G, then M is finite.*

Proof. Let $t \neq 1$ be an element in M. Since t belongs to the product N of finite normal subgroups of G, there exists a finite number n of finite normal subgroups F_i of G with $F_i \subseteqq N$ and

$$t \in \prod_{i=1}^{n} F_i = F\ .$$

Clearly F is a finite normal subgroup of G with t in $F \subseteqq N$. Hence $1 \neq M \cap F$ and because of the minimality of M we have $M = M \cap F \subseteqq F$, so that M is finite.

LEMMA 4.4. *If 1 is the only abelian normal subgroup of G, then the following properties of the product N of finite normal subgroups*

of G are equivalent:

(i) *N is finite.*

(ii) *N contains but a finite number of minimal normal subgroups of G.*

(iii) *The product P of all the minimal normal subgroups of G, contained in N, is the direct product of finitely many minimal normal subgroups of G.*

Proof. Evidently, (ii) is a consequence of (i) and it follows from Lemma 4.2 that P is a direct product of minimal normal subgroups of G, therefore (iii) is a consequence of (ii).

Assume that (iii) holds. Then application of Lemma 4.3 shows that

(') the normal subgroup P of G is finite .

Since P is finite, the group of automorphisms, induced in P by G, which is essentially the same as $G/\mathfrak{c}_G P$, is finite, too. Since $\mathfrak{z}P = P \cap \mathfrak{c}_G P$ is an abelian normal subgroup of G and since, by hypothesis, 1 is the only abelian normal subgroup of G, we get

(") $$P \cap \mathfrak{c}_G P = 1 \quad \text{and} \quad G/\mathfrak{c}_G P \text{ is finite .}$$

If $N \cap \mathfrak{c}_G P \neq 1$, then there would exist a finite number of finite normal subgroups $E_1, \cdots, E_n$ of G with $E_i \subseteqq N$ and

$$W = \mathfrak{c}_G P \cap \prod_{i=1}^{n} E_i \neq 1 .$$

It follows that W is a finite normal subgroup, not 1, of G. Hence W contains a minimal normal subgroup V of G. Since

$$V \subseteqq W \subseteqq \prod_{i=1}^{n} E_i \subseteqq N ,$$

this minimal normal subgroup V is part of P. Hence

$$1 \subset V \subseteqq P \cap W \subseteqq P \cap \mathfrak{c}_G P = 1 ,$$

by (") which is a contradiction. Therefore

$$N \cap \mathfrak{c}_G P = 1$$

which implies that

$$N = N/(N \cap \mathfrak{c}_G P) \cong N\mathfrak{c}_G P/\mathfrak{c}_G P \subseteqq G/\mathfrak{c}_G P$$

is finite, by (").

Thus (i) is a consequence of (iii), proving Lemma 4.4.

5. **Proof of the Main Theorem.** The proof of the equivalence

of (1), (2), and (3) was given in a course of lectures held by R. Baer at New Mexico State University in the fall term 1967.

Proof of the equivalence of (1) *and* (2). Evidently (1) implies (2.a) and (2.c). Furthermore every epimorphic image of an almost abelian group is almost abelian, which implies (2.b).

Assume conversely the validity of conditions (2.a-c). Then Lemma 4.1 shows

(d) $\mathfrak{h}_a G$ is artinian and soluble,

and

(e) 1 is the only hyperabelian subnormal subgroup of $G/\mathfrak{h}_a G$.

Combine (e) and (2.b) to get

(f) $G/\mathfrak{h}_a G$ is finite.

Therefore, by (d) and (f),

(g) G is artinian.

Since $\mathfrak{h}_a G$ is almost abelian by (d), cf. ([4], p. 345, Satz 6.1 and its proof), there exists an abelian characteristic subgroup C of $\mathfrak{h}_a G$ with finite $\mathfrak{h}_a G/C$. Then C is an abelian characteristic subgroup of G and G/C is finite, by (f). Thus

(h) G is almost abelian

and we have proven the equivalence of (1) and (2).

Proof of the equivalence of (1) *and* (3). If G is artinian and almost abelian then every subgroup of G is artinian and every epimorphic image of G is almost abelian, proving that (3) is a consequence of (1).

Assume the validity of (3.a) and (3.b). Then, by (3.a), every cyclic subgroup of G is finite. Hence

(c) G is a torsion group.

Lemma 4.1 shows

(d) $\mathfrak{h}_a G$ is artinian and soluble,

and

(e) 1 is the only abelian subnormal subgroup of $H = G/\mathfrak{h}_a G$.

If A is an abelian subgroup of H then there exists a subgroup B of G with $\mathfrak{h}_a G \subseteq B$ and $A = B/\mathfrak{h}_a G$. By (d), B is soluble and hence hyperabelian and by (3.a) every abelian subgroup of B is artinian. Hence, by Lemma 1.3. (b), every torsion group of automorphisms of every abelian subgroup of B is finite. Because of (c) we may apply ([4], p. 345, Satz 6.1, equivalence of (1) and (4)) on B. This shows that B is artinian. Hence A is artinian, too, proving

(f) Abelian subgroups of H are artinian.

Let F be the product of all finite normal subgroups of H and let P be the product of all minimal normal subgroups of H which are contained in F. Application of Lemma 4.2 shows that P is a direct

product of minimal normal subgroups X of H. Since H is a torsion group by (c), we may pick in every X an element $e(X)$ of order a prime. Then the subgroup S generated by these $e(X)$ is a direct product of the cyclic groups $\{e(X)\}$, each of order a prime. Thus S is abelian and hence, by (f), artinian. Therefore and since the elements of S are of squarefree order, S is finite, cf. ([5], p. 68; Exercise 19). Hence

(g) P is a direct product of finitely many minimal normal subgroups of H. Because of (e) and (g) we may apply Lemma 4.4 to show that

(h) F is finite.

Since $H/\mathfrak{c}_H F$ is essentially the same as the group of automorphisms induced by H in F, it is finite, as F is and, by (e), $1=\mathfrak{z}F=F\cap \mathfrak{c}_H F$. Thus

(i) $H/\mathfrak{c}_H F$ is finite and $1 = F \cap \mathfrak{c}_H F$.

Assume by way of contradiction that H is infinite. Then $F\neq H$ because of (h), so that H/F is an epimorphic image, not 1, of G. By (3.b) there exists an almost abelian normal subgroup, not 1, of H/F. Consequently there exists a normal subgroup V of H with

$$F \subset V \text{ and almost abelian } V/F .$$

Apply (i) to see that

$$(') \qquad V/\mathfrak{c}_V F = V/(V \cap \mathfrak{c}_H F) \cong V\mathfrak{c}_H F/\mathfrak{c}_H F \subsetneqq H/\mathfrak{c}_H F$$

is finite and

$$('') \qquad \mathfrak{c}_V F = (V \cap \mathfrak{c}_H F)/(F \cap \mathfrak{c}_H F) \cong F(V \cap \mathfrak{c}_H F)/F \subsetneqq V/F$$

is almost abelian.

Because of ('') $\mathfrak{c}_V F$ possesses an abelian characteristic subgroup C with finite $\mathfrak{c}_V F/C$. As a characteristic subgroup of the normal subgroup $\mathfrak{c}_V F$ of H the group C is normal in H. Hence $C = 1$ by (e), which shows that $\mathfrak{c}_V F$ is finite. Hence V is a finite normal subgroup of H, by ('), whence $V \subsetneqq F \subset V$, a contradiction, proving

(k) $H = G/\mathfrak{h}\mathfrak{a}G$ is finite.

Combination of (d) and (k) shows that

(l) G is artinian and almost abelian.

This completes the proof of the equivalence of (1) and (3) of the Main Theorem.

Proof of the equivalence of (1), (4), *and* (5).

(1) $\rightarrow$ (4). Every artinian group is a torsion group, hence (4.a) is valid. If G is furthermore almost abelian, then there exists an abelian characteristic subgroup A of G with finite G/A. Since A is artinian

as a subgroup of the artinian group G, its torsion groups of automorphisms are finite, by Lemma 1.3. Therefore Lemma 2.2 is applicable, showing that every torsion group Γ of automorphisms of G is almost abelian. Define Δ as in Lemma 2.2.

The set S of all the elements of squarefree order in the artinian abelian group A is a finite characteristic subgroup of A; cf. ([5], p. 68; Exercise 19). Let Σ be the set of all elements of squarefree order in Δ. If σ is an element of order n in Σ and if x is an element in G, then $x^{\sigma-1}$ belongs to A and is therefore a fixed element of σ. Thus $x^{\sigma} = (x^{\sigma-1})x$ and, by complete induction,

$$x^{\sigma^i} = (x^{\sigma-1})^i x \quad \text{for every integer } i \geqq 0 ,$$

so that in particular

$$x = x^{\sigma^n} = (x^{\sigma-1})^n x .$$

This implies

$$(x^{\sigma-1})^n = 1 \quad \text{for every } x \text{ in } G .$$

Hence $x^{\sigma-1}$ is an element of squarefree order in A and belongs therefore to S. Thus

$$G^{\sigma-1} \subseteqq S \quad \text{for all } \sigma \text{ in } \Sigma .$$

Every $\sigma - 1$, with σ in Σ, induces a single valued mapping of G/A into S, as

$$(ax)^{\sigma-1} = ax^{\sigma-1}a^{-1} = x^{\sigma-1} \quad \text{for } a \text{ in } A, x \text{ in } G ,$$

since $a^{\sigma} = a$ and $x^{\sigma-1}$ both belong to the abelian group A. If σ' and σ'' are two elements in Σ, then $\sigma' - 1$ and $\sigma'' - 1$ induce the same mapping of G/A into S if, and only if, $\sigma' = \sigma''$. Since G/A and S are finite there exists only a finite number of single valued mappings of G/A into S. This implies that Σ is finite. As Δ is an abelian torsion group, application of ([5], p. 68; Exercise 19) shows that Δ is artinian.

Since Γ/Δ is finite, Γ is artinian and almost abelian. Hence (4) is a consequence of (1).

(4) $\rightarrow$ (5). Since G is a torsion group, so is $G/\mathfrak{z}G$; and $G/\mathfrak{z}G$ is essentially the same as the group of inner automorphisms of G, hence it is artinian and almost abelian, by (4.b). So (5) is but a weak form of (4).

(5) $\rightarrow$ (1). Because of (5.b) Proposition 3.2 is applicable, which shows

(d) G' is artinian

and

(e) There exists an abelian characteristic subgroup F of G with $F \supseteq {}_3G$ and finite G/F. Thus G is almost abelian.

By (5.a), F is a torsion group, so that it is locally finite, since it is abelian. Hence G is an extension of the locally finite group F by the finite group G/F, showing

(f) G is locally finite.

Since G/F is finite there exists a finitely generated subgroup E of G with $G = FE$. Because of (f), E is finite:

(g) There exists a finite subgroup E of G with $G = FE$.

Consider an abelian direct factor A of G. Every group of automorphisms of A and of every direct factor of A is isomorphic to a group of automorphisms of G. If A would have infinitely many primary components, it would possess an uncountably infinite, elementary abelian 2-group of automorphisms (Lemma 1.4), contradicting (5.c). For each primary component A_p of A we have, by ([5], p. 63; Th. 18.2),

$$A_p = B \otimes C ,$$

where B is radicable and C is reduced. As a radicable torsion group B is the direct product of groups of type p^∞. If there were infinitely many, B would admit an uncountably infinite, elementary abelian 2-group of automorphisms (Lemma 1.4) which again is impossible because of (5.c). Hence, by ([5], p. 65; Th. 19.2), B is artinian. If $C \neq 1$, then there exists a subgroup Z of order p in C. By Proposition 2.1, $\mathrm{Hom}\,(C/Z, Z)$ is isomorphic to a group of automorphisms of C. Now C/Z is reduced, as C is. If C were infinite, C/Z would be infinite and, by Lemma 1.2, $\mathrm{Hom}\,(C/Z, Z)$ would contain an uncountably infinite, elementary abelian p-subgroup, which would be isomorphic to a group of automorphisms of C. This contradiction shows that C is finite. Hence, by ([5], p. 65; Th. 19.2), we have

(h) Every abelian direct factor of G is artinian.

We derive from (d), (e), and (g) that $G'E$ is artinian and almost abelian. Therefore the set ω of primes which are orders of elements in $G'E$ is finite, see for instance ([3]; p. 13, Bemerkung 2.2, (A)).

Denote by F_ω the set of all ω-elements in F and by $F_{\omega'}$ the set of all elements in F of order prime to every prime in ω. Since F is an abelian torsion group,

$$F = F_\omega \otimes F_{\omega'} .$$

Both, F_ω and $F_{\omega'}$ are characteristic subgroups of the characteristic subgroup F of G, hence they are characteristic in G. By its definition, $F_\omega G'E$ is a normal ω-subgroup of G. Furthermore

$$F_{\omega'} \cap F_\omega G'E = 1 ,$$

since the elements in these two groups have relatively prime order. Thus, by (g),

$$G = FE = F_{\omega'}F_{\omega}E = F_{\omega'} \otimes F_{\omega}G'E .$$

Application of (h) shows that $F_{\omega'}$ is artinian, implying the finiteness of ω'. Therefore

(i) There exist only finitely many primes which are orders of elements in G.

The product R of all the radicable subgroups of $\mathfrak{z}G$ is a radicable characteristic subgroup of $\mathfrak{z}G$ and hence of F. As R is a torsion group it is a product of groups of type $Z(p^\infty)$ for various primes p. Let R_1 be maximal among the direct products of those $Z(p^\infty)$ in R, which have a nontrivial intersection with $G'E$. Then clearly the elements of squarefree order of R_1 lie in $G'E$. This implies that R_1 is artinian. Moreover, R_1 is radicable as it is the direct product of radicable groups. Therefore R_1 is a direct factor of R, cf. ([5], p. 62; Th. 18.1),

$$R = R_1 \otimes R_2 ,$$

where R_2, as an epimorphic image of R is likewise radicable. Let x be an element of prime order p in $R_2 \cap G'E$. Then x is contained in a subgroup $V \subseteq R_2$ of type $Z(p^\infty)$ and $x \neq 1$ would imply $V \cap G'E \neq 1$, a contradiction to the maximality of R_1. Therefore

$$R_2 \cap G'E = R_2 \cap (F \cap G'E) = 1 .$$

Hence, by ([5], p. 63), there exists a complement S of R_2 in F which contains $F \cap G'E$. Thus

$$F = R_2 \otimes S .$$

By (g) therefore

$$G = FE = R_2SE = R_2(SG'E)$$

and by Dedekind's Modular Law

$$R_2 \cap SG'E = R_2 \cap F \cap SG'E = R_2 \cap S(F \cap G'E) = R_2 \cap S = 1 .$$

Since R_2 is part of the center of G, it follows that

$$G = R_2 \otimes SG'E$$

and application of (h) shows that R_2 is artinian. Since R_1 is likewise artinian, R is artinian, and we have shown

(j) Radicable subgroups of $\mathfrak{z}G$ are artinian.

Let p be a prime with $(\mathfrak{z}G)_p \neq 1$. Then there exists a subgroup Z of order p in $\mathfrak{z}G$. Consider

$$\begin{aligned}\operatorname{Hom}(G/G'EZF_p^p, Z) &= \operatorname{Hom}(FG'E/G'EZF_p^p, Z)\\ &\cong \operatorname{Hom}(F_pG'E/G'EZF_p^p, Z)\\ &\cong \operatorname{Hom}(F_p/(G'EZF_p^p \cap F_p), Z) = H\ .\end{aligned}$$

By ([5], p. 206; E), H is an elementary abelian p-group and by Proposition 2.1 it is isomorphic to a group of automorphisms of G. Hence, by (5.c), H is countable and therefore—cf. ([5], p. 208; Lemma 54.2) —the elementary abelian p-group

$$F_p/(G'EZF_p^p \cap F_p) \quad \text{is finite}\ .$$

This implies the finiteness of F_p/F_p^p, since otherwise $(G'EZF_p^p \cap F_p)/F_p^p$ would be infinite, but, by Dedekind's Modular Law, this is equal to

$$F_p^p(G'EZ \cap F_p)/F_p^p \cong (G'EZ \cap F_p)/(G'EZ \cap F_p^p)\ ,$$

which is an artinian elementary abelian p-group and hence finite, cf. ([5], p. 65; Th. 19.2). Thus

(k) F_p/F_p^p is finite for every p with $(\mathfrak{z}G)_p \neq 1$.

Let S be the set of all elements x in F_p with x^p in $(\mathfrak{z}G)_p$. Then $S/(\mathfrak{z}G)_p$ is the socle of $F_p/(\mathfrak{z}G)_p$ and

$$F_p/(\mathfrak{z}G)_p = F_p/(F_p \cap \mathfrak{z}G) \cong F_p\mathfrak{z}G/\mathfrak{z}G$$

is artinian as a subgroup of the—by (5.b)—artinian group $F/\mathfrak{z}G$. Hence its socle

$$S/(\mathfrak{z}G)_p \quad \text{is finite}\ ,$$

cf. ([5], p. 65; Th. 19.2). By definition of S we have

$$S^p = (\mathfrak{z}G)_p \cap F_p^p\ .$$

Now

$$(\mathfrak{z}G)_p/S^p = (\mathfrak{z}G)_p/[(\mathfrak{z}G)_p \cap F_p^p] \cong F_p^p(\mathfrak{z}G)_p/F_p^p$$

is finite, because, in case $(\mathfrak{z}G)_p \neq 1$, it is a subgroup of F_p/F_p^p, which is finite, by (k). Therefore, if $(\mathfrak{z}G)_p/(\mathfrak{z}G)_p^p$ were infinite, $S^p/(\mathfrak{z}G)_p^p$ would be infinite. But the mapping σ, defined by the rule

$$\sigma\colon (\mathfrak{z}G)_p x \longrightarrow (\mathfrak{z}G)_p^p x^p \quad \text{for} \quad x \in S$$

is an epimorphism of the finite group $S/(\mathfrak{z}G)_p$ upon $S^p/(\mathfrak{z}G)_p^p$, implying the finiteness of $S^p/(\mathfrak{z}G)_p^p$. Hence

(l) $(\mathfrak{z}G)_p/(\mathfrak{z}G)_p^p$ is finite for all p.

Since $(\mathfrak{z}G)_p$ is an abelian torsion group,

$$(\mathfrak{z}G)_p = A \otimes B\ ,$$

where A is radicable and B is reduced. A is artinian, by (j). B/B^p is finite, by (l), and since B is reduced, B is finite, by Lemma 1.1. Hence

(m) $(\mathfrak{z}G)_p$ is artinian for every p.

By (i), there exists only a finite number of primes which are orders of elements in G. Hence $\mathfrak{z}G$ is the product of finitely many primary components, each of which is artinian, by (m). This implies

(n) $\mathfrak{z}G$ is artinian.

Combination of (5.b), (e), and (n) shows that G is artinian and almost abelian.

Thus (1) is a consequence of (5) which completes the proof of the Main Theorem.

REMARK. (A) It is impossible to weaken condition (3.a) by requiring only that abelian subnormal subgroups are artinian and that G be a torsion group, since both this weaker condition and (3.b) are satisfied by every direct product of any number of nonabelian, finite, simple groups.

(B) The infinite cyclic group is an example of a torsionfree abelian group whose group of automorphisms is finite. This shows that conditions (4.a) and (5.a) are indispensable for the validity of the Main Theorem.

(C) A direct product of infinitely many cyclic groups of order a prime satisfies (5.a) and (5.b) showing the indispensability of condition (5.c).

6. Corollaries of the Main Theorem. The first two corollaries of the equivalence of (1), (4), and (5) of the Main Theorem, one characterizing the artinian groups with finite central quotient group, the other characterizing the finite groups by means of their automorphism groups, which will be useful in § 7, may be found in [2]. We will give a proof of Corollary 6.1, since it is a simple application of the Main Theorem. It is a consequence of the equivalence of (1) and (3) of the Main Theorem that the product of all almost abelian normal subgroups of a group G is almost abelian and artinian, provided its abelian subgroups are artinian (Corollary 6.4).

COROLLARY 6.1. *The following properties of the group G are equivalent*:

(1) *G is artinian with finite $G/\mathfrak{z}G$.*

(2) *G is a torsion group and every torsion group of automorphisms of G is finite.*

Cf. ([2], p. 521, theorem and proof of the equivalence of (i) and (ii)).

Proof. (1) $\rightarrow$ (2). Every artinian group is a torsion group. Since

$\mathfrak{z}G$ is an artinian characteristic subgroup of G we may, because of Lemma 1.3, apply Lemma 2.2. Thus: If Γ is a torsion group of automorphisms of G and if Δ is the subgroup of Γ which fixes every element in $\mathfrak{z}G$ and in $G/\mathfrak{z}G$, then Δ is abelian and has finite index in Γ. The group Δ is a torsion subgroup of the group Σ of all those automorphisms of G which induce the identity in both $\mathfrak{z}G$ and $G/\mathfrak{z}G$. Application of Proposition 2.1 yields

$$\Sigma \cong \mathrm{Hom}\,(G/\mathfrak{z}G, \mathfrak{z}G) \cong \mathrm{Hom}\,(G/G'\mathfrak{z}G, \mathfrak{z}G)\ .$$

Since G is artinian, application of Lemma 1.3 shows that the torsion subgroup of $\mathrm{Hom}\,(G/G'\mathfrak{z}G, \mathfrak{z}G)$ is finite. Hence Δ is finite and Γ, as an extension of the finite group Γ/Δ by the finite group Δ, is finite, too.

(2) $\rightarrow$ (1). If G is a torsion group and if every torsion group of automorphisms of G is finite, then the group of inner automorphisms of G is finite. This is equivalent to the finiteness of $G/\mathfrak{z}G$. Application of the Main Theorem (equivalence of (1) and (4)) shows that G is artinian.

COROLLARY 6.2. *The following properties of the group G are equivalent*:

(1) *G is finite.*

(2) *G is a torsion group and the group of automorphisms of G is finite.*

For a proof see ([2], p. 529, Corollary).

COROLLARY 6.3. *The following properties of the abelian group A are equivalent*:

(1) *A is finite.*

(2) *A is a torsion group and the group of automorphisms of A is countable.*

Proof. Evidently, (2) is a consequence of (1). Assume conversely that (2) holds. As an abelian torsion group, A is the direct product

$$A = B \otimes C$$

of a reduced group B and a radicable group C, see ([5], p. 63), where C is a product of groups of type $Z(p^\infty)$. The automorphism group of a group of type $Z(p^\infty)$ is isomorphic to the group of p-adic integers, prime to p, cf. ([5], p. 211; Exercise 5), which is uncountable. Therefore $C=1$. Application of the Main Theorem, equivalence of (1) and (5), shows that A is artinian. This implies the finiteness of B, cf.

([5], p. 65; Th. 19.2). Hence $A = B$ is finite.

DEFINITION. $\mathfrak{a}\mathfrak{a}G$ = product of all almost abelian, normal subgroups of G. This is a well determined characteristic subgroup of G.

COROLLARY 6.4. *If every abelian subgroup of G is artinian, then*

(a) $\mathfrak{a}\mathfrak{a}G$ *is artinian and almost abelian;*

(b) $\mathfrak{a}\mathfrak{a}G$ *contains every almost hyperabelian subnormal subgroup of G;*

(c) *Every abelian subgroup of $G/\mathfrak{a}\mathfrak{a}G$ is artinian;*

(d) 1 *is the only almost abelian subnormal subgroup of $G/\mathfrak{a}\mathfrak{a}G$;*

(e) $\mathfrak{a}\mathfrak{a}(G/\mathfrak{a}\mathfrak{a}G) = 1$.

Proof. Every epimorphic image of $\mathfrak{a}\mathfrak{a}G$ is a product of almost abelian normal subgroups, as $\mathfrak{a}\mathfrak{a}G$ is. Consequently, every epimorphic image H, not 1, of $\mathfrak{a}\mathfrak{a}G$ possesses an almost abelian normal subgroup, not 1. Therefore conditions (3.a,b) of the Main Theorem are satisfied by $\mathfrak{a}\mathfrak{a}G$ so that $\mathfrak{a}\mathfrak{a}G$ is artinian and almost abelian, proving (a).

Assume that S is an almost hyperabelian, subnormal subgroup of G. Then there exists a hyperabelian subgroup T of S with finite S/T. By hypothesis, every abelian subgroup of T is artinian. Thus it follows from ([4], p. 345, Satz 6.1) that T is artinian, soluble and almost abelian. Consequently T possesses an abelian subgroup A of finite index in T and A has finite index also in S as S/T is finite. Hence S is almost abelian. Thus

(f) Every almost hyperabelian, subnormal subgroup of G is almost abelian.

Since S is subnormal in G there exists a finite chain of subgroups S_i of G such that

$$S = S_0, S_i \text{ is normal in } S_{i+1}, S_n = G .$$

Clearly $S = \mathfrak{a}\mathfrak{a}S_0$ and $\mathfrak{a}\mathfrak{a}S_i$ is a characteristic subgroup of the normal subgroup S_i of S_{i+1}. Since, by (a), $\mathfrak{a}\mathfrak{a}S_i$ is artinian and almost abelian, we have $\mathfrak{a}\mathfrak{a}S_i \subseteqq \mathfrak{a}\mathfrak{a}S_{i+1}$, so that

$$S = \mathfrak{a}\mathfrak{a}S_0 \subseteqq \cdots \subseteqq \mathfrak{a}\mathfrak{a}S_n = \mathfrak{a}\mathfrak{a}G ,$$

proving (b).

If $X/\mathfrak{a}\mathfrak{a}G$ is an almost abelian subgroup of $G/\mathfrak{a}\mathfrak{a}G$, then the subgroup X of G satisfies, by hypothesis and (a), conditions (3.a,b) of the Main Theorem so that X is almost abelian and artinian. Hence $X/\mathfrak{a}\mathfrak{a}G$ is artinian, which proves (c). If $X/\mathfrak{a}\mathfrak{a}G$ is furthermore subnormal in $G/\mathfrak{a}\mathfrak{a}G$, then X is an almost abelian subnormal subgroup of G. Hence $X \subseteqq \mathfrak{a}\mathfrak{a}G$, by (b), which proves (d). It also proves (e), because

of (c) and (a), if X is the well determined characteristic subgroup of G with $X/\mathfrak{a}\mathfrak{a}G = \mathfrak{a}\mathfrak{a}(G/\mathfrak{a}\mathfrak{a}G)$.

7. **Classes of groups.** We are going to give a characterization of the class $\omega(a)$ of all artinian and almost abelian groups and of the class $\omega(f)$ of all finite groups by formal group theoretical properties.

A group which belongs to the class ω is called an ω-group.

THEOREM 7.1. *The class ω of groups is the class $\omega(a)$ of all artinian and almost abelian groups if, and only if, ω satisfies the following properties*:

I. 1 *is an ω-group,*

II. *Subgroups of ω-groups are ω-groups,*

III. *Epimorphic images of ω-groups are ω-groups,*

IV.a *An extension of an ω-group by an ω-group is an ω-group,*

V.a *If G is an ω-group, then G is a torsion group and every torsion group of automorphisms of G is an ω-group,*

VI.a *A torsion group is an ω-group if all its torsion groups of automorphisms are ω-groups,*

VII. *Every ω-group is countable,*

VIII. *Every infinite ω-group possesses an abelian subnormal subgroup, not* 1.

THEOREM 7.2. *The class ω of groups is the class $\omega(f)$ of all finite groups if, and only if, ω satisfies the following conditions*:

I. 1 *is an ω-group,*

II. *Subgroups of ω-groups are ω-groups,*

III. *Epimorphic images of ω-groups are ω-groups,*

IV.f *Direct products of finitely many ω-groups are ω-groups,*

V.f *Every group of automorphisms of an ω-group is an ω-group,*

VI.f *A torsion group is an ω-group, if its automorphism group is an ω-group,*

VII. *Every ω-group is countable,*

VIII. *Every infinite ω-group possesses an abelian subnormal subgroup, not* 1.

Proof of the necessity of conditions I *to* VIII. The class $\omega(a)$ of all artinian and almost abelian groups clearly satisfies I, II, III, VII and VIII. That $\omega(a)$ is extension inherited may be found in ([3], p. 14, Folgerung 2.6) and the validity of conditions V.a and VI.a is contained in the Main Theorem, equivalence of (1) and (4).

It is evident that the class $\omega(f)$ of all finite groups satisfies I, II, III, IV.f, V.f, VII and VIII and the validity of VI.f is contained in Corollary 6.2.

Proof of the sufficiency of conditions I *to* VIII.

(1) Every ω-group is a torsion group (provided II, IV.f, V.f, VII).

Proof. Assume, that G is an ω-group, which is not a torsion group. Then G contains an infinite cyclic subgroup Z, which is an ω-group, by II. It follows, from IV.f, that $F = Z \otimes Z$ is an ω-group. If $F = \{a\} \otimes \{b\}$, then automorphisms σ' and σ'' of F are defined by

$$a^{\sigma'} = ab^2 , \qquad b^{\sigma'} = b$$
$$a^{\sigma''} = a , \qquad b^{\sigma''} = a^2 b .$$

It is a consequence of V.f that $\Sigma = \{\sigma', \sigma''\}$ is an ω-group, so that, by II, its commutator subgroup Σ' is an ω-group, too. By its definition, Σ is a free group of rank 2 and it is well known that Σ' is a free group of infinite rank. If Λ is a free set of generators of Σ' and if Δ is a subset of Λ which is neither vacuous nor Λ, then there exists one and only one automorphism of order 2 of Σ', which leaves every element in Δ fixed and maps every element $\Lambda - \Delta$ upon its inverse. The group of automorphisms generated by these involutions is an ω-group by V.f and it is uncountably infinite by construction. This contradicts VIII and proves (1).

(2) Every finite group is an ω-group (I, IV, V, VI).

Proof. 1 is an ω-group, by I. Since 1 is the only group of automorphisms of the cyclic group $Z(2)$ of order 2 it follows from either of the conditions VI, that $Z(2)$ is an ω-group. By either of the conditions IV, every finite, elementary abelian 2-group is an ω-group. Every finite group is isomorphic to a group of automorphisms of a suitably selected finite elementary abelian 2-group. Because of V therefore every finite group is an ω-group.

(3.a) Every abelian ω-group is artinian (II, V.a, VII).

(3.f) Every abelian ω-group is finite (II, IV.f, V.f, VII).

Proof. (3.a) If A is an abelian ω-group, then it is, by V.a, a torsion group whose torsion groups of automorphisms are ω-groups, and hence, by VII, countable. Application of the Main Theorem, equivalence of (1) and (5), shows that A is artinian.

(3.f) If A is an abelian ω-group, then it is, by (1), a torsion group whose automorphism group is an ω-group, by V.f, and hence countable, by VII. Application of Corollary 6.3 shows that A is finite.

(4.a) Every artinian, almost abelian group is an ω-group (I, IV.a, V.a, VI.a).

Proof. By (2) every finite group is an ω-group. If A is artinian

and abelian, then every torsion group of automorphisms of A is finite (Lemma 1.3) and hence an ω-group. VI.a shows that A is an ω-group. Every artinian and almost abelian group is an extension of an artinian and abelian group by a finite group. Thus, by (2), it is an extension of an ω-group by an ω-group and therefore, by IV.a, an ω-group.

(5.a) If ω satisfies the conditions II, III, V.a, VII, VIII, then every ω-group is artinian and almost abelian.

Proof. If G is an ω-group, then every abelian subgroup A of G is an ω-group by II, and A is artinian, by (3.a). Therefore—if we denote by $\mathfrak{a}\mathfrak{a}G$ the product of all almost abelian, normal subgroups of G—it follows from Corollary 6.4 that

(i) $\mathfrak{a}\mathfrak{a}G$ is artinian and almost abelian, and

(ii) 1 is the only almost abelian, subnormal subgroup of $G/\mathfrak{a}\mathfrak{a}G$.

If G were not artinian and almost abelian, then, by (i), $G \neq \mathfrak{a}\mathfrak{a}G$ and, by (ii), $G/\mathfrak{a}\mathfrak{a}G$ is infinite. Because of III, $G/\mathfrak{a}\mathfrak{a}G$ is an ω-group and VIII shows the existence of an abelian subnormal subgroup, not 1, contradicting (ii). This contradiction shows that G is artinian and almost abelian.

(5.f) If ω satisfies the conditions II, III, IV.f, V.f, VII, VIII, then every ω-group is finite.

Proof. Let G be an ω-group and denote by $\mathfrak{l}\mathfrak{f}G$ the product of all locally finite, normal subgroups of G. According to ([7], p. 135; Satz 28 and p. 141; Satz 40*) we have

(i) $\mathfrak{l}\mathfrak{f}G$ is locally finite, and

(ii) 1 is the only locally finite subnormal subgroup of $G/\mathfrak{l}\mathfrak{f}G$.

Because of III, $G/\mathfrak{l}\mathfrak{f}G$ is an ω-group and it follows from (1) that G and $G/\mathfrak{l}\mathfrak{f}G$ are torsion groups.

Application of VIII and (ii) shows $G/\mathfrak{l}\mathfrak{f}G = 1$, so that $G = \mathfrak{l}\mathfrak{f}G$ is locally finite by (i). Every abelian subgroup of G is an ω-group by II, hence, by (3.a) every abelian subgroup of G is finite. Application of the theorem of Kargapolov-P. Hall-Kulatilaka (see [6], p. 453; 15.4.3) yields the finiteness of G.

Combine (4.a) and (5.a) to get the sufficiency of the conditions in Theorem 7.1; and combination of (3.f) and (5.f) shows the sufficiency of the conditions of Theorem 7.2.

This completes the proof of the two theorems.

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WASHINGTON STATE UNIVERSITY

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ADDITIONAL RESULTS ON MODULES OVER POLYDISC ALGEBRAS

E. L. STOUT

This paper deals with a class $\mathscr{K}_N$ of domains in Stein manifolds and with certain algebras of holomorphic functions naturally associated with them.

The class $\mathscr{K}_N$ consists of those relatively compact domains Δ in N-dimensional Stein manifolds such that for some neighborhood Ω of $\bar{\Delta}$ and some neighborhood W of $\bar{U}^N$, the closure of $U^N = \{(z_1, \cdots, z_N) \in C^N : |z_1|, \cdots, |z_N| < 1\}$, the unit polydisc in C^N, there exists a proper holomorphic map $\Phi: \Omega \to W$ which is nonsingular at every point of $\Phi^{-1}(T^N)$, T^N the distinguished boundary of U^N, and which has, in addition, the property that $\Delta = \Phi^{-1}(U^N)$. The collection of all such maps Φ is denoted by $\mathscr{M}(\Delta, U^N)$, and if $\Delta, \Delta' \in \mathscr{K}_N$, $\mathscr{M}(\Delta, \Delta')$ denotes the set of all maps Ψ: $\Delta \to \Delta'$ such that if $\Phi \in \mathscr{M}(\Delta', U^N)$, then $\Phi \circ \Psi \in \mathscr{M}(\Delta, U^N)$. For $\Delta \in \mathscr{K}_N$ let $\mathscr{A}(\Delta) = \{f \in \mathscr{C}(\bar{\Delta}): f$ is holomorphic in $\Delta\}$, and let $H^\infty(\Delta) = \{f: f$ is holomorphic and bounded in $\Delta\}$. If $\Phi \in \mathscr{M}(\Delta, \Delta')$, then $\mathscr{A}(\Delta)$ is a module over its subalgebra $\Phi^* \mathscr{A}(\Delta') = \{f \circ \Phi: f \in \mathscr{A}(\Delta')\}$, and this paper treats the structure of $\mathscr{A}(\Delta)$ as a $\Phi^* \mathscr{A}(\Delta')$-module. The first section of the paper presents an example to show that $\mathscr{A}(\Delta)$ need not be free over $\Phi^* \mathscr{A}(\Delta')$, and the second section shows that it is a finitely generated, projective $\Phi^* \mathscr{A}(\Delta')$-module. The final section establishes certain conditions sufficient for the freeness of $\mathscr{A}(\Delta)$. Parallel results obtain for $H^\infty(\Delta)$ as a $\Phi H^\infty(\Delta')$-module.

These results supplement results obtained in [7]. In that paper some of these questions were treated for the special case that $\Delta = U^N$. For example, it was shown there that if $\Phi \in \mathscr{M}(\Delta, U^N)$, then $\mathscr{A}(\Delta)$ is a free module over $\Phi^* \mathscr{A}(U^N)$. We refer to this paper for some of the elementary properties of the elements of $\mathscr{K}_N$ and of $\mathscr{M}(\Delta', \Delta)$.

Given $\Delta', \Delta \in \mathscr{K}_N$, and $\Psi \in \mathscr{M}(\Delta; \Delta)$, the triple $(\Delta; \Psi \mid \Delta', \Delta)$ is an analytic cover in the sense of [5] and consequently has a well defined multiplicity λ: λ is that integer such that for all points $\zeta \in \Delta'$ off a variety, the set $\Psi^{-1}(\zeta)$ consists of λ points.

If M is a complex manifold, $\mathscr{S}$ a sheaf on M, and ζ a point of M, then $\mathscr{S}_\zeta$ denotes the stalk of $\mathscr{S}$ at ζ and $\mathscr{O}_M$ denotes the sheaf of germs of functions holomorphic on M. We will usually write $\mathscr{O}_\zeta$ instead of $(\mathscr{O}_M)_\zeta$. If K is a subset of M, $\mathscr{O}(K)$ denotes the sections of $\mathscr{O}_M$ over K.

1. An example. If $\Delta, \Delta' \in \mathscr{K}_1$ and $\Psi \in \mathscr{M}(\Delta', \Delta)$, then [1] $\mathscr{A}(\Delta')$

is a free module over $\Psi^*\mathscr{A}(\Delta)$ whose rank is the multiplicity of Ψ. In higher dimensions the analogous result is not true as the following example shows.

We will show that for $N = 4, 5, 6, \cdots$, there exist $\Delta, \Delta' \in \mathscr{K}_N$ and $\Psi \in \mathscr{M}(\Delta', \Delta)$ such that $\mathscr{A}(\Delta')$ admits no set of generators over $\Psi^*\mathscr{A}(\Delta)$ consisting of λ elements, λ the multiplicity of Ψ. In our example Ψ will be two-to-one and a local homeomorphism at each point of Δ. Denote by $P_N(C)$ and $P_N(R)$ respectively N-dimensional complex and real projective space. In $P_N(C)$, $N \geqq 4$, consider the manifold V consisting of those points with homogeneous coordinates $(z_0, \cdots, z_N)$ such that $z_0^2 + \cdots + z_N^2 \neq 0$. In the case $N = 2$, this manifold was considered in another connection by Forster [3]. The manifold V is connected since it is the complement in $P_N(C)$ of a variety, and, as Forster remarked, it is a Stein manifold. The space $P_N(R)$ is contained in a natural way in V: $P_N(R)$ is the set of all points which admit real homogeneous coordinates, and, moreover, $P_N(R)$ is a deformation retract of V. This was the fact which made V useful for Forster, and it is the essential point in the present example. A deformation of V onto $P_N(R)$ can be given explicitly as follows [3]. If $\zeta \in V$, let $(z_0, \cdots, z_N)$ be homogeneous coordinates for ζ such that $z_0^2 + \cdots + z_N^2 > 0$. Given $t \in [0, 1]$, define $H_t(\zeta)$ to be the point with homogeneous coordinates $(x_0 + ity_0, \cdots, x_N + ity_N)$ if $z_j = x_j + iy_j$. Thus, V and $P_N(R)$ are of the same homotopy type and in particular they have the same fundamental group, Z_2[1]. Consequently, if $\tilde{V}$ denotes the universal covering manifold of V and η: $\tilde{V} \to V$ the natural projection, then η is a local homeomorphism and each fiber $\eta^{-1}(p)$ consists of exactly two points. The manifold $\tilde{V}$ admits a complex structure with respect to which η is a holomorphic map, and when $\tilde{V}$ is endowed with this complex structure, it becomes a Stein manifold. (For the fact that $\tilde{V}$ is Stein, see [8]).

We set $S = \eta^{-1}(P_N(R))$, and we shall show that S is, topologically, the N-sphere. Since η is a local homeomorphism, S is evidently an N-dimensional manifold. A priori it is not clear that S is connected; let S_0 be a component of S. Then η carries S_0 onto $P_N(R)$, and with the projection η, S_0 is a covering space of $P_N(R)$. Let $p_0 \in P_N(R)$ and let $s_0 \in \eta^{-1}(p_0) \cap S_0$. Let i: $(P_N(R), p_0) \to (V, p_0)$ and j: $(S_0, s_0) \to (\tilde{V}, s_0)$ be inclusion maps. We then have induced homomorphisms of the fundamental groups

$$\begin{aligned} i_*&: \ \pi_1(P_N(R), p_0) \to \pi_1(V, p_0), \\ j_*&: \ \pi_1(S_0, s_0) \to \pi_1(\tilde{V}, s_0), \\ \eta_*&: \ \pi_1(\tilde{V}, s_0) \to \pi_1(V, p_0), \\ (\eta \mid S_0)_*&: \ \pi_1(S_0, s_0) \to \pi_1(P_N(R), p_0) \ . \end{aligned}$$

[1] The integers mod 2.

There is the commutativity relation $\eta_* j_* = i_*(\eta \mid S_0)_*$. Since $\tilde{V}$ is the universal covering space of V, $\eta_* = 0$, and since $P_N(R)$ is a deformation retract of V, i_* is an isomorphism. Consequently $(\eta \mid S_0)_* = 0$. From the uniqueness of covering spaces corresponding to a given subgroup of the fundamental group (see, e.g. [6, Th. 6.6.16]) it follows that S_0 is homeomorphic to the universal covering space of $P_N(R)$, i.e., to the N-sphere and that $\eta \mid S_0$ is two-to-one. Since η is two-to-one on $\tilde{V}$, we have that $S = S_0$ and consequently that S is an N-sphere.

Next we show the existence of a $\Delta \in \mathscr{K}_N$ which contains $P_N(R)$ and which is contained in V. For this purpose, define a map Φ: $V \to \boldsymbol{C}^N$ by setting

$$\Phi(\mathfrak{z}) = \left(\frac{z_1^2}{z_0^2 + \cdots + z_N^2}, \cdots, \frac{z_N^2}{z_0^2 + \cdots + z_N^2} \right)$$

if $\mathfrak{z} \in V$ has homogeneous coordinates $(z_0, \cdots, z_N)$. The Φ so defined is proper. If not, there is a sequence $\{\zeta^{(k)}\}_{k=1}^{\infty}$ in $\boldsymbol{C}^N$ which converges to $\zeta^{(0)} \in \boldsymbol{C}^N$ such that for some choice of points $\mathfrak{z}_k \in \Phi^{-1}(\zeta^{(k)})$, $\mathfrak{z}_k \to \mathfrak{z}_0 \in P_N(\boldsymbol{C}) \backslash V$. Let $\mathfrak{z}_0$ have homogeneous coordinates $(z_0^{(0)}, \cdots, z_N^{(0)})$. Then $\Sigma(z_j^{(0)})^2 = 0$. We can choose homogeneous coordinates $(z_0^{(k)}, \cdots, z_N^{(k)})$ for $\mathfrak{z}_k$ so that for fixed j, $0 \leqq j \leqq N$, $z_j^{(k)} \to z_j^{(0)}$. Let $\zeta^{(0)} = (\zeta_1^{(0)}, \cdots, \zeta_N^{(0)})$. Since $\Phi(\mathfrak{z}_k) \to \zeta^{(0)}$, we have, for $1 \leqq j \leqq N$,

$$\zeta_j^{(0)} = \lim_{k \to \infty} (z_j^{(k)})^2 ((z_0^{(k)})^2 + \cdots + (z_0^{(k)})^2)^{-1} ,$$

and since $\mathfrak{z}_k \to \mathfrak{z}_0$ and $\mathfrak{z}_0 \notin V$, this implies that $z_j^{(k)} \to 0$. From $z_j^{(k)} \to z_j^{(0)}$, we conclude that $z_j^{(0)} = 0$ for $1 \leqq j \leqq N$. The fact that $\Sigma(z_j^{(0)})^2 = 0$ implies that $z_0^{(0)} = 0$, so $(z_0^{(0)}, \cdots, z_N^{(0)}) = (0, \cdots, 0)$ which is impossible. Thus Φ is proper. A short calculation shows that with the exception of the points in the variety

$$E = \{(\zeta_1, \cdots, \zeta_N) \in \boldsymbol{C}^N: \quad \zeta_1 \cdots \zeta_N = 0\} ,$$

every point of $\boldsymbol{C}^N$ has exactly 2^N preimages under Φ so the multiplicity of Φ is 2^N. This remark also indicates that Φ is regular at each point of the sets $\Phi^{-1}(\{(\zeta_1, \cdots, \zeta_N): \mid \zeta_1 \mid = \cdots = \mid \zeta_N \mid = R\})$. If $\mathfrak{z} \in V$ has real homogeneous coordinates, the definition of $\Phi(\mathfrak{z})$ shows that $\Phi(\mathfrak{z})$ lies in $\bar{U}^N$, so if $\varepsilon > 0$, then the set

$$\Delta_\varepsilon = \Phi^{-1}\{(z_1, \cdots, z_N) \in \boldsymbol{C}^N: \quad \mid z_j \mid < 1 + \varepsilon \text{ for } j = 1, \cdots, N\}$$

is an element of $\mathscr{K}_N$ with the desired property.

Let $P_N(R) \subset \Delta \subset V$, $\Delta \in \mathscr{K}_N$, and let $\Delta' = \eta^{-1}(\Delta)$. The mapping η: $\tilde{V} \to V$ is a covering map and so is certainly an element of $\mathscr{M}(\Delta', \Delta)$. Assume that $\mathscr{A}(\Delta')$ is generated as a module over $\eta^* \mathscr{A}(\Delta)$ by two elements, F_1 and F_2, so that if $f \in \mathscr{A}(\Delta')$ then for some $g_1, g_2 \in \mathscr{A}(\Delta)$

we have that $f(\zeta) = F_1(\zeta)g_1(\eta(\zeta)) + F_2(\zeta)g_2(\eta(\zeta))$. In particular, the functions F_1 and F_2 must separate points on each of the fibers $\eta^{-1}(\zeta)$, $\zeta \in P_N(R)$. Let S^N be the standard N-sphere in R^{N+1} and denote by $\xi: S^N \to P_N(R)$ the usual covering map which identifies antipodal points. If τ: $S^N \to S$ is a homeomorphism such that $\xi = \eta \circ \tau$, then the mapping $S^N \to C^2$ given by $\zeta \to (F_1(\tau(\zeta)), F_2(\tau(\zeta)))$ is continuous and separates antipodal points. Since C^2 is, topologically, R^4, and since we have assumed $N \geqq 4$, we have obtained a contradiction to the Borsuk-Ulam Theorem [6, Corollary 4.3.7]. Consequently, $\mathscr{A}(\Delta')$ is not generated by two elements over $\eta^*\mathscr{A}(\Delta)$. This argument shows, in fact, that any set of generators for $\mathscr{A}(\Delta')$ must contain more than $N/2$ generators.

2. $\mathscr{A}(\Delta')$ as a module over $\Psi^*\mathscr{A}(\Delta)$. Complementing the previous example, we have the following result.

THEOREM 2.1. *If $\Delta', \Delta \in \mathscr{K}_N$ and $\Psi \in \mathscr{M}(\Delta', \Delta)$, then $\mathscr{A}(\Delta')$ and $H^\infty(\Delta')$ are finitely generated projective modules over $\Psi^*\mathscr{A}(\Delta)$ and $\Psi^*H^\infty(\Delta)$ respectively.*

Proof. It is easy to see that $\mathscr{A}(\Delta')$ is finitely generated over $\Psi^*\mathscr{A}(\Delta)$ and that a similar result obtains concerning $H^\infty(\Delta')$. Let $\Psi \in \mathscr{M}(\Delta, U^N)$. Then $\Phi \circ \Psi \in \mathscr{M}(\Delta', U^N)$, and consequently $\mathscr{A}(\Delta')$ is finitely generated over $(\Phi \circ \Psi)^*\mathscr{A}(U^N)$: We know by [7, Th. I. 4] that for some $B_1, \cdots, B_m \in \mathscr{O}(\bar{\Delta}')$, each $f \in \mathscr{A}(\Delta)$ is of the form

$$f = \sum B_j f_j \circ \Phi \circ \Psi$$

for some choice of f_j in $\mathscr{A}(U^N)$. Since $f \circ \Phi$ is in $\mathscr{A}(\Delta)$, this shows that $\mathscr{A}(\Delta')$ is finitely generated over $\Psi^*\mathscr{A}(\Delta)$. The case of $H^\infty(\Delta')$ can be treated in the same way. Somewhat more is required to show that these modules are projective.

Since $\Psi \in \mathscr{M}(\Delta', \Delta)$, there is a neighborhood Ω' of $\bar{\Delta}'$ which is mapped properly onto a neighborhood Ω of $\bar{\Delta}$ by Ψ. We know [7, Lemma 1. 2] that the direct image sheaf $\Psi * \mathscr{O}_{\Omega'}$ is locally free of rank λ, λ the multiplicity of Ψ on Ω. Let W be a relatively compact open set which is a Stein manifold and which satisfies $\Omega \supset \bar{W} \supset W \supset \bar{\Delta}$. By Cartan's Theorem A and compactness, there exist

$$\tilde{F}_1, \cdots, \tilde{F}_q \in \Gamma(\Omega, \Psi * \mathscr{O}_{\Omega'})$$

such that if $\zeta \in \bar{W}$, then the germs $(\tilde{F}_j)_\zeta$ generate $(\Psi * \mathscr{O}_{\Omega'})_\zeta$[2]. Thus, if H is the sheaf homomorphism $\mathscr{O}^q \to \Psi_* \mathscr{O}_{\Omega'}$ given by $H(f_1, \cdots, f_q) = \sum f_j(\tilde{F}_j)_\zeta$ for all $f_j \in \mathscr{O}_\zeta$ then

[2] By a result of Forster and Ramspott [4, Satz 2], we can take $q = \lambda + [N/2]$.

$$(1) \qquad \mathcal{O}^q \xrightarrow{H} \Psi_* \mathcal{O}_{\Omega'} \longrightarrow 0$$

is exact over W.

We now need a lemma which is surely well known but for which we are unable to provide a reference. (Remarks of the referee have enabled us to abbreviate our discussion of this lemma).

LEMMA 2.2. *If* $\mathcal{S}_1 \xrightarrow{H} \mathcal{S}_2 \to 0$ *is an exact sequence of locally free sheaves over a complex manifold* M, *then* $\ker H$ *is locally free.*

Proof. The sheaves $\mathcal{S}_1$ and $\mathcal{S}_2$ are locally free and the question is local, so we can suppose $\mathcal{S}_1 = \mathcal{O}^p$, $\mathcal{S}_2 = \mathcal{O}^q$. Since $\ker H$ is coherent, it suffices to prove that for each $\zeta \in M$, the stalk $(\ker H)_\zeta$ is a free $\mathcal{O}_\zeta$-module. (See Lemma 1.2 of [7]). We have the exact sequence

$$0 \longrightarrow \ker H \longrightarrow \mathcal{O}^p \longrightarrow \mathcal{O}^q \longrightarrow 0$$

so we can invoke [2, I. 2.5, Proposition 5 and II. 5.2, Corollary 2] to conclude that $(\ker H)_\zeta$ is a projective $\mathcal{O}_\zeta$-module. Since projective modules over local rings are free, we can conclude that $(\ker H)_\zeta$ is free as desired.

To continue with the proof of the theorem, we apply the lemma to the sequence

$$(2) \qquad 0 \longrightarrow \ker H \longrightarrow \mathcal{O}^q \xrightarrow{H} \Psi * \mathcal{O}_{\Omega'} \longrightarrow 0$$

over W obtained from (1), and we find that this is an exact sequence of locally free sheaves. Consequently [4, Th. VIII C 7], the sequence (2) splits, and in particular there is a sheaf isomorphism L of $\Psi_* \mathcal{O}_{\Omega'}$ into $\mathcal{O}^q$ such that $H \circ L$ is the identity on $\Psi_* \mathcal{O}_{\Omega'}$ and $L \circ H$ is a projection of $\mathcal{O}^q$ onto the range of L. Apply this to the spaces of sections over $\bar{\Delta}$ and pull the resulting statement back to $\bar{\Delta}'$ by way of the map Ψ. We find that there are exact sequences of $\Psi^* \mathcal{O}(\bar{\Delta})$-module homomorphisms

$$0 \longrightarrow \mathcal{O}(\bar{\Delta}') \xrightarrow{L'} (\Psi^* \mathcal{O}(\bar{\Delta}))^q$$

and

$$(\Psi^* \mathcal{O}(\bar{\Delta}))^q \xrightarrow{H'} \mathcal{O}(\bar{\Delta}') \longrightarrow 0 .$$

Here $H'(f_1 \circ \Psi, \cdots, f_q \circ \Psi) = \sum F_j f_j \circ \Psi$ where $F_j \in \mathcal{O}(\bar{\Delta}')$ corresponds to the section $\tilde{F}_j$ of $\Psi_* \mathcal{O}_{\Omega'}$, $L' \circ H'$ projects $\Psi^* \mathcal{O}(\bar{\Delta})$ onto the range of L', and $H' \circ L'$ is the identity on $\mathcal{O}(\bar{\Delta}')$. This establishes $\mathcal{O}(\bar{\Delta}')$ as a finitely generated, projective $\Psi^* \mathcal{O}(\bar{\Delta})$-module.

To treat $\mathcal{A}(\Delta')$ and $H^\infty(\Delta')$, note that since L is a sheaf isomorphism, L' extents to an isomorphism L'' of $\mathcal{O}(\Delta')$ into $(\Psi^* \mathcal{O}(\Delta))^q$ and

that H' extends to a homomorphism H'' of $(\Psi^* \mathscr{O}(\Delta))^q$ to $\mathscr{O}(\Delta')$. The form of H shows that H'' carries $(\Psi^* \mathscr{A}(\Delta))^q$ into $\mathscr{A}(\Delta')$ and $(\Psi^* H^\infty(\Delta))^q$ into $H^\infty(\Delta')$. In fact H'' carries $(\Psi^* \mathscr{A}(\Delta))^q$ and $(\Psi^* H^\infty(\Delta))^q$ onto $\mathscr{A}(\Delta')$ and $H^\infty(\Delta')$ respectively. As we noted at the beginning of the proof, if $f \in \mathscr{A}(\Delta')$, then $f = \sum B_j f_j \circ \Psi$ for some $f_j \in \mathscr{A}(\Delta)$ and some $B_j \in \mathscr{O}(\bar{\Delta}')$. We have $B_j = H'(\tilde{B}_j)$ for some $\tilde{B}_j \in (\Psi^* \mathscr{O}(\bar{\Delta}))^q$. On $(\Psi^* \mathscr{A}(\Delta))^q$, H'' acts as a $\Psi^* \mathscr{A}(\Delta)$-module homomorphism, so we have $f = H''(\sum \tilde{B}_j f_j \circ \Psi)$. The case that f lies in $H^\infty(\Delta')$ may be treated in a similar way.

Also, L'' carries $\mathscr{A}(\Delta')$ into $(\Psi^* \mathscr{A}(\Delta))^q$ and $H^\infty(\Delta')$ into $(\Psi^* H^\infty(\Delta))^q$. If $f \in \mathscr{A}(\Delta')$, we write $f = \sum B_j f_j \circ \Psi$ as above. Then $L''f = \sum f_j \circ L'' B_j$. We have $L'' B_j \in (\Psi^* \mathscr{O}(\bar{\Delta}))^q \subset (\Psi^* \mathscr{A}(\Delta))^q$. Thus $L''f$ is in $(\Psi^* \mathscr{A}(\Delta))^q$ as asserted. The H^∞ case follows in the same way.

The operator $L'' \circ H''$ acts on $(\Psi^* \mathscr{A}(\Delta))^q$ as a projection with range the range of L'' on $\mathscr{A}(\Delta')$. To see this, note first that the range of $L'' \circ H''$ is $L''(\mathscr{A}(\Delta'))$, for H'' carries $(\Psi^* \mathscr{A}(\Delta))^q$ onto $\mathscr{A}(\Delta')$. If $f \in \mathscr{A}(\Delta')$, then since $H'' \circ L''$ is the identity, we find that $L'' \circ H''(L''f) = L''f$, so $L'' \circ H''$ is a projection. Since L'' takes $\mathscr{A}(\Delta')$ isomorphically into $(\Psi^* \mathscr{A}(\Delta))^q$, we have proved that $\mathscr{A}(\Delta')$ is a projective $\Psi^* \mathscr{A}(\Delta)$-module. In the same way, it follows that $H^\infty(\Delta')$ is a projective $\Psi^* H^\infty(\Delta)$-module, and the proof of the theorem is concluded.

3. Criteria for the freeness of $\mathscr{A}(\Delta')$ over $\Psi^* \mathscr{A}(\Delta)$. There are certain cases in which $\mathscr{A}(\Delta')$ is necessarily free over $\Psi^* \mathscr{A}(\Delta)$. To introduce some of these we need to consider products. Suppose $\Delta_1 \in \mathscr{K}_{N_1}$, $\Delta_2 \in \mathscr{K}_{N_2}$, and let $\Phi_j \in \mathscr{M}(\Delta_j, U^{N_j})$. If Ω_j is a neighborhood of $\bar{\Delta}_j$ which is mapped properly into the neighborhood W_j of $\bar{U}^{N_j}$, $j = 1, 2$, then the map Ψ: $\Omega_1 \times \Omega_2 \to W_1 \times W_2$ given by

$$\Psi(\zeta_1, \zeta_2) = (\Phi_1(\zeta_1), \Phi_2(\zeta_2)) \in W_1 \times W_2$$

is proper, and $\Delta_1 \times \Delta_2 = \Psi^{-1}(U^{N1+N2})$. Moreover, Ψ is a local homeomorphism at each point of $\Psi^{-1}(T^{N1+N2})$. Thus $\Delta_1 \times \Delta_2 \in \mathscr{K}_{N1+N2}$, and $\Psi \in \mathscr{M}(\Delta_1 \times \Delta_2, U^{N1+N2})$. Similarly, if we are given Δ_1' and Δ_2' in $\mathscr{K}_{N1}$ and $\mathscr{K}_{N2}$ respectively and if $\Phi_j \in \mathscr{M}(\Delta_j', \Delta_j)$, $j = 1, 2$, then the map $\Phi_1 \times \Phi_2$ from $\Delta_1' \times \Delta_2'$ to $\Delta_1 \times \Delta_2$ defined by $\Phi_1 \times \Phi_2(\zeta_1, \zeta_2) = (\Phi_1(\zeta_1), \Phi_2(\zeta_2))$ is an element of $\mathscr{M}(\Delta_1' \times \Delta_2', \Delta_1 \times \Delta_2)$. If Φ_j has multiplicity λ_j, then $\Phi_1 \times \Phi_2$ has multiplicity $\lambda_1 \lambda_2$.

THEOREM 3.1. *If for $j = 1, 2$, $\Delta_j, \Delta_j' \in \mathscr{K}_{N_j}$, if $\Phi_j \in \mathscr{M}(\Delta_j', \Delta_j)$ is of multiplicity λ_j, and if $\mathscr{A}(\Delta_j')$ is free of rank λ_j over $\Psi^* \mathscr{A}(\Delta_j)$, then $\mathscr{A}(\Delta_1' \times \Delta_1')$ is free of rank $\lambda_1 \lambda_2$ over $(\Phi_1 \times \Phi_2)^* \mathscr{A}(\Delta_1 \times \Delta_2)$.*

Before giving the proof of this theorem, let us mention that by Theorem 2.3 of [7], if $\Phi \in \mathscr{M}(U^N, U^N)$, then in an obvious extension

of the above notation, $\Phi = \varphi_1 \times \cdots \times \varphi_N$ where each $\varphi_j \in \mathscr{M}(U, U)$ is a finite Blaschke product.

Proof. Let $\{F_1^{(1)}, \cdots, F_{\lambda_1}^{(1)}\}$ be a free basis for $\mathscr{A}(\Delta_1')$ over $\Phi_1^* \mathscr{A}(\Delta_1)$ and let $\{F_1^{(2)}, \cdots, F_{\lambda_2}^{(2)}\}$ be one for $\mathscr{A}(\Delta_2')$ over $\Phi_2^* \mathscr{A}(\Delta_2)$. We assert that the set $\{F_j^{(1)} F_k^{(2)}\colon \ 1 \leqq j \leqq \lambda_1, 1 \leqq k \leqq \lambda_2\}$ is a free basis for $\mathscr{A}(\Delta_1' \times \Delta_2')$ over $(\Phi_1 \times \Phi_2)^* \mathscr{A}(\Delta_1 \times \Delta_2)$.

Define $E_1(\mathfrak{z})$ for $\mathfrak{z} \in \overline{\Delta}_1'$ to be the set $\Phi_1^{-1}(\Phi_1(\mathfrak{z}))$, and define $E_2(\mathfrak{z})$ for $\mathfrak{z} \in \overline{\Delta}_2'$ in an analogous way. In general $E_1(\mathfrak{z})$ will consist of λ_1 points. Since $\{F_k^{(1)}\}$ is a free basis for $\mathscr{A}(\Delta_1')$ over $\Phi_1^* \mathscr{A}(\Delta_1)$, each $f \in \mathscr{A}(\Delta_1')$ has a unique expression in the form

$$f(\mathfrak{z}) = F_1^{(1)}(\mathfrak{z})\, f_1(\Phi_1(\mathfrak{z})) + \cdots + F_{\lambda_1}^{(1)}(\mathfrak{z})\, f_{\lambda_1}(\Phi_1(\mathfrak{z}))$$

with $f_1, \cdots, f_{\lambda_1} \in \mathscr{A}(\Delta_1)$. The functions $f_j \circ \Phi_1$ can be computed explicitly by Cramer's rule: $f_j(\Phi_1(\mathfrak{z})) = D_j(\mathfrak{z}) D^{-1}(\mathfrak{z})$ where, for $\mathfrak{z} \in \overline{\Delta}'$ such that $E_1(\mathfrak{z})$ consists of λ_1 distinct points, say $E_1(\mathfrak{z}) = \{\mathfrak{z}_1, \cdots, \mathfrak{z}_{\lambda_1}\}$, we have

$$D(\mathfrak{z}) = \det\,(F_j^{(1)}(\mathfrak{z}_k))_{1 \leqq j,k \leqq \lambda_1}\ ,$$

and $D_j(\mathfrak{z})$ is obtained from $D(\mathfrak{z})$ by replacing the j^{th} column by the column vector $(f(\mathfrak{z}_1), \cdots, f(\mathfrak{z}_{\lambda_1}))$.

If we are given an element $G \in \mathscr{A}(\Delta_1' \times \Delta_2')$, then for fixed $w \in \overline{\Delta}_2'$, the preceding remarks may be applied to the element $G(\cdot, w)$ of $\mathscr{A}(\Delta_1')$:

$$G(\mathfrak{z}, w) = \sum_{j=1}^{\lambda_1} F_j^{(1)}(\mathfrak{z})\, f_j(\Phi_1(\mathfrak{z}), w)$$

where, for fixed w, $f_j(\cdot, w) \in \mathscr{A}(\Delta_1)$. The expression for f_j as a certain quotient of determinants shows that $f_j(\Phi_1(\mathfrak{z}), w)$ is in fact an element of $\mathscr{A}(\Delta_1' \times \Delta_2')$. Thus, for fixed $\mathfrak{z}$, we can write

$$f_j(\Phi_1(\mathfrak{z}), w) = \sum_{k=1}^{\lambda_2} F_k^{(2)} w g_{j,k}(\mathfrak{z}, \Phi_2(w))\ .$$

Again, we can compute the functions $g_{j,k}$ explicitly: If $E_2(w) = \{w_1, \cdots, w_{\lambda_2}\}$, then $g_{j,k}(\mathfrak{z}, \Phi_2(w)) = \tilde{D}_k(\mathfrak{z}, \Phi_2(w))\, \tilde{D}(\mathfrak{z}, \Phi_2(w))^{-1}$ where, as before,

$$\tilde{D}(\mathfrak{z}, \Phi_2(w)) = \det\,(F_k^{(2)}(w_m))_{1 \leqq k,m \leqq \lambda_2}$$

and $\tilde{D}_k(\mathfrak{z}, \Phi_2(w))$ is obtained by replacing the k^{th} column of $\tilde{D}$ by the column vector $(f_j(\Phi_1(\mathfrak{z}), w_1), \cdots, f_j(\Phi_1(\mathfrak{z}), w_{\lambda_2}))$. This representation for $g_{j,k}$ shows that for fixed w, $g_{j,k}(\mathfrak{z}, \Phi_2(w))$ is, as a function of $\mathfrak{z}$, constant on the set $E_1(\mathfrak{z})$. Thus, we can write $g_{j,k}(\mathfrak{z}, \Phi_2(w)) = h_{j,k}(\Phi_1(\mathfrak{z}), \Phi_2(w))$ for some suitably chosen $h_{j,k} \in \mathscr{A}(\Delta_1 \times \Delta_2)$. We now have the representation

$$G(\mathfrak{z}, w) = \Sigma F_j^{(1)}(\mathfrak{z}) F_k^{(2)}(w) h_{j,k}(\Phi_1(\mathfrak{z}), \Phi_2(w))$$

for G. Thus, $\{F_j^{(1)}F_k^{(2)}\}$ is a set of generators for $\mathscr{A}(\Delta_1' \times \Delta_2')$ over $(\Phi_1 \times \Phi_2)^* \mathscr{A}(\Delta_1 \times \Delta_2)$.

That $F_j^{(1)}F_k^{(2)}$ are free generators is now clear: If there were a nontrivial relation

$$0 = \Sigma F_j^{(1)}(\zeta)F_k^{(2)}(w)h_{j,k}(\Phi_1(\zeta), \Phi_2(w)) .$$

Then for some fixed choice of w we could regard this as a nontrivial relation among the functions $F_1^{(1)}, \cdots, F_{\lambda_1}^{(1)}$. But since $\{F_j^{(1)}\}$ is a free basis for $\mathscr{A}(\Delta_1')$, no such relation can exist, and the theorem is proved.

It is clear that a similar result obtains for products of more than two elements of $\mathscr{K}_N$ and that an analogous theorem holds for bounded functions.

We saw in [7] that if $\Delta \in \mathscr{K}_N$ and $\Phi \in \mathscr{M}(\Delta, U^N)$, then $\mathscr{A}(\Delta)$ is a free module over $\Phi^*\mathscr{A}(U^N)$. The essential ingredient of this proof is the fact that for some neighborhood Ω of $\bar{\Delta}$ and some neighborhood W of U^N, the sheaf $\Phi_*\mathscr{O}_\Omega$ is a free sheaf over W. The relation between the freeness of $\mathscr{A}(\Delta)$ over $\Phi^*\mathscr{A}(U^N)$ and the freeness of the sheaf $\Phi_*\mathscr{O}_\Omega$ is one which persists in more general settings.

THEOREM 3.2. *If $\Delta, \Delta' \in \mathscr{K}_N$, if $\Phi \in \mathscr{M}(\Delta', \Delta)$, and if $\mathscr{A}(\Delta')$ is free of rank λ, λ the multiplicity of Φ, over $\Phi^*\mathscr{A}(\Delta)$, then for some neighborhood Ω of $\bar{\Delta}'$ on which Φ is defined, the sheaf $\Phi_*\mathscr{O}_\Omega$ is free over $\mathscr{O}_{\Phi(\Omega)}$.*

Before proving the theorem, a simple preliminary observation is needed.

LEMMA 3.3. *If $\Delta, \Delta' \in \mathscr{K}_N$, if $\Phi \in \mathscr{M}(\Delta', \Delta)$, and if $\mathscr{A}(\Delta')$ is free over $\Phi^*\mathscr{A}(\Delta)$, then there exists a set of free generators for $\mathscr{A}(\Delta')$ which consists of functions holomorphic on a neighborhood of $\bar{\Delta}'$.*

Proof. By hypothesis there exists an isomorphism

$$L: \quad \Phi^*\mathscr{A}(\Delta)^q \to \mathscr{A}(\Delta');$$

it is of the form

$$L(f_1 \circ \Phi, \cdots, f_q \circ \Phi) = \sum_{j=1}^{q} F_j f_j \circ \Phi$$

for some fixed elements $F_1, \cdots, F_q \in \mathscr{A}(\Delta')$. The operator L is continuous and so it has a continuous inverse L^{-1}. If S: $\Phi^*\mathscr{A}(\Delta)^q \to \mathscr{A}(\Delta')$ is near L in the norm topology of $\mathscr{L}(\Phi^*\mathscr{A}(\Delta)^q, \mathscr{A}(\Delta'))$,[3] then $S \circ L^{-1}$

[3] We use $\mathscr{L}(X, Y)$ to denote the continuous linear operators from the Banach space X to the Banach space Y.

is near the identity of $\mathscr{L}(\mathscr{A}(\Delta'), \mathscr{A}(\Delta'))$ and so is invertible. Thus, if S is near L, S is also an isomorphism. Therefore if we choose functions $G_1, \cdots, G_q$ holomorphic on a neighborhood of $\bar{\Delta}'$ so that G_j is uniformly near F_j on $\bar{\Delta}'$ and if we define

$$S: \quad \Phi^*\mathscr{A}(\Delta)^q \to \mathscr{A}(\Delta')$$

by $S(f_1 \circ \Phi, \cdots, f_q \circ \Phi) = \Sigma G_j f_j \circ \Phi$, then S is a $\Phi^*\mathscr{A}(\Delta)$-module isomorphism, i.e., $\{G_1, \cdots, G_q\}$ is a free basis for $\mathscr{A}(\Delta')$ over $\Phi^*\mathscr{A}(\Delta)$. That the desired approximating functions exist is contained in [7, Corollary I. 6].

Proof of Theorem 3.4. Let $F_1, \cdots, F_\lambda \in \mathscr{O}(\bar{\Delta}')$ be a free basis for $\mathscr{A}(\Delta')$ over $\Phi^*\mathscr{A}(\Delta)$, and let Ω be a neighborhood of $\bar{\Delta}$ such that $F_1, \cdots, F_\lambda$ are all holomorphic in $\Omega' = \Phi^{-1}(\Omega)$. We have a homomorphism $\mathscr{F}: \ \mathscr{O}_\Omega^\lambda \to \Phi_*\mathscr{O}_{\Omega'}$ defined by $\mathscr{F}(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_\lambda) = \Sigma(\tilde{F}_j)_\zeta \boldsymbol{f}_j$ for all

$$(\boldsymbol{f}_1, \cdots, \boldsymbol{f}_\lambda) \in \mathscr{O}_\zeta^\lambda, \zeta \in \Omega' .$$

Here $\tilde{F}_j$ is the section of $\Phi_*\mathscr{O}_{\Omega'}$ corresponding to F_j, and $(\tilde{F}_j)_\zeta$ is its germ at ζ. We shall show that $\mathscr{F}$ is a sheaf isomorphism at least when we restrict attention to some, possibly smaller neighborhood of $\bar{\Delta}$.

Consider a point $\zeta \in \bar{\Delta}$. We shall show that the stalk map given by $\mathscr{F}$ is an isomorphism in the stalk over ζ. Since $\Phi_*\mathscr{O}_{\Omega'}$ is locally free of rank λ, there is an isomorphism $L: \ \mathscr{O}_\zeta^\lambda \to (\Phi_*\mathscr{O}_{\Omega'})_\zeta$. Denote by $\boldsymbol{e}_j$ the element $(0, \cdots, 0, 1, 0, \cdots, 0)$ (1 in the j^{th} place) of $\mathscr{O}_\zeta^\lambda$, and let $\boldsymbol{g}_j \in (\Phi_*\mathscr{O}_{\Omega'})_\zeta$ be the image of $\boldsymbol{e}_j$ under L. There is a neighborhood W of ζ in which all the germs $\boldsymbol{g}_j$ can be represented by sections of $\Phi_*\mathscr{O}_{\Omega'}$; call these sections g_j. Thus, $g_j \in \mathscr{O}(\Phi^{-1}(W))$. If $V \subset W$ is a small neighborhood of ζ, then on $(\Phi^{-1}(V))^-$, the functions g_j will admit uniform approximation by functions $G_j \in \mathscr{O}(\bar{\Delta})$. We can choose the approximating functions G_j so that in the expression

$$G_j = \Sigma F_k h_k^{(j)} \circ \Phi \qquad (h_k^{(j)} \in \mathscr{A}(\Delta))$$

the functions $h_k^{(j)}$ lie in $\mathscr{O}(\bar{\Delta})$. The G_j give sections $\tilde{G}_j$ of $\Phi_*\mathscr{O}_{\Omega'}$ which lie in the range of $\mathscr{F}$. Moreover, for any choice of G_j, we obtain a homomorphism $L': \ \mathscr{O}_\zeta^\lambda \to (\Phi_*\mathscr{O}_{\Omega'})_\zeta$ by setting $L'(\boldsymbol{e}_j) = (\tilde{G}_j)_\zeta$ If the functions G_j approximate the functions g_j sufficiently well, the L' so obtained will be an isomorphism since L is. Fix a choice of the G_j so that L' is an isomorphism.

Using L', we can see that $\mathscr{F}$ is onto (in the stalk over ζ), for the range of $\mathscr{F}$ contains $\{\tilde{G}_1, \cdots, \tilde{G}_\lambda\}$ and so it contains the module generated by this set. Since L' is onto, it follows that this module is the whole of $(\Phi_*\mathscr{O}_{\Omega'})_\zeta$.

The fact that $\mathscr{F}$ is one-to-one in the stalk over ζ follows from

Cartan's Theorem A. If $\mathscr{T}(f_1, \cdots, f_\lambda) = 0$, $(f_1, \cdots, f_\lambda) \in \mathscr{O}^\lambda_\zeta$, then $\mathscr{R}$, the sheaf of relations among $\tilde{F}_1, \cdots, \tilde{F}_\lambda$ is nontrivial over $\bar{\Delta}$. Thus by Cartan's Theorem A, there is a nontrivial section of $\mathscr{K}$ over $\bar{\Delta}$, i.e., there exist $h_1, \cdots, h_\lambda \in \mathscr{O}(\bar{\Delta})$ not all of which are the zero function, such that $\Sigma h_j \tilde{F}_j = 0$, i.e., $\Sigma F_j h_j \circ \Phi = 0$. This is impossible since $\{F_1, \cdots, F_\lambda\}$ is a free basis for $\mathscr{A}(\Delta')$ over $\Phi^* \mathscr{A}(\Delta)$.

Thus, for all $\zeta \in \Delta$, $\mathscr{T}$ carries $\mathscr{O}^\lambda_\zeta$ isomorphically onto $(\Phi^* \mathscr{O}_{\Omega'})_\zeta$. Consequently, the same assertion holds for all ζ in a neighborhood of $\bar{\Delta}$, and the theorem is proved.

Note added in proof. My colleague S.J. Sidney has observed that with a somewhat more careful use of the Borsuk-Ulam theorem, it is possible to show that the example of Section I is valid in dimensions two and three as well as in higher dimensions. Consider, in the notation of that section, a $\Delta \in \mathscr{K}_N$, $N \geqq 2$ with $P_N(R) \subset \Delta \subset V$, and let $\Delta' = \eta^{-1}(\Delta)$. Assume that F and G generate $A(\Delta')$ as a module over $\eta^* \mathscr{A}(\Delta)$. If f is any element of $\mathscr{A}(\Delta')$, we can write $f = f_e + f_0$, $f_e, f_0 \in \mathscr{A}(\Delta')$ where f_e is *even* in the sense that it is constant on the fiber $\eta^{-1}(\zeta)$, $\zeta \in \Delta$, and f_0 is *odd* in that if $\eta^{-1}(\zeta) = (\zeta', \zeta'')$, then $f_0(\zeta') = -f_0(\zeta'')$. To obtain such a decomposition, write $f_e(\zeta) = \frac{1}{2}(f(\zeta') + f(\zeta''))$ where $\eta^{-1}(\eta(\zeta)) = \{\zeta', \zeta''\}$, and define f_0 to be $f - f_e$. It is clear that f_e is, in fact, even, and that f_0 is odd. It is easily verified that this decomposition of f into a sum of even and odd parts is unique.

By hypothesis, if $h \in \mathscr{A}(\Delta')$, we have $h = fF + gG$ for some choice of $f, g \in \eta^* \mathscr{A}(\Delta)$. The functions f and g are both even, and it follows that if $h = h_e + h_0$, then

$$h_e = fF_0 + gG_e$$

and

$$h_0 = fF_0 + gG_0 .$$

By suitable choice of $h \in \mathscr{A}(\Delta)$, we can arrange that the pair $(h_e(\zeta), h_0(\zeta))$ be any point of C^2, so it follows that the

$$\text{determinant } \mathfrak{d}(\zeta) = \begin{vmatrix} F_e(\zeta) & G_e(\zeta) \\ F_0(\zeta) & G_0(\zeta) \end{vmatrix} \text{ is zero for no choice of } \zeta.$$

However, $\mathfrak{d}$ is an odd function. Thus, continuing with the notation of Section I, $\mathfrak{d} \circ \xi$ is a C-valued function on S^N which is zero-free and odd in that if p and q are antipodal points in S^N, then $\mathfrak{d} \circ \xi(p) = -\mathfrak{d} \circ \xi(q)$. Since C is topologically R^2 and since $N \geqq 2$, the Borsuk-Ulam theorem implies that such an odd function has a zero, and the desired contradiction has been reached.

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YALE UNIVERSITY

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A CHARACTERISTIC PROPERTY OF THE SPHERE

L. TAMÁSSY

It was proven by P. Funk that the only symmetrical star-shaped body, all of whose intersections with planes through its midpoint have surface area π, is the unit sphere. In this paper the same conclusion is deduced from a materially weaker hypothesis and an application of the result is given.

We consider the set Π of those planes through the midpoint of a symmetrical body in the three-space, whose normals make an angle $<\eta$ with a fixed plane Γ. η is an arbitrary positive constant.

THEOREM. *The only symmetrical star-shaped body with smooth boundary, which is intersected by elements of Π in figures of surface-area π, is the unit sphere.*

Star-shapeness means, that each closed ray from the midpoint out meets the boundary of the body at most at one point.

The property expressed in the theorem obviously holds for the unit sphere.

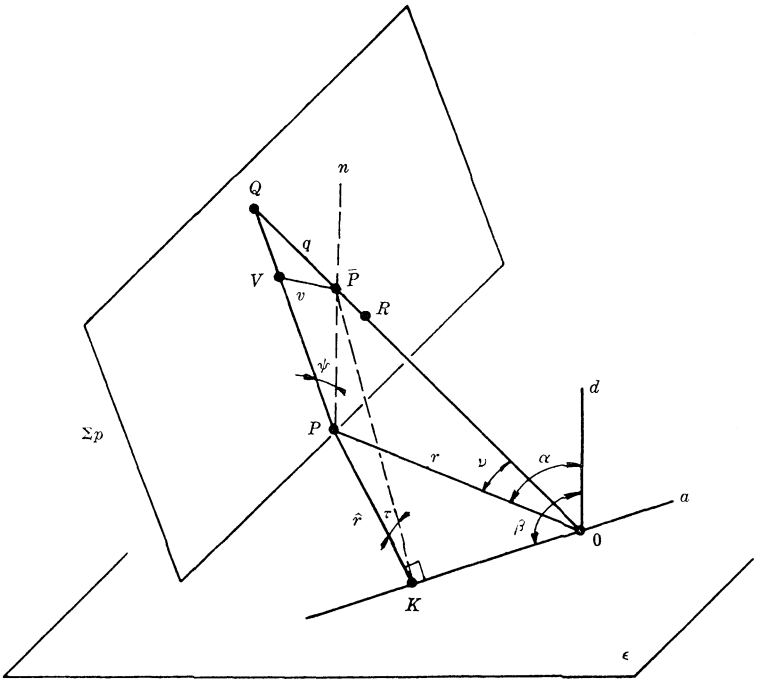
Let B be a symmetric starshaped body with smooth boundary and center 0. Let $\vec{n}$ be a unit vector. We denote the plane through 0 and perpendicular to $\vec{n}$ by $P(\vec{n})$, and the surface-area of the intersection of $P(\vec{n})$ and of B by $A(\vec{n})$: $A(\vec{n}) = \text{Area}\ (B \cap P(\vec{n}))$. Let 0 be the origin of a cartesian coordinate system and Γ the (y, z) plane.

We suppose $A(\vec{n}) = \pi$ for $\sphericalangle(\vec{n}, \Gamma) < \eta$, and we start to prove the uniqueness of B.

Let P be a point of the boundary ∂B of B in the (x, y) plane ε. We denote the tangent plane to ∂B in P by Σ_p. Let a be a line in ε through 0 and σ a plane through a (see the figure). We draw the normal n to ε through P and we denote the intersection of n and σ by $\bar{P}$; the intersection of ∂B and $\bar{P}0$ by R; and the intersection of Σ_p and $0R$ by Q. K denotes the foot of the normal from P to a. We draw the perpendicular to n through $\bar{P}$ in the plane of 0 and n; let V be the point of intersection of this line and of QP. Finally we denote

$$\overline{Q\bar{P}} = q;\ \overline{P0} = r;\ \overline{PK} = \hat{r};\ \sphericalangle QP\bar{P} = \psi;\ \sphericalangle PK\bar{P} = \tau;\ \sphericalangle P0\bar{P} = \nu$$

and the angles measured in the positive sense from an arbitrary direction d to $0P$ (resp. to a) by α (resp. β). τ and ν are regarded positive if $\bar{P}$ lies in the positive halfspace of ε, otherwise they are negative. q is positive if n separates Q and 0 (in the plane of n, 0),



otherwise it is negative. ψ has the same sign as q, and r and $\hat{r}$ are not negative. We note that Q and R are uniquely determined by α, β and τ for a given body B.

Let now σ turn about a into ε. Then τ and ν tend to zero and $R(\nu)$ moves in a curve in the plane of 0 and n having the tangent QP at P. Thus we must have

$$\lim_{\nu \to 0} \frac{\overline{0Q} - \overline{0R}}{\nu} = 0 . \tag{1}$$

Since $\nu \to 0$ follows from $\tau \to 0$ and inversely, and since $|\tau| \geqq |\nu|$, we get from (1)

$$\lim_{\tau \to 0} \frac{\overline{0Q} - \overline{0R}}{\tau} = 0 .$$

Multiplying by $\overline{0Q} + \overline{0R}$ and emphasizing the dependence of Q and R on α, β, τ, we get

$$\lim_{\tau \to 0} \frac{\overline{0Q}^2(\alpha, \beta, \tau) - \overline{0R}^2(\alpha, \beta, \tau)}{\tau} = 0 . \tag{2}$$

Let $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n-1} < \alpha_n = \pi$ be a subdivision of the interval $(0, \pi)$ of α. We consider (2) for the different $\alpha_i (i = 1, 2, \cdots, n)$, and we put $\Delta\bar{\alpha}_i = \sphericalangle Q(\alpha_{i-1}, \beta, \tau) 0 Q(\alpha_i, \beta, \tau)$. Now, multiplying by $\Delta\bar{\alpha}_i$, sum-

ming for i form 1 to n, and then carrying out an appropriate limit process, we obtain

$$\lim_{\tau\to 0}\frac{\int_0^{\pi}\overline{0\bar{Q}}^2 d\bar{\alpha}-\int_0^{\pi}\overline{0Q}^2 d\bar{\alpha}}{\tau}=0\,. \tag{3}$$

where the second integral of the numerator gives the surface-measure of the intersection of B and σ and, thus, its value equals π according to our condition, provided that $|\tau|<\eta$. (In this case the normal of σ makes with Γ a smaller angle than η).

Furthermore

$$\overline{0\bar{Q}}=\overline{0\bar{\bar{P}}}+\overline{\bar{\bar{P}}\bar{Q}}=r+q+o(\tau)$$

where o denotes the small ordo, introduced by Landau.[1] According to an elementary computation,

$$\Delta\bar{\alpha}=\Delta\alpha+o(\tau)\Delta\alpha$$

and

$$\begin{aligned}\int_0^{\pi}\overline{0\bar{Q}}^2 d\bar{\alpha}=&\int_0^{\pi}r^2 d\alpha+2\int_0^{\pi}rq\,d\alpha+\int_0^{\pi}q^2 d\bar{\alpha}+\int_0^{\pi}(r^2+2rq)o(\tau)d\alpha\\&+\int_0^{\pi}[2r+2q+o(\tau)]o(\tau)d\bar{\alpha}\,.\end{aligned} \tag{4}$$

Substituting (4) into (3) and changing the order of the integration and of the limit process in some terms we obtain

$$\begin{aligned}&\lim_{\tau\to 0}\frac{\int_0^{\pi}r^2 d\alpha-\int_0^{\pi}\overline{0\bar{R}}^2 d\bar{\alpha}}{\tau}+2\int_0^{\pi}\Big[\lim_{\tau\to 0}\frac{rq}{\tau}\Big]d\alpha\\&+\int_0^{\pi}\Big[\lim_{\tau\to 0}\frac{q^2}{\tau}\Big]d\bar{\alpha}+\int_0^{\pi}\lim_{\tau\to 0}\frac{[r^2+2rq]o(\tau)}{\tau}\,d\bar{\alpha}\\&+\int_0^{\pi}\lim_{\tau\to 0}\frac{[2r+2q+o(\tau)]o(\tau)}{\tau}\,d\bar{\alpha}=0\,.\end{aligned} \tag{5}$$

The first term vanishes since $\int_0^{\pi}r^2 d\alpha=\text{Area}\,(B\cap\varepsilon)=\pi$. The two last terms vanish because of $\lim_{\tau\to 0}\{o(\tau)/\tau\}=0$.

We show that $\lim_{\tau\to 0}(q^2/\tau)=0$. From the triangle $PQ\bar{P}_{\Delta}$ we have $q:rtg\nu=\sin\psi:\sin(\pi/2-\psi-\nu)$. Thus, $q=rtg\nu\,\sin\psi/\cos(\psi+\nu)$, and, because of $|\tau|\geqq|\nu|$

[1] $\lim_{\tau\to 0}\frac{\theta(\tau)}{\tau}=0\,.$

$$\left|\frac{q^2}{\tau}\right| \leqq \left|\frac{q^2}{\nu}\right| = r^2 \left|\frac{tg\nu}{\nu}\right| \left|tg\nu\right| \frac{\sin^2\psi}{\cos^2(\psi+\nu)} \to 0 .$$

if $\tau \to 0$ and, thus $\nu \to 0$.

Now, we transform the only remaining term $\int_0^\pi [\lim_{\tau\to 0}(rq/\tau)]d\alpha$ of (5). An elementary computation shows that $q = v + o(\tau)$, where $v = \overline{VP}$. (The sign of v equals that of q.) So (5) gives

$$(6) \qquad \int_0^\pi \left[\lim_{\tau\to 0} \frac{rv}{\tau}\right] d\alpha = 0 .$$

Since Q is a function of α, β, τ and P a function of α, v is a function of α, β, τ. If $a \perp 0P$, then $\beta = \alpha \pm (\pi/2)$ $0 \leqq \beta < \pi$., and then $\nu = \tau$. We put $v(\alpha, \alpha \pm \pi/2, T) \equiv v_0^*(\alpha)$. T being a fixed positive value of τ which satisfies the condition $T' < \eta$. The subscript "0" denotes the perpendicularity of a and $0P$, and the asterisk denotes $\tau = T$. Thus, $v^*: v_0^* = \overline{P\bar{P}}^*: \overline{P\bar{P}}_0^* = \hat{r}\, tg\, T: r\, tg\, T$ and $v^* = v_0^*(\hat{r}/r)$. Furthermore, $v^*: v = \hat{r}\, tgT: \hat{r}\, tg\tau$. Hence $v^* = v(tg\, T/tg\, \tau)$. Comparing this to the previous expression of v^* and taking in account the relation $\hat{r} = r\,|\sin(\alpha - \beta)|$, we have

$$v = v_0^* \frac{tg\,\tau}{tg\,T} |\sin(\alpha - \beta)| .$$

Substituting this into (6), and taking in account that r and v_0^* are functions of the α alone, we get

$$\int_0^\pi r(\alpha) v_0^*(\alpha) \left[\lim_{\tau\to 0} \frac{tg\,\tau}{\tau}\right] \frac{1}{tg\,T} |\sin(\alpha - \beta)|\, d\alpha = 0 .$$

Performing the limit process and reducing by the constant $1/tg\, T$, we obtain

$$(7) \qquad \int_0^\pi r(\alpha) v_0^*(\alpha) |\sin(\alpha - \beta)|\, d\alpha = 0 .$$

B must be shaped so that this equation is fulfilled for any β between 0 and π.

We show that (7) can hold only for $v_0^*(\alpha) \equiv 0$. Put $r(\alpha)v_0^*(\alpha) \equiv \varphi(\alpha)$. Then (7) has the form

$$(8) \qquad \int_0^\pi \varphi(\alpha) |\sin(\alpha - \beta)|\, d\alpha = 0 \qquad \text{for all } \beta \in (0, \pi) .$$

Let us suppose for the moment that $\varphi(\alpha)$ has a finite number of sign changes between 0 and π and is not identically zero. Let these be $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then

$$(9) \qquad \int_0^\pi \varphi(\alpha) \sum_{i=1}^{r} [a_i \,|\sin(\alpha - \alpha_i)| + b_i \,|\sin(\alpha - \beta_i)|]d\alpha = 0$$

for any a_i, b_i, β_i, since (9) is a linear combination of (8) for some special β which satisfy $\alpha_j < \beta_j < \alpha_{j+1} (j = 1, 2, \cdots, r-1)$; $\beta_r = \pi$. We consider the functions

$$\begin{aligned}
f_1(\alpha) &= -a_1 \sin(\alpha - \alpha_1) - b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
\tilde{f}_1(\alpha) &= +a_1 \sin(\alpha - \alpha_1) - b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
f_2(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) - a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
\tilde{f}_2(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) + a_2 \sin(\alpha - \alpha_2) \\
&\quad - b_2 \sin(\alpha - \beta_2) - \cdots - b_r \sin(\alpha - \beta_r) \\
&\vdots \\
\tilde{f}_r(\alpha) &= +a_1 \sin(\alpha - \alpha_1) + b_1 \sin(\alpha - \beta_1) + a_2 \sin(\alpha - \alpha_2) \\
&\quad + b_2 \sin(\alpha - \beta_2) + \cdots + b_r \sin(\alpha - \beta_r) .
\end{aligned}$$

We denote by $\Phi(\alpha)$ the sum under the integral sign in (9). Then

$$\begin{aligned}
f_i(\alpha) &\equiv \Phi(\alpha) \qquad \beta_{i-1} \leqq \alpha \leqq \alpha_i \qquad (\beta_0 = 0) \\
\tilde{f}_i(\alpha) &\equiv \Phi(\alpha) \qquad \alpha_i \leqq \alpha \leqq \beta_i \qquad\qquad (i = 1, 2, \cdots, r) .
\end{aligned}$$

and

$$(10) \qquad f_i(\alpha_i) = \tilde{f}_i(\alpha_i)$$

as well as

$$\tilde{f}_j(\beta_j) = f_{j+1}(\beta_j) \qquad (j = 1, 2, \cdots, r-1) .$$

Each of the $f_i(\alpha)$ and of the $\tilde{f}_i(\alpha)$ has the form $A \sin\alpha + B\cos\alpha$. Thus each of them has only one zero between 0 and π defined by $tg\,\alpha = B/A$, and $f_i(\alpha)$ esp. $\tilde{f}_i(\alpha)$ changes sign at this point.

Making use of the $3n - 1$ parameters a_i, b_i, β_j we can achieve

$$(11) \qquad f_i(\alpha_i) = 0 .$$

and

$$(12) \qquad \operatorname{sgn} \frac{df_i}{d\alpha}(\alpha_i) = \operatorname{sgn} \frac{d\tilde{f}_i}{d\alpha}(\alpha_i) .$$

Then $\tilde{f}_i(\alpha_i) = 0$ is a consequence of (10). Thus the $\alpha_i (i = 1, 2, \cdots, r)$

are all the zeros and sign changes of $\Phi(\alpha)$ between 0 and π. $\varphi(\alpha)$ is continuous, since both $r(\alpha)$ and $v_0^*(\alpha)$ are so. Because of (12) and the continuity of $f(\alpha)$, $\Phi(\alpha)$ and $\varphi(\alpha)$ have the same sign over $(0, \pi)$, and they do not vanish identically. This implies a contradiction to (9). Therefore, $\varphi(\alpha)$ cannot have a finite number of sign changes. Hence, the sign changes of $\varphi(\alpha)$ have some accumulation points. Let us suppose that these accumulation points are not everywhere dense. Since $\varphi(\alpha)$ is continuous, cutting out the accumulation points (each with a suitable small neighborhood) changes the integral (8) by less than an arbitrary small number. Performing the previous process on the remaining domain of α we arrive again at a contradiction. Hence, the sing changes of $\varphi(\alpha)$ must be everywhere dense. This is impossible unless $\varphi(\alpha)$ vanishes identically:

$$\varphi(\alpha) = r(\alpha)v_0^*(\alpha) \equiv 0 \,. \tag{13}$$

If $r(\alpha)$ were zero for any α, then B would reduce to the point 0 by the starshapeness and the symmetry of B, and we would have Area $(B \cap \varepsilon) = 0 \neq \pi$. Thus we get $r(\alpha) \neq 0$, and from (13) $v_0^*(\alpha) \equiv 0$. This means that ε is perpendicular to the tangent planes to ∂B at the points of $\partial B \cap \varepsilon$.

Until now we used $A(\vec{n})$ for planes whose angle with ε is smaller than η. Then we have used $A(\vec{n})$ only for those $\vec{n}$ for which $(\vec{n}, z) < \eta$, z being the coordinate axis perpendicular to ε. The endpoints of these vectors $\vec{n}$ fill a small circle C on the unit sphere around "north pole" (resp. around its south pole).

Now we want to show that $\Sigma_p \perp 0P$. Instead of ε we choose a plane ε' through $0P$ whose angle with ε is $\omega (0 < \omega < T)$. Let us perform the foregoing consideration for this ε' replacing T by a T' such that $0 < T' < T - \omega$. These considerations lead to $\Sigma_p \perp \varepsilon'$. But from $\Sigma_p \perp \varepsilon, \varepsilon'$ it follows that Σ_p is perpendicular to the intersection of ε and ε', i.e., $\Sigma_p \perp 0P$. In this part of the proof we have used $A(\vec{n})$ again just for the $\vec{n}$ whose endpoints fill a circle C' on the unit sphere with a center inside C (because of $\omega < T$), and with a radius such that $C' \subset C$ (because of $T' < T - \omega$). Hence, we were able to proove $\Sigma_p \perp 0P$; $P \in \partial B \cap \varepsilon$, using $A(\vec{n})$ merely for $\vec{n}$ belonging to C.

Let us consider a pencil of planes through the x axis. Let us perform our whole proof for all the planes of the pencil. This leads us to the result that $\Sigma_p \perp 0P$ for each $P \in \partial B$. But a continuous surface with this property cannot be other than a sphere whose radius must be 1 because of $A(\vec{n}_0) = \pi$, $\vec{n}_0 \perp \varepsilon$. During this last step of the proof we used $A(\vec{n})$ for all planes of the pencil in a small circular neighborhood of the normal of the plane of the pencil on the unit sphere. The union of these circles forms a small strip around the great circle

in the (y, z) plane of the unit sphere, as stated in the theorem.

We show an immediate application of the theorem. In a Minkowskian geometry M_n with indicatrix S, the surface area $|F|$ of a plane figure F is defined as the euclidean surface area $\lambda(F)$ of this figure F, multiplied by an appropriate factor. This factor is π divided by the euclidean surface-area of the intersection of the interior B of S and of the plane σ through the midpoint 0 of S and parallel to F:

$$|F| = \frac{\pi}{\lambda(B \cap \sigma)} \lambda(F)^2 . \tag{14}$$

The first factor on the right hand side of (14) is a function of the plane position only: $\pi/m(\sigma)$. H. Busemann [2] investigated affine measures where this factor is not deduced from a convex symmetric body B but it is rather arbitrarily given. From the theorem of this paper follows again an answer to the question treated by H. Busemann, whether or not an arbitrary continuous $m(\sigma)$ can always be derived from a symmetric star-shaped body B as surface area of its intersections with planes through its midpoint.[3] Namely, taking a differentiable function $m(\sigma)$ over the positions of planes in 3-space, (a) having value π on the planes, whose normals make an angle with the y, z plane smaller than η, and (b) having values differing from π elsewhere, we get a function which cannot be derived from any symmetrical body in the above sense (i.e., so that $m(\sigma) \equiv$ Area $(B \cap \sigma)$). Namely the body B ought to be a sphere because of (a), according to our theorem; on the other hand it cannot be a sphere, since no sphere satisfies (a) and (b) simultaneously. Thus this $m(\sigma)$ presents a simple and concrete negative example. Negative examples for higher dimensional spaces are any functions over the position of planes in n-space that are differentiable extensions of the previous function.

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POLYTECHNIC INSTITUTE OF BROOKLYN, NEW YORK
AND
UNIVERSITY OF DEBRECEN, HUNGARY

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SOME ASPECTS OF GOLDIE'S TORSION THEORY

MARK L. TEPLY

Goldie's torsion class $\mathscr{G}$ is a class of left R-modules closed under taking submodules, factor modules, extensions, arbitrary direct sums, and injective envelopes. The corresponding Goldie torsionfree class $\mathscr{F}$ is precisely the class of left R-modules possessing zero singular submodule. It is shown that $\mathscr{G}$ is closed under taking direct products if and only if nonzero left ideals in $\mathscr{F}$ have nonzero socles. Another theorem gives four conditions equivalent to the following: Any direct sum of torsionfree injective modules is injective. One of these four conditions is that the ring R is an essential extension of a finite direct sum $\mathscr{G}(R) \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where each L_i is a uniform left ideal of R. It is natural to ask when R actually equals this direct sum. A sufficient condition for this to happen is given. Rings in which every torsionfree principal left ideal is projective are studied. Particular attention is paid to those rings whose Goldie torsion filters possess a cofinal subset of finitely generated left ideals.

In this paper all rings R are associative with unit, and all modules are unitary left R-modules.

Before proceeding, we review some results from [2]. The Goldie torsion class is the smallest class of modules which is closed under taking factor modules, extensions, and arbitrary direct sums, and which contains all factor modules B/A, where A is an essential submodule of B. Then the class $\mathscr{F} = \{F \in {}_R\mathscr{M} \mid \mathrm{Hom}_R(G, F) = 0 \text{ for all } G \in \mathscr{G}\}$ is a torsionfree class in the sense of [5]. Moreover, $(\mathscr{G}, \mathscr{F})$ is a torsion theory in the sense of [5], $\mathscr{G} \cap \mathscr{F} = 0$, and every $M \in {}_R\mathscr{M}$ has a (necessarily unique) maximal torsion submodule $\mathscr{G}(M)$ such that $M/\mathscr{G}(M) \in \mathscr{F}$. $\mathscr{F}$ is precisely the class of R-modules which have zero singular submodule; moreover, $\mathscr{F}$ is closed under taking submodules, direct products, extensions, and injective envelopes. In particular, a left ideal I of R is in $\mathscr{F}$ if and only if I has zero singular submodule when considered as a left R-module. In case R is an integral domain, then $\mathscr{G}$ coincides with class of modules which are torsion in the usual sense.

Associated with $\mathscr{G}$ there is a filter of left ideals $F(\mathscr{G}) = \{L \mid R/L \in \mathscr{G}\}$. In [1] J. S. Alin shows $L \in F(\mathscr{G})$ if and only if there exists L' essential in R such that $L \subseteq L'$ and $(L: x)$ is essential in R for all $x \in L'$. In particular, every essential left ideal of R is in $F(\mathscr{G})$. As in [14], $F(\mathscr{G})$ is said to have a cofinal subset of finitely generated left ideals if, given $L \in F(\mathscr{G})$, there exists $I \subseteq L$ such that I is finitely

generated and $I \in F(\mathscr{G})$.

When we say a module G is "torsion," then we mean $G \in \mathscr{G}$; when we say a module F is torsionfree, we mean $F \in \mathscr{F}$.

An R-module M will be called uniform if, for any pair L, N of nonzero submodules of M, $L \cap N \neq 0$. A left ideal of R is called uniform if it is uniform as a left R-module.

1. Products of Goldie torsion modules. In [9] J. P. Jans investigated classes of modules which are closed under submodules, homomorphic images, extensions, and direct products. Such classes are called torsion-torsionfree classes (TTF classes). It is clear that the Goldie torsion class will be a TTF class if and only if $\mathscr{G}$ is closed under direct products. R. S. Pierce has pointed out ([9], Th. 2.1) that $\mathscr{G}$ is closed under products if and only if $I = \bigcap_{L \in F(\mathscr{G})} L \in F(\mathscr{G})$. In that case I is a two-sided idempotent ideal of R. In studying the simple torsion class $\mathscr{S}$ of S. E. Dickson [5], J. S. Alin has shown [1] that if $\mathscr{S}$ is closed under direct products, then $\mathscr{G} \cap \mathscr{S}$ is closed under direct products. If nonzero modules have nonzero socles, then $\mathscr{S} = {}_R\mathscr{M}$, and hence Alin's results shows $\mathscr{G}$ is closed under direct products. This motivates the main result of this section (Th. 1.3).

Since the ideal $I = \bigcap_{L \in F(\mathscr{G})} L$ plays a key role in examining Goldie torsion classes closed under direct products, we begin this section by examining I.

PROPOSITION 1.1. *Let* $I = \bigcap_{L \in F(\mathscr{G})} L$. *Then* $I \in \mathscr{F}$.

Proof. By Zorn's lemma, there is a left ideal J maximal with respect to $J \cap \mathscr{G}(I) = 0$. Then $J + \mathscr{G}(I)$ is essential in R. Since $\mathscr{G}$ is closed under extensions, then the exact sequence

$$0 \to \frac{J + \mathscr{G}(I)}{J} \to \frac{R}{J} \to \frac{R}{J + \mathscr{G}(I)} \to 0$$

yields $R/J \in \mathscr{G}$, i.e., $J \in F(\mathscr{G})$. Thus by the definition of I, $J \supseteq I \supseteq \mathscr{G}(I)$, and hence $\mathscr{G}(I) = 0$.

LEMMA 1.2. *Let* $I = \bigcap_{L \in F(\mathscr{G})} L$, *and suppose* $\mathscr{G}$ *is closed under direct products. Then*:

(1) *If* J *is a left ideal of* R *and* $J \subseteq I$, *then* $I = J \oplus K$ *for some left ideal* K.

(2) *If* $I \neq 0$, *then* $I = \bigoplus \sum_{\alpha \in \mathfrak{A}} S_\alpha$ *where* S_α *is simple and* $\mathfrak{A}$ *is an index set.*

Proof. Let $J \subseteq I$ be a left ideal of R. There exists a left ideal

K of R maximal with respect to the properties $K \cap J = 0$ and $K \subseteq I$. Then $K \oplus J$ is essential in I and hence $I/K \oplus J \in \mathscr{G}$. By [9] Th. 2.1, $I \in F(\mathscr{G})$; so the exact sequence

$$0 \to I/J \oplus K \to R/J \oplus K \to R/I \to 0$$

yields $R/J \oplus K \in \mathscr{G}$, i.e., $J \oplus K \in F(\mathscr{G})$. It follows from the definition of I that $J \oplus K \supseteq I$, and hence $I = J \oplus K$.

(2) follows from (1) and [4], Theorem 15.3.

THEOREM 1.3. *The following are equivalent:*

(1) $\mathscr{G}$ *is closed under direct products.*

(2) *Nonzero modules in* $\mathscr{F}$ *have nonzero socles.*

(3) *Nonzero left ideals in* $\mathscr{F}$ *have nonzero socles.*

Proof. (1) $\Rightarrow$ (2): Suppose $F \in \mathscr{F}$ and $0 \neq x \in F$. By [9] Theorem 2.1 and (1), $I \in F(\mathscr{G})$. Claim $Ix \neq 0$; for otherwise $I \in F(\mathscr{G})$ implies $Rx \in \mathscr{G}$, and $Rx \subseteq F$ implies $Rx \in \mathscr{G}$. Thus $Rx \in \mathscr{G} \cap \mathscr{F} = 0$, a contradiction. By Lemma 1.2, it then follows that $S_\alpha x \neq 0$ for some simple left ideal S_α. Then $S_\alpha x$ is a nonzero homomorphic image of S_α, and hence is a simple submodule of F. Thus F has nonzero socle.

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): If $R \in \mathscr{G}$, then $\mathscr{G} = {}_R\mathscr{M}$, and so the result is trivially true. Suppose $R \notin \mathscr{G}$. Since $\mathscr{G}$ is closed under essential extensions, there exists $K \subseteq R$ such that $K \in \mathscr{F}$. By (3) there exists $S \subseteq K$ with S simple.

Claim that $L \in F(\mathscr{G})$ implies $S \subseteq L$. For if $L \cap S = 0$, then $S \cong S + L/L \subseteq R/L \in \mathscr{G}$; and hence $S \in \mathscr{G}$, contradicting $S \in \mathscr{F}$. But then $L \cap S \neq 0$ implies $S \subseteq L$. Since $L \in F(\mathscr{G})$ was arbitrarily chosen, then $S \subseteq \bigcap_{L \in F(\mathscr{G})} L = I$. Therefore $K \cap I \neq 0$.

Let J be a left ideal of R maximal with respect to $J \cap I = 0$. By the previous paragraph J contains no left ideals in $\mathscr{F}$; hence $\mathscr{G}(J)$ is essential in J. Therefore $J \in \mathscr{G}$. So the exact sequence

$$0 \to \frac{J+I}{I} \to \frac{R}{I} \to \frac{R}{I+J} \to 0$$

yields $R/I \in \mathscr{G}$, i.e., $I \in F(\mathscr{G})$. Hence $\mathscr{G}$ is closed under direct products by [9], Theorem 2.1.

In [1] J. S. Alin points out that every simple module in $\mathscr{F}$ is projective. Hence if R has no projective simples and $\mathscr{G}$ is closed under directs products, Proposition 1.1 and Lemma 1.2 imply $I = 0$. Thus we obtain:

PROPOSITION 1.4. *Suppose R has no projective simples. Then $\mathscr{C}$ is closed under direct products if and only if $R \in \mathscr{C}$.*

It is known [2] that the following are equivalent: (1) $R = \mathscr{C}(R) + S$ (ring direct sum) where S is semisimple with minimum condition; (2) $\mathscr{F}$ is closed under homomorphic images; and (3) the Goldie global dimension of R (see [2]) is zero. Thus it is of interest to examine these rings in relation to the condition: $\mathscr{C}$ is closed under direct products.

PROPOSITION 1.5. *$R = \mathscr{C}(R) + S$ (ring direct sum) where S is semisimple with minimum condition if and only if the following conditions are satisfied:*

(1) *$\mathscr{C}$ is closed under products.*

(2) *$I = \bigcap_{L \in F(\mathscr{C})} L$ is finitely generated as a left ideal.*

(3) *There are no nonzero nilpotent left ideals in $\mathscr{F}$.*

Proof. ($\Leftarrow$): If R has no projective simple modules, then we are done by Proposition 1.4. Let $R = S_1 \oplus M_1$ with S_1 simple. If $I = \bigcap_{L \in F(\mathscr{C})} L$, then $S_1 \subseteq I$ follows from (1). Hence $I = S_1 \oplus (M_1 \cap I)$. If $M_1 \cap I$ contains a simple summand S_2 of R, then it follows that $R = S_1 \oplus S_2 \oplus (M_1 \cap M_2)$ where $R = S_2 \oplus M_2$. Proceeding by induction $R = S_1 \oplus S_2 \oplus \cdots \oplus S_n \oplus (\bigcap_{i=1}^{n} M_i)$. Now this induction process must stop after a finite number of steps by (2), say

$$R = S_1 \oplus \cdots \oplus S_N \oplus \left(\bigcap_{i=1}^{N} M_i\right)$$

where $I \cap (\bigcap_{i=1}^{N} M_i)$ contains no simple summands of R. Set $\bigcap_{i=1}^{N} M_i = G$. We claim that $I \cap G = 0$. For otherwise (1) and Theorem 1.3 imply there exists a simple module $S \subseteq I \cap G$. $S^2 \neq 0$ by (3) and Proposition 1.1. Let $x, y \in S$ such that $xy \neq 0$. Then y generates S, and $(0: y) \cap S = 0$ since S is simple. But $(0: y)$ is maximal, and hence $(0: y) \oplus S = R$, contradicting $I \cap G$ contains no simple summands of R. Hence $I \cap G = 0$ as claimed, and so $I = S_1 \oplus \cdots \oplus S_N$.

Now observe that $G = \mathscr{C}(R)$ as follows: Clearly $\mathscr{C}(R)$ cannot properly contain G. On the other hand, $G = I + G/I \subseteq R/I \in \mathscr{C}$ by (1), and hence $G \in \mathscr{C}$. Therefore, $G = \mathscr{C}(R)$.

Since $R = G \oplus I$ and since G and I are two-sided ideals, then $R = G + I$ (ring direct sum.)

($\Rightarrow$): If $R = \mathscr{C}(R) + S$, where S is semisimple with minimum conditions, then [2], Theorem 3.1. and the remark following [2], Corollary 3.4, show $\mathscr{C}$ is closed under direct products.

2. **Direct sums of torsionfree injectives.** The main theorem

of this section points out the relationship between cofinal subsets of finitely generated left ideals in $F(\mathscr{G})$ and properties of sums of torsionfree injective modules. This relationship is also studied in [13]. An example in [13] shows that, for more general torsion theories [5] than the Goldie theory, the analogue of Theorem 2.1 (1) $\Rightarrow$ (5) is not always true. In [7], Theorem 5.6, a ring with $\mathscr{G}(R) = 0$ and the ascending chain condition is represented as an essential extension of a finite direct sum. Condition (3) of Theorem 2.1 represents more general rings in this way.

THEOREM 2.1. *The following are equivalent:*

(1) *$F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals.*

(2) *R contains no infinite direct sum of torsionfree left ideals.*

(3) *R is an essential extension of $\mathscr{G}(R) \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n$, where each L_i is a uniform left ideal.*

(4) *$R/\mathscr{G}(R) \cong K$ where K is an essential extension of a direct sum of finitely many uniform torsionfree left ideals.*

(5) *Any direct sum of torsionfree injective modules is injective.*

REMARK. [13], Theorem 1.2, gives eight additional conditions which are equivalent to (5) (above) for more general hereditary torsion theories in the sense of [1], [9], and [13].

Proof. (1) $\Rightarrow$ (2): Let $\oplus \sum_{\alpha \in w} L_\alpha$ be a direct sum of nonzero torsionfree left ideals of R. Then there exists a left ideal L of R such that $L \cap (\sum_{\alpha \in w} L_\alpha) = 0$ and $V = L \oplus (\sum_{\alpha \in w} L_\alpha)$ is essential in R. Then $V \in F(\mathscr{G})$, so (1) implies that there exists $N \subseteqq V$ with $N \in F(\mathscr{G})$ and N finitely generated. Now each generator N has a nonzero representation in only finitely many coordinants of $V = L \oplus (\oplus \sum_{\alpha \in w} L_\alpha)$. Since N is finitely generated, it follows that there exists $u \subseteqq w$ such that $|u| < \aleph_0$ and $N \subseteqq L \oplus \sum_{\alpha \in u} L_\alpha$. Suppose $|u| < |w|$. Then

$$\oplus \sum_{\alpha \in w-u} L_\alpha \cong \frac{V}{L \oplus (\oplus \sum_{\alpha \in u} L_\alpha)} \subseteqq \frac{R}{L \oplus (\oplus \sum_{\alpha \in u} L_\alpha)} \in \mathscr{G}$$

since $N \subseteqq L \oplus (\oplus \sum_{\alpha \in u} L_\alpha)$ implies $L \oplus (\oplus \sum_{\alpha \in u} L_\alpha) \in F(\mathscr{G})$. Hence $\oplus \sum_{\alpha \in w-u} L_\alpha \in \mathscr{G} \cap \mathscr{F} = 0$, contradicting $L_\alpha \neq 0$. Hence $|w| = |u| < \aleph_0$, establishing (2).

(2) $\Rightarrow$ (5): Consider the diagram:

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{i} & R \\ & & \phi \downarrow & & \\ & & \oplus \sum_{\alpha \in w} F_\alpha & & \end{array} \qquad \text{(c)}$$

where I is a left ideal of R and $\{F_\alpha\}_{\alpha \in w}$ is a set of torsionfree injective R-modules. Note that $\phi(\mathscr{G}(I))$ 0. If I is an essential extension of $\mathscr{G}(I)$, then $I = \mathscr{G}(I)$ since $\mathscr{G}$ is closed under essential extensions; but then the zero map from R to $\bigoplus \sum_{\alpha \in w} F_\alpha$ makes (c) commute.

If I is not an essential extension of $\mathscr{G}(I)$, then by (2) there exists a finite direct sum $\bigoplus \sum_{i=1}^{n} Rx_i$ contained in I maximal with respect to $\mathscr{G}(I) \cap (\bigoplus \sum_{i=1}^{n} Rx_i) = 0$. Then $U = \mathscr{G}(I) \bigoplus (\bigoplus \sum_{i=1}^{n} Rx_i)$ has the properties: $I/U \in \mathscr{G}$ and $\phi(U)$ is contained in a direct sum of finitely many F'_αs ($\alpha \in w$). So by the injectivity of the F'_αs ($\alpha \in w$), there exists $f: R \to \bigoplus \sum_{\alpha \in w} F_\alpha$ such that $f \mid U = \phi \mid U$. Since $(f - \phi)(U) = 0$, there is an induced map $g: I/U \to \bigoplus \sum_{\alpha \in w} F_\alpha$ via $g(x + U) = (f - \phi)(x)$. Since $I/U \in \mathscr{G}$ and $F_\alpha \in \mathscr{F}\ \forall\ \alpha$, then $g = 0$, and hence $f \mid I = \phi$. Hence f makes (c) commute, and thus $\bigoplus \sum_{\alpha \in w} F_\alpha$ is injective.

(5) $\Rightarrow$ (1): [13], Theorem 1.5.

(2) $\Rightarrow$ (3): By (2) let $\bigoplus \sum_{i=1}^{n} L_i$ be a maximal direct sum of *torsionfree left ideals of* R. *From (2) it follows that each torsionfree* left ideal contains a uniform left ideal. So by breaking the L_i apart into direct subsums, we may assume each L_i is uniform. By the maximal property of $\bigoplus \sum_{i=1}^{n} L_i$, it follows that $\mathscr{G}(R) \bigoplus (\bigoplus \sum_{i=1}^{n} L_i)$ is essential in R.

(3) $\Rightarrow$ (4): If $M \not\subseteq \mathscr{G}(R)$, we claim $M \cap \sum_{i=1}^{n} L_i \neq 0$. For suppose $M \not\subseteq \mathscr{G}(R)$ and $M \cap (\sum_{i=1}^{n} L_i) = 0$. Then let

$$0 \neq x = g + r_1 + r_2 + \cdots + r_n \in \left(\mathscr{G}(R) \bigoplus \left(\sum_{i=1}^{n} L_i\right)\right) \cap M$$

by (3). Then $g \neq 0$. If $(0: g)x \neq 0$, then $M \cap \sum_{i=1}^{n} L_i \neq 0$, a contradiction to our assumption. Hence $(0: g)\, x = 0$, and so $x \in \mathscr{G}(M)$. Thus $M \cap (\sum_{i=1}^{n} L_i \bigoplus \mathscr{G}(R)) \in \mathscr{G}$. From (3), $M \cap (\sum_{i=1}^{n} L_i \bigoplus \mathscr{G}(R))$ is essential in M, and hence $M \in \mathscr{G}$ by $\mathscr{G}$ closed under essential extensions. Thus $M \subseteq \mathscr{G}(R)$, which is a contradiction to our choice of M.

From the claim, it follows that $R/\mathscr{G}(R) \cong K$ is an essential extension of $(\bigoplus \sum_{i=1}^{n} L_i) \bigoplus \mathscr{G}(R)/\mathscr{G}(R) \cong \bigoplus \sum_{i=1}^{n} L_i$ with L_i uniform.

(4) $\Rightarrow$ (1): Assume $R \notin \mathscr{G}$. Let K be an essential extension of $\bigoplus \sum_{i=1}^{n} L_i$, L_i uniform left ideals; $K = R/\mathscr{G}(R)$ by (4). If $I \in F(\mathscr{G})$, then $I \cap L_i \neq 0$ since $L_i \in \mathscr{F}$. Choose $0 \neq x_i \in I \cap L_i$ for $i = 1, 2, \cdots, n$. Since L_i is uniform, Rx_i is essential in L_i. Hence

$$\frac{\bigoplus \sum_{i=1}^{n} L_i}{\bigoplus \sum_{i=1}^{n} Rx_i} \cong \bigoplus \sum_{i=1}^{n} L_i/Rx_i \in \mathscr{G}\ .$$

So the exact sequence

$$0 \to \frac{\mathscr{G}(R) \oplus \sum_{i=1}^{n} L_i}{\oplus \sum_{i=1}^{n} Rx_i} \to \frac{R}{\oplus \sum_{i=1}^{n} Rx_i} \to \frac{R}{\mathscr{G}(R) \oplus \sum_{i=1}^{n} L_i} \to 0$$

yields $R/\sum_{i=1}^{n} Rx_i \in \mathscr{G}$, i.e., $\sum_{i=1}^{n} Rx_i \in F(\mathscr{G})$.

The following result of F. L. Sandomierski ([12], Th. 2.5) is an immediate corollary of Theorem 2.1 (2) $\Rightarrow$ (5):

COROLLARY 2.2. *If R has no infinite direct sums of left ideals, then any direct sum of injective R-modules with zero singular submodule is injective.*

EXAMPLE. To see that R can have an infinite direct sum of left ideals without having an infinite direct sum of torsionfree left ideals, consider the ring $R = [\prod_{\alpha \in w} P^{(\alpha)}] + N$ (ring direct sum) where $|w| \geqq \aleph_0$, $P^{(\alpha)} = Z/(p^2)$ (Z = integers, p = prime), and N is a commutative Noetherian ring with zero singular ideal. Then $\mathscr{G}(R) = \prod_{\alpha \in w} P^{(\alpha)}$, and R clearly has infinite direct sums of left ideals. But since R is commutative and $N \in \mathscr{T}$ is Noetherian, R contains no infinite direct sums of torsionfree left ideals. Thus Theorem 2.1 gives a proper generalization of Sandomierki's result.

If R has zero singular ideal, then $F(\mathscr{G}) = \{I \mid I$ is an essential left ideal of $R\}$. Hence the following result of C. Walker and E. A. Walker ([14], Th. 4.20 (b) $\Leftrightarrow$ (c)) is an immediate corollary of Theorem 2.1 (1) $\Leftrightarrow$ (2).

COROLLARY 2.3. *Let R have zero singular ideal, and let $F(\mathscr{G})$ be the filter of all essential left ideals. Then the following are equivalent*:

(1) *$F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals.*

(2) *R has no infinite direct sums of left ideals.*

3. ($\mathscr{T}PP$) rings. From Theorem 2.1 we see that the condition, $F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals, is equivalent to R being an essential extension of the finite direct sum

$$\mathscr{G}(R) \oplus L_1 \oplus L_2 \oplus \cdots \oplus L_n ,$$

where each L_i is a uniform left ideal of R. It is a natural question to ask when R is actually equal to this direct sum. The condition that torsionfree principal left ideals are projective plays an interesting role as a sufficient condition for equality. The main purpose of this section is to examine this role and hence to obtain generalizations of some results of L. Levy [11] and A. Hattori [8].

A ring R is called $(\mathscr{T}PP)$ if every torsionfree principal left ideal is projective. This is a generalization of Hattori's concept of (PP) ring [8]. There a ring is called (PP) if every principal left ideal is projective. It is easily seen that a ring R is (PP) if and only if R is $(\mathscr{T}PP)$ and $\mathscr{G}(R) = 0$.

THEOREM 3.2. *Let R be an $(\mathscr{T}PP)$ ring. Then the following are equivalent:*

(1) *$F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals.*

(2) *$R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n \oplus A$, where each Re_i is a uniform left ideal and where A is an essential extension of*

$$\mathscr{G}(R) \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_m .$$

$N_i^2 = 0$ for $i = 1, 2, \cdots, m$, and each N_i is R-isomorphic with some $Re_{j(i)}$ where $j(i) \in \{1, 2, \cdots, n\}$. Moreover, $1 = e_1 + e_2 + \cdots + e_n + a$, $a \in A$, and $aN_i = 0$ for $i = 1, 2, \cdots, m$.

Proof. (1) $\Rightarrow$ (2): If $R = \mathscr{G}(R)$, then there is nothing to prove. If $R \neq \mathscr{G}(R)$, then there exists a left ideal I such that $I \cap \mathscr{G}(R) = 0$ since $\mathscr{G}$ is closed under essential extensions. By (1) and Theorem 2.1 (2) we may assume $I = Rx$ is a uniform left ideal in $\mathscr{T}$. Now the exact sequence

$$0 \longrightarrow (0: x) \longrightarrow R \longrightarrow Rx \longrightarrow 0$$

must split since R is $(\mathscr{T}PP)$. Hence $R = D_1 \oplus A_1$ where $D_1 \cong Rx$. Write $1 = e_1 + a_1$ where $e_1 \in D$, $a_1 \in A_1$. Then $D_1 = Re_1$. Since $\mathscr{G}$ is closed under homomorphic images and $D_1 \in \mathscr{T}$, then $\mathscr{G}(R) \subseteq A_1$.

We proceed by induction to define $Re_2, Re_3, \cdots, Re_n$ as follows: Suppose $Re_1, \cdots, Re_u$ has been constructed such that

$$R = Re_1 \oplus \cdots \oplus Re_u \oplus A_u ,$$

$\mathscr{G}(R) \subseteq A_u$, each Re_i is a uniform left ideal, and

$$1 = e_1 + e_2 + \cdots + e_u + a_u , \qquad a_u \in A_u .$$

If $A_u = \mathscr{G}(R)$, we are done. If $A_u \neq \mathscr{G}(R)$, then let $Rx \neq 0$ be a torsionfree uniform left ideal contained in A_u. If $Re_i x = 0$ for all $i = 1, 2, \cdots, u$, then $A_u x = Rx \in \mathscr{T}$. So the exact sequence

$$0 \longrightarrow (0: x) \cap A_u \longrightarrow A_u \longrightarrow A_u x \longrightarrow 0$$

splits since $A_u x$ is projective. Hence $A_u = D_{u+1} \oplus A_{u+1}$ where $D_{u+1} \cong A_u x$. Write $a_u = e_{u+1} + a_{u+1}$ with $e_{u+1} \in D_{u+1}$, $a_{u+1} \in A_{u+1}$. Then $R = Re_1 \oplus \cdots \oplus Re_u \oplus Re_{u+1} \oplus A_{u+1}$, and $\mathscr{G}(R) \subseteq A_{u+1}$.

By (1) and Theorem 2.1 (2), it follows that the above process

must stop after finitely many steps (i.e., eventually we cannot assume $Re_i x = 0$ for all i as above). Hence we may assume that we have constructed uniform torsionfree left ideals $Re_1, Re_2, \cdots, Re_n$ with the properties:

(i) $R = Re_1 \oplus \cdots \oplus Re_n \oplus A$.

(ii) $1 = e_1 + e_2 + \cdots + e_n + a$, $a \in A$.

(iii) If $Rx \subseteq A$ is a nonzero torsionfree uniform left ideal, then $Re_i x \neq 0$ for some $i \in \{1, 2, \cdots, n\}$.

(iv) $\mathscr{G}(R) \subseteq A$.

Suppose $0 \neq Rx \in \mathscr{F}$, $Rx \subseteq A$, and Rx uniform. It is easily seen that $e_i e_j = 0$ for $i \neq j$ and $e_i a = 0 \; \forall \; i = 1, 2, \cdots, n$. So for $y \in Rx$, $Re_i ay = 0$. Since $Ray \subseteq Rx$, it follows from (iii) above that $ay = 0$. Let q be the least integer such that $Re_q x \neq 0$, which exists by (iii). It is easily verified that if $v \in \bigoplus \sum_{i \neq q} Re_i \oplus A$, then $ve_q x = 0$. It follows that if $(0 : e_q x) \cap Re_q \neq 0$, then $(0 : e_q x)$ is essential in R, and hence $(0 : e_q x) \in F(\mathscr{G})$. But then $Re_q x \in \mathscr{G} \cap \mathscr{F} = 0$, a contradiction. Therefore, $(0 : e_q x) \cap Re_q = 0$.

Note that the conditions

(a) $(0 : e_q x) \supseteq (\bigoplus \sum_{i \neq q} Re_i) \oplus A$

(b) $(0 : e_q x) \cap Re_q = 0$

(c) $R = Re_1 \oplus \cdots \oplus Re_n \oplus A$

imply that $(0 : e_q x) = (\bigoplus \sum_{i \neq q} Re_i) \oplus A$, and therefore $Re_q x \cong Re_q$. Set $N_1 = Re_q x$. Then $N_1^2 \subseteq (Rxa) \cdot Rx = Rx \cdot (aRx) = Rx \cdot 0 = 0$.

If A is not an essential extension of $N_1 \oplus \mathscr{G}(R)$, then repeat the process used to obtain N_1 to get $N_2 \subseteq A$ such that $N_2^2 = 0$ and $Re_p \cong N_2$ for some $p \in \{1, 2, \cdots, n\}$. By induction we construct $N_3, N_4, \cdots$ with the desired properties. Moreover, by (1) and Theorem 2.1 this induction process stops after finitely many steps, say m. Thus A is an essential extension of $N_1 \oplus N_2 \oplus \cdots \oplus N_m \oplus \mathscr{G}(R)$ as desired.

(2) $\Rightarrow$ (1): This is an immediate consequence of Theorem 2.1 (3) $\Rightarrow$ (1).

The next theorem is the main result of this section.

THEOREM 3.3 *The following are equivalent*:

(1) $R = \mathscr{G}(R) \oplus R_1 \oplus R_2 \oplus \cdots \oplus R_m$, *where* $R_i R_j = 0$ *for* $i \neq j$, R_i *is* $(\mathscr{F}PP)$, R_i *is a direct sum of uniform left ideals, and* R_i *has a simple classical left quotient ring with minimum conditions for* $i = 1, 2, \cdots, m$.

(2) (i) R *is* $(\mathscr{F}PP)$.

(ii) $F(\mathscr{G})$ *has a cofinal subset of finitely generated left ideals.*

(iii) *There are no nonzero torsionfree nilpotent left ideals in* R.

(1) $\Rightarrow$ (2): Write $R_i = Rf_i$, where $1 = f_1 + f_2 + \cdots + f_m + g$

and $g \in \mathscr{G}(R)$. We claim that any principal left ideal in $\mathscr{T}$ is R-isomorphic to a principal left ideal in $Rf_1 + Rf_2 + \cdots + Rf_m$. For if $x = r_1f_1 + \cdots + r_mf_m + rg$ generates a principal left ideal in $\mathscr{T}$, then the mapping of Rx induced by

$$x \longrightarrow r_1f_1 + \cdots + r_mf_m$$

has kernel $K = \{trg \mid t \in R, trg \in Rx\}$. But then $Rtrg \in \mathscr{G} \cap \mathscr{T} = 0$. Thus $K = 0$ and the claim is established.

Thus to show (i), it is sufficient to show that each principal left ideal in $Rf_1 + \cdots + Rf_m$ is R-projective. Since each $Rf_i = R_i$ is $(\mathscr{T}PP)$ and $f_icf_j = 0$ for $i \neq j$ $(c \in R)$, this is easily verified using the fact that each R_i is $(\mathscr{T}PP)$.

Condition (ii) is immediate from Theorem 2.1 (3).

Looking at the projections of any torsionfree nilpotent left ideal into the R_i's, we see that the images of these projections all must be nilpotent left ideals. Since each R_i has a simple classical left quotient ring with minimum conditions, then it follows from [10], Theorem (p. 268), that each image is zero. It follows that zero is the only nilpotent torsionfree left ideal.

(2) $\Rightarrow$ (1): By Theorem 3.2, we have

$$R = Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n \oplus A\ ,$$

where A is an essential extension of $\mathscr{G}(R) \oplus N_1 \oplus N_2 \oplus \cdots N_m$ with $N_i^2 = 0$ for $i = 1, 2, \cdots, m$. From (iii) it follows that $A = \mathscr{G}(R)$. Hence $R = Re_1 \oplus \cdots \oplus Re_n \oplus \mathscr{G}(R)$.

Define $i \sim j$ if either $e_iRe_j \neq 0$ or $e_jRe_i \neq 0$ for $1 \leqq i, j \leqq n$. It is easily verified that $\sim$ is an equivalence relation. Let $S(1), S(2), \cdots, S(m)$ be the distinct equivalence classes of $\{1, 2, \cdots, n\}$ under $\sim$, and let $R_i = \sum_{j \in S(i)} Re_j$. Using Goldie's Theorem ([10], p. 268), the reader can verify that the R_i have the required properties.

Note that "$(\mathscr{T}PP)$" can be replaced by "finitely generated torsionfree left ideals are projective" or "torsionfree left ideals are projective" in the statement of Theorem 3.3, and the result remains true with only trivial modifications in the proof. In case R has a semisimple left classical quotient ring with minimum conditions, then $\mathscr{G}(R) = 0$, R has no infinite direct sums of left ideals, and R has no nonzero nilpotent left ideals. Hence the following result of L. Levy ([11], Th. 4.3) is a special case of Theorem 3.3.

COROLLARY 3.4. *Let R be a hereditary ring with semisimple left classical quotient ring S. Then R is a direct sum of hereditary rings $\{R_i \mid i = 1, 2, \cdots, n\}$ which have simple left classical quotient rings with minimum condition. When considered as a set of left*

ideals of R, $\{R_i \mid i = 1, 2, \cdots, n\}$ *constitutes a minimal set of annihilator ideals of* R, *and hence the quotient rings of the* R_i*s are the simple components of* S.

Since the concept of torsion defined by L. Levy [11] coincides with the Goldie torsion concept for rings possessing a semisimple left classical quotient ring with minimum conditions, the next corollary is a generalization of [11], Theorem 6.1.

COROLLARY 3.5. *Suppose that the conditions of Theorem* 3.3 *hold with "*$(\mathscr{F}PP)$*" replaced by "finitely generated left ideals in* $\mathscr{F}$ *are projective." Suppose* $R/\mathscr{G}(R)$ *has the ascending chain condition on annihilator right ideals, and suppose that* R *contains no infinite direct sum of right ideals. Then every finitely generated* R*-module* M *is a direct sum of* $\mathscr{G}(M)$ *and finitely many left ideals of* R.

Proof. Under these hypotheses $R/\mathscr{G}(R)$ has a two-sided semisimple classical quotient ring with minimum conditions. Note that $R/\mathscr{G}(R)$ also possesses a Goldie torsion theory, which coincides with Levy's torsion theory for $R/\mathscr{G}(R)$. Let M be a finitely generated R-module. Since $M/\mathscr{G}(M)$ is a $R/\mathscr{G}(R)$-module, it follows from [11], Theorem 5.2, that $M/\mathscr{G}(M)$ is isomorphic to a submodule of a free $R/\mathscr{G}(R)$-module. But by Theorem 3.3, $R/\mathscr{G}(R) = R_1 + R_2 + \cdots + R_m$ (ring direct sum), where each R_i is semi-hereditary. So [3] Theorem I.6.1 yields $M/\mathscr{G}(M) \cong \bigoplus \sum_{i=1}^{n} I_i$, where I_i is a finitely generated left ideal of $R/\mathscr{G}(R)$. By Theorem 3.3, each $x \in R$ can be written uniquely as $r_1 + r_2 + \cdots + r_m + g$, where $r_i \in R_i$ and $g \in \mathscr{G}(R)$. Hence each $x' = x + \mathscr{G}(R) \in R/\mathscr{G}(R)$ can be written uniquely as

$$r_1 + r_2 + \cdots + r_m + \mathscr{G}(R) .$$

Thus each I_i is R-isomorphic to a finitely generated left ideal of R contained $\sum_{i=1}^{m} R_i$ via

$$r_1 + r_2 + \cdots + r_m + \mathscr{G}(R) \longrightarrow r_1 + r_2 + \cdots + r_m .$$

So by Theorem 3.3, each I_i is R-projective. Therefore,

$$M \cong \mathscr{G}(M) \oplus I_1 \oplus I_2 \oplus \cdots \oplus I_n .$$

An element n of a left ideal N of R is said to be *in the center modulo* $\mathscr{G}(R)$ if the image of n under the natural homomorphism $\rho: R \to R/\mathscr{G}(R)$ is a nonzero element of the center of the ring $R/\mathscr{G}(R)$. Equivalently, an element n is in the center modulo $\mathscr{G}(R)$ if, for each $x \in R$, there exists an element $t_x \in \mathscr{G}(R)$ such that $xn = nx + t_x$.

PROPOSITION 3.6. *If R is $(\mathscr{F}PP)$ and every nonzero nilpotent left ideal in $\mathscr{F}$ contains an element in the center modulo $\mathscr{G}(R)$, then in fact R has no nilpotent left ideals in $\mathscr{F}$.*

Proof. Assume otherwise. Then there exists $n \in R$ with the following properties:

(1) $Rn \in \mathscr{F}$.

(2) $(Rn)^2 = 0$.

(3) For each $x \in R$, there exists $t_x \in \mathscr{G}(R)$ such that $nx = x + t_x$.

By $(\mathscr{F}PP)$, there is an isomorphism $\lambda: Rn \to Re$, where $e^2 = e \neq 0$. Let $r \in R$ be such that $\lambda(rn) = e$. Then $n \in (0: rn) = (0: e)$, so that $ern = ner + t = t$ for some $t \in \mathscr{G}(R)$. So (1) implies that $e \in (0: rn) = (0: e)$, a contradiction.

COROLLARY 3.7. *Let R be commutative. Then the following are equivalent:*

(1) *R is $(\mathscr{F}PP)$ and $F(\mathscr{G})$ has a cofinal subset of finitely generated left ideals.*

(2) *$R = \mathscr{G}(R) + R_1 + R_2 + \cdots + R_n$ (ring direct sum), where each R_i is an integral domain.*

Proof. Apply Proposition 3.6 and Theorem 3.3. Then note that a commutative prime ring is an integral domain. Conversely, (2) $\Rightarrow$ (1) follows from Theorem 3.3.

Recall that R is (PP) if and only if R is $(\mathscr{F}PP)$ and $\mathscr{G}(R) = 0$. Hence the following result of A. Hattori ([8], Lemma 3) is a special case of Corollary 3.7.

COROLLARY 3.8. *Let R be a commutative ring having no infinite direct sum of left ideals. Then R is a (PP) ring if and only if R is a direct sum of integral domains.*

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UNIVERSITY OF FLORIDA
GAINESVILLE FLORIDA

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EXTREMAL STRUCTURE OF STAR-SHAPED SETS

F. E. TIDMORE

It is shown that the convex kernel of a compact star-shaped subset S of a finite-dimensional linear topological space L_n is determined by the $(n-1)$-extreme points of S. The cardinality of the set of k-extreme points is determined for compact star-shaped sets of dimension greater than two. Also given is the result that any compact star-shaped subset S of L_n contains a countable set of $(n-1)$-extreme points which determines the convex kernel of S. Another result is that a compact nonconvex star-shaped set S in a locally convex space L is determined by the convex kernel of S and the subset of points that are extreme in S relative to the convex kernel of S.

The convex kernel of a star-shaped set S will be denoted by ckS, the line segment $\{\alpha x + (1-\alpha)y\colon \alpha \in [0,1]\}$ will be denoted by xy, the ray $\{\beta y + (1-\beta)x\colon \beta \geqq 1\}$ will be denoted by xy^{∞} and $L(x,y)$ will denote the line containing x and y, $x \neq y$. The convex hull of a set S will be denoted by conv S. The notation intv S will denote the interior of S relative to the minimal flat that contains S. The set $\{x\colon f(x) = \alpha\}$, where f is a linear functional, will be denoted $[f\colon \alpha]$. Set-theoretic difference will be denoted by \, and the closure of a set S will be denoted by cl S.

The concept of k-extreme point was introduced by Asplund [1].

DEFINITION 1. If S is a subset of a linear space L, a point $x \in S$ is a k-extreme point of S if no k-simplex Δ exists such that $x \in \text{intv } \Delta \subset S$.

For a subset S of a linear space L, S_x will denote the x-star of S determined by the point $x \in S$; that is, the set of points y such that $xy \subset S$. If S is a closed (compact) subset of a linear topological space L, then for any $x \in S$, S_x is a closed (compact) set. If $T \subset S$, let

$$S_T = \bigcap_{x \in T} S_x .$$

A point p belongs to the convex kernel of S if, and only if, $xp \subset S$ for all $x \in S$, which is true if, and only if, $p \in S_x$ for all $x \in S$. Thus $ckS = S_S$, which motivates the following definition.

DEFINITION 2. In a linear space L a subset T of a star-shaped

set S is said to star-generate the convex kernel of S if $ckS = S_T$. Such a subset T is said to be a star-generating set for ckS.

THEOREM 1. *Let S be a compact star-shaped subset of L_{k+1}. Then the set $S(k)$ of k-extreme points of S is a star-generating set for ckS.*

Proof. Without loss of generality, suppose that $0 \in ckS$. If $S = ckS$, then S is convex and $S_x = S$ for each $x \in S$ and the result follows since $\varnothing \neq S(1) \subset S(k)$. Let $p \in S \backslash ckS$. Then there exists a point $y \in S$ such that $py \not\subset S$. Since S is compact, y can be chosen such that $S \cap \text{intv } py^{\infty} = \varnothing$. Since $py \not\subset S$, there exists a point $z \in (\text{intv } py) \backslash S$. If $y \in S(k)$, then $p \notin S_y$ implies $p \notin S_{S(k)}$. If $y \notin S(k)$ there exists a k-simplex Δ such that $y \in \text{intv } \Delta \subset S$. Consider the convex cone $C = \{\beta y + (\lambda - \beta + 1)\, z\colon \beta, \lambda \geqq 0\}$, which has vertex z and is contained in the subspace L' with basis $\{p, y\}$. Since $S \cap \text{intv } py^{\infty} = \varnothing$, Δ must intersect L' in some line other than $L(p, y)$; thus, $S \cap \text{intv } C \neq \varnothing$. There exists a linear functional f defined on L_{k+1} such that $f(q) = 1$ for every $q \in L(p, y)$; clearly $0 \notin L(p, y)$ since $py \not\subset S$ and $0 \in ckS$. The continuous linear functional f_1, the restriction of f to L', attains a maximum on the compact set $C \cap S$ at some point $w \in \text{intv } C$. Let $H = [f\colon f(w)]$. Since $H \cap C \cap S$ is a compact subset of the 1-dimensional set $H \cap L'$, there exists a minimal closed line segment in intv C which contains $H \cap C \cap S$. Each endpoint of this segment, which may be degenerate, must be a point in $S(k)$. Let v be one of these endpoints. The points p, y, z and v are in L'. If $pv \subset S$, then the fact that $0 \in ckS$ implies that $z \in \text{conv } \{0, p, v\} \subset S$, a contradiction. Hence, $pv \not\subset S$ and $p \notin S_{S(k)}$. Therefore, $S \backslash ckS \subset S \backslash S_{S(k)}$, which gives the desired equality, since clearly $ckS \subset S_{S(k)}$.

It is not always sufficient to consider only the set of familiar extreme points $S(1)$ as a star-generating set for ckS. For example, in E_3 let S be the union of three closed faces of a 3-simplex. In some cases, proper subsets of $S(k)$ exist which will star-generate ckS. However, characterizing such subsets may be very difficult, as indicated by the following example.

EXAMPLE 1. In the plane E_2 let B_u be the upper closed unit half-disc, B_r the right closed unit half-disc. Let

$$T_1 = \text{conv } [\{-2e_1\} \cup (B_r + (2e_1 + e_2))] ,$$
$$T_2 = \text{conv } [\{-2e_2\} \cup (B_u + (2e_2 - e_1))] ,$$
$$S = T_1 \cup T_2 \cup (-T_1) \cup (-T_2) .$$

Then any star-generating subset of $S(1)$ must contain four distinct

sequences of carefully chosen extreme points.

THEOREM 2. *If S is a compact star-shaped set in L_n, and* dim $(S) \geqq 3$, *then $S(n-1)$ is an uncountable set.*

Proof. Without loss of generality, it can be assumed that $0 \in ckS$. Since dim $(S) \geqq 3$ there exists some point $x \in S, x \neq 0$. If $\beta x \in S(n-1)$ for every $\beta \in (0,1)$, then $S(n-1)$ is uncountable. Otherwise, consider some $w = \beta x$ such that $w \notin S(n-1)$. Then there exists an $(n-1)$-simplex Δ such that $w \in \text{intv}\, \Delta \subset S$. Since $n-1 \geqq 2$ there exists a nondegenerate line segment $zw \subset \Delta$ such that $zw \cap 0x = \{w\}$. There exists a linear functional f on L_n such that

$$f(w) = f(z) = 1 .$$

There exists a point $y \in [f: 0]$ such that the set $\{y, z, w\}$ is linearly independent. For each $\lambda \in [0,1]$ consider the subspace $L(\lambda)$ of L_n with basis $\{y, \lambda z + (1-\lambda)w\}$. Let f_λ be the restriction of f to $L(\lambda)$. The set $L(\lambda) \cap S$ is compact; hence; f_λ attains a maximum on $L(\lambda) \cap S$ at some point $u, f_\lambda(u) \geqq 1$. Since dim $(L(\lambda) \cap [f: f(u)]) = 1$ and

$$L(\lambda) \cap S \cap [f: f(u)]$$

is compact, there exists a minimal closed line segment in $L(\lambda)$ which contains $L(\lambda) \cap [f: f(u)] \cap S$. This line segment must have at least one endpoint, which must belong to $S(n-1)$. For each pair of distinct real numbers λ, μ in $[0,1]$, $L(\lambda) \cap L(\mu) \subset [f; 0]$. There exists points $p_\lambda \in L(\lambda) \cap S(n-1)$, $p_\mu \in L(\mu) \cap S(n-1)$ such that $f(p_\lambda) \geqq 1, f(p_\mu) \geqq 1$, which implies that $p_\lambda \neq p_\mu$. Thus, the set $S(n-1)$ is uncountable.

THEOREM 3. *Let S be a closed subset of a linear topological space L and let T be a subset of S that star-generates ckS, which may be empty. If M is a dense subset of T, then M star-generates ckS.*

Proof. Since $M \subset T$ then clearly $S_T \subset S_M$. Suppose that M is a proper subset of T and ckS is a proper subset of S_M. Then there exists a point $q \in S_M \backslash S_T$. But $S_T = S_M \cap S_{T \backslash M}$; thus $q \notin S_{T \backslash M}$. This implies that $q \notin S_x$ for some $x \in T \backslash M$. Since $q \in S_M, M \subset S_q$, which is closed. Hence, $x \in T \subset \text{cl}\, M \subset S_q$, which implies that $xq \subset S$ and that $q \in S_x$, a contradiction. Therefore, $ckS = S_M$.

THEOREM 4. *If S is a compact star-shaped subset of a normed linear space L, then any subset T of S which star-generates the convex kernel of S contains a countable subset M which also star-generates the convex kernel of S.*

Proof. The norm of L induces a metric on L. The compact set S can be considered as a compact metric space, where space is now used in the topological sense. The compact metric space is separable, which implies that S is second countable [2]. Any nonempty subset T of S is a second countable topological space with the relative topology, which implies that T is separable. There exists a countable subset M of T such that $T \subset \operatorname{cl} M$. Theorem 3 implies that M star-generates ckS and the theorem is proved.

COROLLARY. *Let S be a compact star-shaped subset of L_{k+1}. Then there exists a countable subset of $S(k)$ which star-generates ckS.*

Klee [3] introduced the concept of relative extreme point.

DEFINITION 3. If S and T are subsets of a linear space L, then $x \in S$ is said to be extreme in S relative to T if there do not exist points $y \in S, z \in T$ such that $x \in \operatorname{intv} yz$.

If S is a star-shaped set, exk S will denote the points of S which are extreme relative to ckS, and $E_S = (\operatorname{exk} S) \backslash ckS$.

THEOREM 5. *Let S be a compact nonconvex star-shaped set in a locally convex space L. Then $C = S$, where*

$$C = \bigcup_{y \in E_S} \operatorname{conv} (ckS \cup \{y\}) .$$

Proof. Since $E_S \subset S$, conv $(ckS \cup \{y\}) \subset S$ for each $y \in E_S$. Thus, $C \subset S$. Consider $z \in ckS \cup \operatorname{exk} S$; since $E_S \neq \varnothing$, as shown below, $z \in C$. Let $K = ckS$. Suppose that $z \in S \backslash (ckS \cup \operatorname{exk} S)$ and without loss of generality, suppose that $z = 0$. Since K is compact and convex, K^* and $-K^*$ are closed convex cones with vertex 0, where $K^* = \{\lambda x : x \in K, \lambda \geqq 0\}$. Since $z \notin \operatorname{exk} S$ there exist points $x \in K$ and $w \in S$ such that $0 \in \operatorname{intv} xw$. Clearly $w \in -K^* \backslash \{0\}$, $S \cap (-K^* \backslash \{0\}) \neq \varnothing$ and $S \cap (-K^*)$ is compact. Let u be an arbitrary point in $-K^* \backslash \{0\}$; since L is locally convex and K^* is closed and convex, there exists a closed hyperplane $H = [f : f(u)]$ such that $u \in H$ and $H \cap K^* = \varnothing$, where f is a continuous linear functional. It can be assumed that $f(K^*) \leqq 0$, which implies that $f(u) > 0$. The functional f then attains a maximum on $S \cap (-K^*)$ at some point $v \in S \cap (-K^*)$. Suppose that $v \notin \operatorname{exk} S$. There exist points $p \in K, q \in S$ such that $v \in \operatorname{intv} pq$. Since $v \in -K^*$, $v = -\lambda p'$, $p' \in K, \lambda > 0$, and

$$v = \alpha p + (1 - \alpha) q , \qquad 0 < \alpha < 1 .$$

Therefore, $v = -\lambda p' = \alpha p + (1 - \alpha) q$ and $q = \tau q'$, where $\tau < 0$ and $q' \in K$. Thus, $q \in S \cap (-K^*)$. But it can be easily shown that

$f(q) > f(v)$, which contradicts the fact that $f(v) \geqq f(x)$ for each $x \in S \cap (-K^*)$. Hence, $v \in (\text{exk } S) \cap (-K^*)$ and $0 \in C$, which implies that $S \subset C$. This inclusion, along with the one given earlier, implies that $S = C$.

The following result shows that the set E_S is minimal in its use in Theorem 5.

THEOREM 6. *Let S be a compact nonconvex star-shaped set in a locally convex space L. If T is a proper subset of E_S then*

$$C(T) = \bigcup_{y \in T} \text{conv } (ckS \cup \{y\})$$

is a proper subset of S.

Proof. Consider any proper subset T of E_S; there exists some point $x \in E_S \backslash T$. If $x \in C(T)$ there exists some $y \in T$ such that $x \in$ conv $(ckS \cup \{y\})$. Hence, $x = \lambda z + (1 - \lambda)y$, where $\lambda \in [0, 1], z \in ckS$. But $\lambda \in (0, 1)$ since $x \notin ckS \cup T$. This implies that $x \notin \text{exk } S$, a contradiction. Thus, $x \notin C(T)$, which must be a proper subset of S.

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OKLAHOMA STATE UNIVERSITY
TEXAS TECHNOLOGICAL COLLEGE

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UNCONDITIONAL AND SHRINKING BASES IN LOCALLY CONVEX SPACES

L. J. WEILL

Let E be a locally convex space with an unconditional Schauder basis $\{x_k\}$ and let $\{f_k\}$ be the sequence of coefficient functionals biorthogonal to $\{x_k\}$. Owing to works of R. C. James and S. Karlin it is known that if E is a Banach space then each of the three conditions which follow is necessary and sufficient for $\{f_k\}$ to be a basis for E^* in the strong (norm) topology.

(1) E has no subspace topologically isomorphic to the space l^1.

(2) E^* is separable in the strong topology.

(3) E^* is weakly $(w(E^*, E^{**}))$ sequentially complete.

The primary purpose of this paper is to show that in certain spaces which are more general than Frechet spaces and hence than Banach spaces, each of the above three conditions is necessary and sufficient for

(0) $\{f_k\}$ is a strong basis for E^*.

Specifically if E is a complete barrelled space, each of (1) and (2) is sufficient for (0). In any locally convex space (2) implies (1) (even if E has no basis) and so each of (1) and (2) is necessary for (0). If E is a space having the property that weak* bounded subsets of E^* are strongly bounded (complete locally convex spaces and barrelled spaces have this property) then (3) is sufficient for (0). (3) is necessary for (0) if E is both barrelled and metrizable.

Besides the papers of James [8] and Karlin [9], related matter of importance is contained in the works of M. M. Day [2, Ch. 4] and C. Bessaga and A. Pelczynski [1]. E. L. Dubinsky and J. R. Retherford [5] using Köthe sequence space techniques have proved the (1) $\leftrightarrow$ (2) part of Theorem 2.12.

1. **Preliminarie and fundamental theorems.** Since the main results of this paper depend upon many theorems which are not widely known in their more general settings, the author thought it wise to include this section which introduces, in addition to basic theorems, some difinitions, terminology, and notation. No proofs are given for known results.

If E is a locally convex space, then π will be used for the canonical mapping from E to E^{**}, the space of strongly continuous functions on E^*. For convenience, a subscript 2 will be added to the word "space" to designate that the Hausdorff axiom is satisfied.

A sequence $\{x_k\}$ in a linear topological space$_2$ is called a basis if for each element x in the space there is a unique sequence of scalars $\{f_k(x)\}$ such that $x = \sum_{k=1}^{\infty} f_k(x)x_k$. The resulting linear functionals $\{f_k\}$ are called the coefficient functionals. If the coefficient functionals are continuous, then $\{x_k\}$ (or $\{x_k, f_k\}$) is called a Schauder basis. The n-th partial sum operator s_n is defined by $s_n(x) = \sum_{k=1}^{n} f_k(x)x_k$; the n-th remainder is defined by $R_n(x) = \sum_{k=n+1}^{\infty} f_k(x)x_k$. We shall use Φ for the collection of all nonempty finite subsets of the positive integers ω. If $\sigma \in \Phi$, then the σ-partial sum operator S_σ is defined by $S_\sigma(x) = \sum_{k \in \sigma} f_k(x)x_k$; the σ-remainder is defined by $R_\sigma(x) = x - S_\sigma(x)$. Note that $R_\rho \circ R_\sigma(x) = R_{\rho \cup \sigma}(x)$, and if $\rho \cap \sigma \neq \varnothing$ then $S_\rho \circ S_\sigma(x) = S_{\rho \cap \sigma}(x)$; if $\rho \cap \sigma = \varnothing$, then $S_\rho \circ S_\sigma(x) = \theta$ for every x.

PROPOSITION 1.1. *Let E be a linear topological space$_2$. If $\{x_k, f_k\}$ is a Schauder basis or a weak Schauder basis for E, then $\{f_k, \pi(x_k)\}$ is a weak* Schauder basis for E^*. It follows that if $\{f_k\}$ is also a strong basis then it is a strong Schauder basis with $\{\pi(x_k)\}$ as the coefficient functionals. The σ-partial sum for $\{f_k\}$ is then the adjoint of S_σ; i.e., $S_\sigma^*(f) = \sum_{k \in \sigma} \pi(x_k)(f)f_k = \sum_{k \in \sigma} f(x_k)f_k$. The σ-remainder is R_σ^* and of course $R_\sigma^* f = f - S_\sigma^* f$.*

As a kind of converse we have the following.

PROPOSITION 1.2. *Let E be a locally convex space$_2$. If $\{f_k F_k\}$ is a weak* Schauder basis for E^*, then $\{x_k, f_k\}$ is a weak Schauder basis for E, where for each k, x_k is that unique element of E such that $\pi(x_k) = F_k$.*

The next two propositions require the Barrel Theorem for their proofs.

PROPOSITION 1.3. *Let E be a barrelled space$_2$. If $\{x_k, f_k\}$ is a Schauder basis for E then $\{f_k\}$ is a strong basis for the closed linear span, $\overline{[f_k]}$, of $\{f_k\}$.*

Actually, in Proposition 1.3, "Schauder basis" may be replaced by "weak Schauder basis" as the following reveals.

PROPOSITION 1.4. *In a barrelled space a weak Schauder basis is a Schauder basis.*

A Schauder basis $\{x_k, f_k\}$ is called a shrinking basis if $\{f_k\}$ is a

basis for E^* (in the strong topology). A basis $\{x_k\}$ is boundedly complete if for each sequence $\{t_k\}$ of scalars such that $\{\sum_{k=1}^{n} t_k x_k\}_{n=1}^{\infty}$ is bounded, the series $\sum_{k=1}^{\infty} t_k x_k$ is convergent. The next proposition due to J. Dieudonné [3, Prop. 6] strengthens Proposition 1.1 in the case where E is barrelled.

PROPOSITION 1.5. *If $\{x_k, f_k\}$ is a Schauder basis in a barrelled space$_2$ E, then $\{f_k\}$ is a weak* boundedly complete basis for E^*. Hence, if $\{x_k, f_k\}$ is a shrinking basis then $\{f_k\}$ is a boundedly complete basis for E^*.*

The next proposition was proved for Banach spaces by R. C. James [8] and generalized by J. R. Retherford [13].

PROPOSITION 1.6. *Let $\{x_k, f_k\}$ be a Schauder basis for a barrelled space$_2$ E. Then E is reflexive if and only if $\{x_k\}$ is both shrinking and boundedly complete.*

If $(E, \|\cdot\|)$ is a Banach space with basis $\{x_k\}$, then a new norm related to the basis can be formed: $\|x\|' = \sup_n \{\|S_n(x)\|\}$. Both norms have the same topology but the new norm is more useful. A more general device has been developed by C. W. McArthur [14]. In the fundamental lemma which follows a part of this device is abstracted even further in order that the result may be applied to Theorem 1.18.

LEMMA 1.7. *Let (E, τ) be a locally convex space and $\{g_\alpha: \alpha \in A\}$ a net* [10, p. 28] *of pointwise bounded linear operators on E. Let $\mathscr{V}$ be the local base consisting of all barrelled neighborhoods of E. For $V \in \mathscr{V}$, define V' to be $\{x \in E: g_\alpha(x) \in V$ for all $\alpha \in A\}$. Then we have the following.*

(1) *$\mathscr{V}' \equiv \{V': V \in \mathscr{V}\}$ is a local base for a locally convex topology τ' on E.*

(2) *If there is a subnet $\{g_{n(\beta)}: \beta \in B\}$ such that, for each $x \in E$, $\{g_{n(\beta)}(x): \beta \in B\}$ is convergent to x then τ' is stronger than τ.*

(3) *If in addition to* (2) *each g_α is continuous and E is barrelled, then $\tau' = \tau$.*

Proof. By the criterion 6.5 of [10, p. 47], $\mathscr{V}'$ forms a local base for a locally convex topology if the three conditions below are met.

(i) Each $V' \in \mathscr{V}'$ is convex, circled, and radial at zero.

(ii) For each U' and V' of $\mathscr{V}'$ there is a $W' \in \mathscr{V}'$ such that

$W' \subset U' \cap V'$.

(iii) tV' belongs to $\mathscr{Y}'$ for each $V' \in \mathscr{Y}'$ and each scalar t.

Proof of (i). Let s and t be nonnegative scalars such that $s + t = 1$. If $x, y \in V$; then $g_\alpha(x)$, $g_\alpha(y) \in V$ for all $\alpha \in A$. Since V is convex, $sg_\alpha(x) + tg_\alpha(y) = g_\alpha(sx + ty) \in V$ for all $\alpha \in A$ and hence $sx + ty \in V'$ so V' is convex. Let $|t| \leqq 1$ $x \in V'$. $g_\alpha(x) \in V$ for all α and since V is circled, $tg_\alpha(x) = g_\alpha(tx) \in V$ for all α; hence $tx \in V'$. Therefore V' is circled. Let $x \in E$. Since g_α is pointwise bounded, there is a positive scalar s such that if $0 < t < s$ then $tg_\alpha(x) \in V$ for all α. Hence $tx \in V'$ so V' is radial at zero.

Proof of (ii). There exists a $W \in \mathscr{Y}$ such that $W \cup U \cap V$. Hence $W' \subset (U \cap V)' = U' \cap V'$.

Proof of (iii). For each scalar t, $tV \in \mathscr{Y}$. Hence $(tV)' = tV' \in \mathscr{Y}'$.

To prove (2) we show that $V' \subset V$ for each $V \in \mathscr{Y}'$. If $x \in V'$, then $g_\alpha(x) \in V$ for all α and in particular $g_{n(\beta)}(x) \in V$ for all $\beta \in B$. Thus since V is closed V is closed $x \in V$.

To prove (3) we show that V' is τ-closed and hence a τ-barrel. Let $\{y_\beta : \beta \in B\}$ be a net in V' converging to y. Let $\alpha \in A$; then since g_α is continuous, $g_\alpha(y_\beta)$ converges with β to $g_\alpha(y)$. Since $g_\alpha(y_\beta) \in V$ for each $\beta \in B$, $g_\alpha(y) \in V$; α was arbitrary, so $y \in V'$.

Part (1) and (2) of the next proposition follow from the lemma.

PROPOSITION 1.8. *Let (E, τ) be a locally convex space$_2$ with a basis $\{x_k\}$ and let $\mathscr{Y}$ be as in Lemma* 1.7. *For $V \in \mathscr{Y}$, define $V' = \{x : S_n(x) \in V$ for each $n \in \omega\}$.*

(1) *$\mathscr{Y}' = \{V' : V \in V\}$ is a local base for a locally convex topology τ' which is stronger than τ.*

(2) *If E is barrelled and $\{x_k\}$ is a Schauder basis then $\tau = \tau'$.*

(3) *If (E, τ) is complete, then so is (E, τ').*

The last propositions of this section are slight modifications of results due to Retherford and McArthur [14],

PROPOSITION 1.9. *Let E be a complete locally convex space$_2$ and $\{x_k\}$ a sequence of nonzero elements of E. If for each continuous seminorm P on E there is a continuous seminorm Q such that $P(\sum_{k=1}^{p} a_k x_k) \leqq Q(\sum_{k=1}^{q} a_k x_k)$ for $p \leqq q$ and arbitrary scalars $a_1, \cdots, a_q$, then $\{x_k\}$ is a Schauder basic sequence (i.e., $\{x_k\}$ is a Schauder basis for its closed linear span $\overline{[x_k]}$).*

PROPOSITION 1.10. *Let E be a barrelled space$_2$ with a Schauder basis $\{x_k\}$. Then for each continuous seminorm P there is a continuous seminorm Q such that for $p \leqq q$ and arbitrary scalars*

$a_1, \cdots, a_q$, *we have* $P(\sum_{k=1}^{p} a_k x_k) \leqq Q(\sum_{k=1}^{q} a_k x_k)$.

Let $\sum_{k=1}^{\infty} y_k$ be a (formal) series in a linear topological space E. The series is called unordered bounded if the set of unordered partial sums $\{\sum_{k \in \sigma} y_k : \sigma \in \phi\}$ is bounded. The series is called unconditionally (or unordered) convergent if the net of unordered partial sums (with inclusion as the order relation in ϕ) is convergent to an element of E. It is well known that a series $\sum_{k=1}^{\infty} y_k$ in a linear topological space$_2$ is unconditionally convergent if and only if for each permutation p of ω, the set of positive integers, the series $\sum_{k=1}^{\infty} y_p(k)$ is convergent (each series converges to the same element regardless of the permutation).

The series $\sum_{k=1}^{\infty} y_k$ is subseries convergent if for every subsequence $\{y_{n_k}\}$ of $\{y_k\}$ the series $\sum_{k=1}^{\infty} y_{n_k}$ is convergent. $\sum_{k=1}^{\infty} y_k$ is bounded-multiplier convergent if for each bounded sequences of scalars $\{t_k\}$ the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent. $\sum_{k=1}^{\infty} y_k$ is absolutely convergent in a locally convex space E if the series is convergent and for each continuous seminorm P on E, the series $\sum_{k=1}^{\infty} P(y_k)$ is convergent. In any locally convex space, absolute convergence implies unconditional convergence.

PROPOSITION 1.11. *In a locally convex space a series is weakly absolutely convergent if and only if it is weakly unconditionally convergent.*

Define a to be the set of sequences $\{\alpha_m\}$ such that $\alpha_m = \pm 1$ for each m; define b to be the set of sequences $\{\beta_m\}$ such that $\beta_m = 0$ or 1 for each m; and define e to be the set of all complex sequences $\{\varepsilon_m\}$ such that $|\varepsilon_m| \leqq 1$ for each m.

PROPOSITION 1.12. *If $\sum_{k=1}^{\infty} y_k$ is a series in a locally convex space E, then the statements below are equivalent.*

(1) $\sum y_k$ *is unordered bounded.*
(2) $\sum y_k$ *is weakly unordered Cauchy.*
(3) *The set* $\{\sum_{k=1}^{n} \alpha_k y_k : \{\alpha_k\} \in a$ *and* $n \in \omega\}$ *is bounded.*
(4) *The set* $\{\sum_{k=1}^{n} \beta_k y_k : \{\beta_k\} \in b$ *and* $n \in \omega\}$ *is bounded.*
(5) *The set* $\{\sum_{k=1}^{n} \varepsilon_k y_k : \{\varepsilon_k\} \in e$ *and* $n \in \omega\}$ *is bounded.*

COROLLARY 1.13. *In a weakly sequentially complete locally convex space a series is unordered bounded if and only if it is weakly unconditionally convergent (weakly absolutely convergent).*

The Orlicz-Pettis theorem for Banach spaces on subseries convergence has been shown valid for locally convex spaces by Grothendieck

and McArthur. For a proof of the theorem, which follows, see [12].

THEOREM 1.14. *In a locally convex space$_2$ a series is subseries convergent if and only if it is weakly subseries convergent.*

It is well known that in a sequentially complete locally convex space, bounded-multiplier convergence and subseries convergence are each equivalent to unconditional convergence [2]. The next theorem adds three other conditions which are equivalent to unconditional convergence. A proof of the equivalence of (1), (2), (3), and (4) can be found in [15]. (1) is proved equivalent to (2) in [11] by use of a certain convergence criterion.

THEOREM 1.15. *Let $\{y_k\}$ be a sequence in a sequentially complete locally convex space E. The following six statements are equivalent.*

(1) *The series $\sum_{k=1}^{\infty} y_k$ converges unconditionally.*

(2) *For any equicontinuous subset A of E^*, the series $\sum_{k=1}^{\infty} |f(y_k)|$ converges uniformly for $f \in A$.*

(3) *The series $\sum_{k=1}^{\infty} \alpha_k y_k$ converges uniformly for $\{\alpha_k\} \in a$.*

(4) *The series $\sum_{k=1}^{\infty} \varepsilon_k y_k$ converges uniformly for $\{\varepsilon_k\} \in e$.*

(5) *The series $\sum_{k=1}^{\infty} y_k$ is bounded-multiplier convergent.*

(6) *The series $\sum_{k=1}^{\infty} y_k$ is subseries convergent.*

Since a weakly (sequentially) complete locally convex space is (sequentially) complete, Theorem 1.14 and 1.15 imply that in a weakly sequentially complete locally convex space a series is unconditionally convergent if and only if it is weakly unconditionally convergent. Thus similarly to Corollary 1.13 we have the following.

COROLLARY 1.16. *In a weakly sequentially complete locally convex space$_2$, a series is unordered bounded if and only if it is unconditionally convergent.*

A basis $\{x_k, f_k\}$ in a linear topological space$_2$ is called an unconditional basis if for each x in the space the series $\sum_{k=1}^{\infty} f_k(x)x_k$ is unconditionally convergent to x. Hence, $\{x_k, f_k\}$ is an unconditional Schauder basis if and only if for each permutation p of ω, $\{x_{p(k)}, f_{p(k)}\}$ is a Schauder basis. It follows that each of Propositions 1.1 and 1.5 has an analog for unconditional bases. For instance we have the following.

THEOREM 1.17. *If $\{x_k, f_k\}$ is an unconditional Schauder basis for a linear topological space$_2$ E, then $\{f_k, \pi(x_k)\}$ is a weak* unconditional Schauder basis for E^*.*

If $\{x_k, f_k\}$ is a basis and $\alpha = \{\alpha_k\}$ is a sequence of scalars, define $S_{n,\alpha}(x) = \sum_{k=1}^{n} \alpha_k f_k(x) x_k$. Let (E, τ) be a locally convex space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. Let $\mathscr{V}^{\wedge}$ be the local base consisting of all barrelled neighborhoods in E. If $V \in \mathscr{V}^{\wedge}$, define $V_b = \{x \in E: S_\sigma(x) \in V \text{ for all } \sigma \in \Phi\}$, $V_a = \{x \in E: S_{n,\alpha}(x) \in V \text{ for all } n \in \omega \text{ and all } \alpha \in a\}$, and $V_e = \{x \in E: S_{n,\varepsilon}(x) \in V \text{ for all } n \in \omega \text{ and all } \varepsilon \in e\}$. Let $\mathscr{V}^{\wedge}_b = \{V_b: V \in \mathscr{V}^{\wedge}\}$, $\mathscr{V}^{\wedge}_a = \{V_a: V \in \mathscr{V}^{\wedge}\}$, and $\mathscr{V}^{\wedge}_e = \{V_e: V \in \mathscr{V}^{\wedge}\}$.

THEOREM 1.18. *Le (E, τ) be as above. The families $\mathscr{V}^{\wedge}_b$, $\mathscr{V}^{\wedge}_a$, and $\mathscr{V}^{\wedge}_e$ form local bases for locally convex Hausdorff topologies on E, say τ_b, τ_a, and τ_e. Each of these topologies is stronger than τ and if E is barrelled then $\tau = \tau_b = \tau_a = \tau_e$.*

Proof. According to Lemma 1.7, $\mathscr{V}^{\wedge}_b$, $\mathscr{V}^{\wedge}_a$, and $\mathscr{V}^{\wedge}_e$ form local bases if the families of operators $\{S_\sigma: \sigma \in \Phi\}$, $\{S_{n,\alpha}: (n, \alpha) \in \omega \times a\}$, and $\{S_{n,\varepsilon}: (n, \varepsilon) \in \omega \times e\}$ are pointwise bounded. Since $\{x_k, f_k\}$ is an unconditional Schauder basis it is a weak unconditional Schauder basis. Thus $\sum_{k=1}^{\infty} f_k(x) x_k$ is weakly unordered Cauchy for each x and hence by Proposition 1.12 the three families above are pointwise bounded.

Let d be either the family of sequences a or e. The set $\omega \times d$ is a directed set under the order relation $<$ defined by: $(n, \alpha) < (m, \beta)$ if and only if $n \leqq m$. Thus $\{S_{n,\alpha}: (n, \alpha) \in \omega \times d\}$ is a net. Let α^1 be the sequence $(1, 1, 1, \cdots)$ and not that the subnet $\{S_{n,\alpha^1}(x): n \in \omega\}$ converges to x for each $x \in E$. Thus by part (2) of 1.7. τ_b, τ_a, and τ_e are stronger than τ.

Let $\mathscr{V}'^{\wedge}$ be $\mathscr{V}^{\wedge}_b$, $\mathscr{V}^{\wedge}_a$, or $\mathscr{V}^{\wedge}_e$ then $\bigcap_{V' \in \mathscr{V}'} V' \subset \bigcap_{V \in \mathscr{V}} V$. Since τ is a Hausdorff topology, the latter intersection is empty and hence the other topologies are also Hausdorff.

For each $n \in \omega$ and α in b, a, or e, the operator $S_{n,\alpha}$ is continuous so if E is barrelled then $\tau = \tau_b = \tau_a = \tau_e$.

The next result, its variations, and its corollary shall be referred to as the Generalized Gurevic Theorem (see [7]). We use d to mean either of the family of sequences b, a, or e (actually the theorem is valid if d satisfies $a \subset d \subset e$ or $b \subset d \subset e$).

THEOREM 1.19. *Let $\{x_k\}$ be a sequence of nonzero elements in a locally convex space$_2$ E. Call the following condition*: given a neighborhood of zero U there is a neighborhood of zero V such that for an arbitrary sequence $\{a_n\}$ of scalars arbitrary p and q in $\{k: k \text{ is a positive integer or } k = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$, we have $\sum_{n=1}^{q} a_n x_n \in V$ implies $\sum_{n=1}^{p} \delta_n a_n x_n \in U$.*

(1) *If E is sequentially complete, then $*$ implies $\{x_k\}$ is unconditionally Schauder basic.*

(2) *If E is barrelled and $\{x_k\}$ is an unconditional Schauder basis for E, then $*$ holds.*

VARIATION 1.20. In the theorem, condition $*$ can be replaced by condition $*_1$: given an equicontinuous set $A \subset E^*$, there is an equicontinuous set $B \subset E^*$ such that for an arbitrary positive number M and arbitrary sequence $\{a_n\}$ of scalars arbitrary p and q in $\{k: k$ is a positive interger or $k = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$, we have $|\sum_{n=1}^{q} a_n g(x_n)| \leqq M$ for all $g \in B$ implies $|\sum_{n=1}^{p} \delta_n a_n f(x_n)| \leqq M$ for all $f \in A$.

VARIATION 1.21. In the theorem condition $*$ can be replaced by condition $*_2$: given a continuous seminorm P there is a continuous seminorm Q such that for arbitrary sequence $\{a_n\}$ of scarlars arbitrary p and q in $\{k: k$ is a positive integer or $K = \infty\}$ with $q \geqq p$, and arbitrary sequence $\{\delta_n\} \in d$ we have $P(\sum_{n=1}^{p} \delta_n a_n x_n) \leqq Q(\sum_{n=1}^{q} a_n x_n)$.

Proof. First we show that $*$ and $*_1$ are equivalent. Since $\{A_0: A$ equicontinuous$\}$ forms a local base in E, $*$ is easily seen to be equivalent to the following. Given an equicontinuous A there is an equicontinuous B such that for $\cdots$ we have $\sum_{n=1}^{q} a_n x_n \in B_0$ implies $\sum_{n=1}^{p} \delta_n a_n x_n \in A_0$. But this is equivalent to $*_1$.

Now we show that $*$ and $*_2$ are equivalent. Suppose that $*_2$ holds. Let U be a barrelled neighborhood of zero and let P be the Minkowski functional of U (U equals U_P, the closed unit ball of P). In accordance with $*_2$ corresponding to P there is a continuous seminorm Q such that $P(\sum^p) \leqq Q(\sum^q)$. If $\sum^q \in U_Q$ then $Q(\sum^q) \leqq 1$ and hence $P(\sum^p) \leqq 1$; that is, $\sum^p \in U_P$. To show that $*$ implies $*_2$ let P be a continuous seminorm and U_P its unit ball. There exists a U_Q (i.e., a barrelled neighborhood of zero with Minkowski functional Q) such that $\sum^q \in U_Q$ implies $\sum^p \in U_P$; or $Q\sum^q \leqq \varepsilon$ implies $P\sum^p \leqq \varepsilon$. If $Q\sum^q \equiv Q\sum_{k=1}^{q} a_{n_k} x_{n_k}$ is zero then $P\sum^p \leqq \varepsilon$ for every $\varepsilon > 0$ so that $P\sum^p = 0$. Hence $Q\sum^q = P\sum^p = 0$. If $Q\sum^q \neq 0$ then let $t = 1/Q\sum^q$. $Q(t\sum^q) = 1$; hence $P(t\sum^p) \leqq 1$. Upon substitution for t we get $P(\sum^p) \leqq Q(\sum^q)$.

To prove part (1) of the theorem, observe that $*_2$ is a stronger condition than that in Proposition 1.9 and hence $\{x_k\}$ is a Schauder basic sequence. To show that $\{x_k\}$ is unconditional, let U be a neighborhood of zero in E. In accordance with $*$ let V be the neighborhood corresponding to U. Let $\{f_k\} \subset \overline{[x_k]}^*$ be the family of coefficient functionals for $\{x_k\}$. Let $x \in \overline{[x_k]}$; then for n and m large enough $\sum_{k=m}^{n} f_k(x) x_k \in V$. Hence if $\{\delta_k\}$ is any element of d,

$\sum_{k=m}^{n}\delta_k f_k(x)x_k \in U$. It follows from Theorem 1.15 and the completeness of E that $\sum_{k=1}^{\infty} f_k(x)x_k$ is unconditionally convergent.

Part (2) is an immediate consequence of Theorem 1.18. Given a barrelled neighborhood of zero U, then $U_d \equiv \{x \in E: S_{n,\delta}(x) \in U$ for all $n \in \omega$ and $\delta \in d\}$ is also a neighborhood of zero, and U_d satisfies the requirements for V in $*$.

COROLLARY 1.22. *Let E be a barrelled space$_2$ with unconditional Schauder basis $\{x_k, f_k\}$ and let $\{y_k\}$ be any sequence in E and $\{\sigma_k\}$ a sequence in Φ such that $\sigma_n \cap \sigma_m = \varnothing$ if $m \neq n$. Define w_k to be the σ_k-partial sum of y_k; i.e., $w_k = \sum_{j\in\sigma_k} f_j(y_k)x_j$. Then given a neighborhood of zero U there is a neighborhood of zero V such that for arbitrary scalars t_k, arbitrary $\{\delta_k\} \in d$, and arbitrary n and m in $\{k: k \in \omega$ or $k = \infty\}$ with $n \geqq m$ we have $\sum_{k=1}^{n} t_k w_k \in V$ implies $\sum_{k=1}^{m}\delta_k t_k w_k \in U$. Thus if E is also sequentially complete and for each k, $w_k \neq 0$ then the sequence $\{w_k\}$ is unconditionally basic in E.*

Proof. Simply observe that if $a_{jk} = t_k f_j(y_k)$, then $\sum_{k=1}^{n} t_k w_k = \sum_{j\in\sigma_k, k\in\{1,2,\cdots,n\}} a_{jk}x_j$ and in the expansion the j's are all distinct by the null intersection requirement of the hypothesis. Then apply Theorem 1.19.

The Generalized Gurevic Theorem, which we have just proved, is of fundamental importance. We shall apply it often in the next section.

2. **Unconditional and shrinking bases.** As pointed out in the introductory remarks, we shall show that most of the results of James and Karlin are valid for complete barrelled spaces, barrelled spaces, or complete locally convex spaces. Even unqualified locally convex spaces are sufficient for a few of the more simple results. The first theorem follows easily from the remarks made after the definition of unconditional basis, page 12, and Proposition 1.3.

THEOREM 2.1. *Let E be a barrelled space with an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then $\{f_k\pi(x_k)\}$ is an unconditional Schauder basis for E^*.*

Hereafter we shall use "ω-complete" for "sequentially complete".

LEMMA 2.2. *Let E be a locally convex space$_2$ and $\{y_k\}$ a bounded sequence in E.*

(1) *If E is ω-complete, then the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent for each $t = \{t_k\} \in l^1$.*

(2) *If F is a subspace of l^1 such that for each $t \in F$ the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent, then the linear mapping $T: F \to E$ defined by $T(t) = \sum_{k=1}^{\infty} t_k y_k$ is continuous.*

Proof. To see that $\sum_{k=1}^{\infty} t_k y_k$ is convergent if E is ω-complete, let A_0 be a neighborhood of zero (where A is an equicontinuous subset of E^*). Since $\{y_k\}$ is bounded there is a positive scalar s such that $y_k \in sA_0$ for each k. For n and m large enough $\sum_{k=m}^{n} |t_k| < 1/s$. Since $|f(y_k)| \leqq s$ for each $f \in A$ and $k \in \omega$ it follows that $\sum_{k=m}^{n} t_k y_k \in A_0$. Hence the series is convergent.

Let A be equicontinuous and define $M = \sup\{|f(y_k)| : f \in A, k \in \omega\}$. If $t \in F$ and $\|t\| < 1/M + 1$ then for $f \in A$, $|fT(t)| = |\sum_{k=1}^{\infty} t_k f(y_k)| \leqq \sum_{k=1}^{\infty} |t_k| \cdot |f(y_k)| < 1$. Therefore $T(t) \in A_0$ so T is continuous.

LEMMA 2.3. *Let T be a linear transformation of a normed space E into a locally convex space F. If there exists a continuous functional f on F such that $f(\theta) = 0$ and a positive number m such that $|fT(x)| \geqq m\|x\|$ for each $x \in E$, then T is one-to-one and relatively open.*

Proof. If $T(x)$ is zero then so is $f(T(x))$ and hence by the given inequality, $x = \theta$; so T is one-to-one. To prove that T is relatively open we show that the inverse map $T^{-1}: T(E) \to E$ is bounded on a neighborhood of zero, vis. $f^{-1}[-1, 1] \cap T(E)$. Let $x \in T^{-1}(f^{-1}[-1, 1])$ then $T(x) \in f^{-1}[-1, 1]$ so that $|fT(x)| \leqq 1$. Using the given inequality we get $m\|x\| \leqq |fT(x)| \leqq 1$. Thus $\|x\| \leqq 1/m$ for all $x \in T^{-1}(f^{-1}[-1, 1])$; i.e., $T^{-1}(f^{-1}[-1, 1])$ is bounded and hence $T^{-1}: T(E) \to E$ is continuous.

LEMMA 2.4. *Let E be a locally convex space and F a subspace of E. If E^* is separable, then so is F^* (in the $s(F^*, F)$ topology).*

Proof. Let $F^1 = \{f \in E^*: f(x) = 0 \text{ for each } x \in F\}$. The continuous image of a separable topological space is separable. Thus each quotient space (with quotient topology) of a separable space is separable. Hence E^*/F^1 is separable. The canonical map $[f] \to f/F$ is an isomorphism mapping E^*/F^1 onto F^* (see [10, p. 120]). Furthermore this mapping is continuous (where E^*/F^1 has the quotient topology and F^* the $s(F^*, F)$ topology) as we now show. It suffices to show that the mapping $\phi: E^* \to F^*$ defined by $\phi(f) = f/F$ is continuous since this map is the composition of the quotient map $f \to [f]$ and the mapping $[f] \to f/F$ (see [10, Th. 5.7, p. 39]). If $A \subset F$ then denote by $A^{\circ F^*}$ the polar of A in F^* and use $A^{\circ E^*}$ for the polar of A in E^*. Let B be a bounded subset of F. Then it is not hard to see that $\phi^{-1}[B^{\circ F^*}] = B^{\circ E^*}$ and hence ϕ is continuous. Thus the canonical

mapping from E^*/F^1 onto F^* is continuous and therefore F^* is separable.

Define a non-l^1 space (non-c_0 space) to be a locally convex space which has no subspace topologically isomorphic to l^1 (c_0).

COROLLARY 2.5. *If E is a locally convex space and E^* is separable, then E is a non-l^1 space.*

Proof. If E contains a subspace F topologically isomorphic to l^1, then F^* is topologically isomorphic to m, the nonseparable space of bounded sequences. Thus by Lemma 2.4, E^* could not be separable.

The following is an immediate consequence of the corollary.

THEOREM 2.6. *Let E be a locally convex space$_2$ with a basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E is a non-l^1 space.*

To get a converse of Theorem 2.6, the space and basis must be strengthened.

THEOREM 2.7. *Let E be a ω-complete barrelled space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. If E is a non-l^1 space, then $\{x_k, f_k\}$ is a shrinking basis.*

Proof. Suppose that $\{x_k, f_k\}$ is not a shrinking basis. Then for some $g \in E^*$ the series $\sum_{k=1}^{\infty}[\pi(x_k)](g)f_k = \sum_{k=1}^{\infty} g(x_k)f_k$ is not strongly convergent. Thus since the series is weak* convergent (Proposition 1.1) it can not be strongly Cauchy. Therefore there is a strong neighborhood of zero B° (B is bounded in E) such that for each positive integer j there are integers n_j and m_j, $n_j > m_j > j$, with the property that $\sum_{k=m_j}^{n_j} g(x_k)f_k \notin B^0$; furthermore, the sequence $\{n_j\}_{j=1}^{\infty}$ and $\{m_j\}_{j=1}^{\infty}$ can be chosen so that $m_{j+1} > n_j$ for each j. This means that for each j there is a $z_j \in B$ with the property that $|\sum_{k=m_j}^{n_j} g(x_k)f_k(z_j)| > 1$; or, if we let $[m_j, n_j]$ denote just the integers between m_j and n_j then $|gS(z_j)_{[m_j,n_j]}| > 1$. Since the family $\{S_{[m_j,n_j]}\}_{j=1}^{\infty}$ is pointwise bounded (Proposition 1.12) it is equicontinuous because E is barrelled. Hence since $\{z_j\}$ is bounded, the set $\{S(z_j)_{[m_j,n_j]}\}_{j=1}^{\infty}$ is bounded in E. If we let $y_j = S(z_j)_{[m_j,n_j]}$ then we have shown that for each j, $|g(y_j)| > 1$ and the sequence $\{y_j\}$ is bounded. Let $t = \{t_j\} \in l^1$ and $t \neq 0$ and chooose the scalars ε_j so that $|t_j g(y_j)| = \varepsilon_j t_j g(y_j)$. By Lemma 2.2 the series $\sum_{k=1}^{\infty} t_k y_k$ is convergent and by the Generalized Gurevic Theorem (Corollary 1.22) the

sequence $\{y_j\}$ is unconditionally basic. Therefore the series $\sum_{j=1}^{\infty}\varepsilon_j t_j y_j$ is convergent (Theorem 1.12). Now we have $|g\sum_{j=1}^{\infty}\varepsilon_j t_j y_j| = \sum_{j=1}^{\infty}|t_j|\cdot|g(y_j)| > \sum_{j=1}^{\infty}|t_j|$. In accordance with Variation 1.20 let A be the equicontinuous subset of E^* which depends upon the given equicontinuous set $\{g\}$. Then via Variation 1.20 the inequality just shown implies that $|f_t\sum_{j=1}^{\infty}t_j y_j| > \sum_{j=1}^{\infty}|t_j|$ for some $f_t \in A$. If we define $T(t) = \sum_{j=1}^{\infty}t_j y_j$ then we have

(i) for each $t \in l^1$ there is an $f_t \in A$ such that $|f_t T(t)| \geqq ||t||_{l^1}$.

If Q is the Minkowski functional of A_0 then $|f(x)| \leqq Q(x)$ for each $x \in E$ and all $f \in A$. Now from (i) it follows that $|QT(t)| \geqq ||t||_{l^1}$. T is continuous by Lemma 2.2 and relatively open and one-to-one by Lemma 2.3. Thus the subspace $T(l^1)$ of E is topologically isomorphic to l^1.

From Theorem 2.7 and Corollary 2.5 we get the following.

COROLLARY 2.8. *Let E be a ω-complete barrelled space$_2$ with an unconditional Schauder basis. Then if E^* is separable, the basis is shrinking.*

THEOREM 2.9. *Let E be either a ω-complete locally convex space$_2$ or a barrelled space$_2$ (or any locally convex space$_2$ with the property that weak* bounded subsets of E^* are strongly bounded) and let E have an unconditional Schauder basis $\{x_k, f_k\}$. If E^* is weakly ω-complete (i.e., $w(E^*, E^{**})$-ω-complete) then $\{x_k, f_k\}$ is a shrinking basis.*

Proof. $\{f_k\}$ is a weak* unconditional basis so for each $f \in E^*$ the set of unordered partial sums $\{S_\sigma^*(f): \sigma \in \Phi\}$ is weak* bounded and consequently strongly bounded because of the conditions on E. E^* is weakly ω-complete so by Corollary 1.16, $\sum_{k=1}^{\infty}f(x_k)f_k$ is strongly unconditionally convergent; and since we know the series is weak* convergent to f, it must be strongly unconditionally convergent to f.

To get a converse (Theorem 2.12 or 2.13) of Theorem 2.9 we impose on E (or E^*) more restrictions which are provided by Theorem 2.11, a generalization of Day's Theorem 2 [2, p. 74].

LEMMA 2.10. *Let E be a locally convex space$_2$. If E is weakly sequentially complete, then E is a non-c_0 space.*

Proof. Suppose that E is weakly ω-complete but that there is a subspace F which is topologically isomorphic to c_0. As a complete subset of the Hausdorff space E, F is closed and, since it is convex,

weakly closed. Thus F is weakly ω-complete which is impossible since c_0 is not weakly ω-complete.

THEOREM 2.11. *Let E be a barrelled space, with an unconditional Schauder basis $\{x_k, f_k\}$. Then the statements below are equivalent.*

(1) *The basis is boundedly complete.*

(2) *E is weakly ω-complete.*

(3) *E is a ω-complete non-c_0 space.*

Proof. (1) $\rightarrow$ (2). Let $\{y_n\}$ be a weakly Cauchy sequence in E. Then $\{y_n\}$ is bounded and $\{f(y_n)\}_{n=1}^{\infty}$ is convergent for every $f \in E^*$. For each $k \in \omega$ let $a_k = \lim_{n\to\infty} f_k(y_n)$. Since $S_\sigma(y_n) - \sum_{k\in\sigma} a_k x_k = \sum_{k\in\sigma}(f_k(y_n) - a_k)x_k$, $s_\sigma(y_n)$ converges with n to $\sum_{k\in\sigma} a_k x_k$ (given a neighborhood of zero V choose the circled neighborhood W such that $W + \cdots + W \subset V$ where there are as many W's as the cardinal number of σ, and then pick $\varepsilon > 0$ so that $\varepsilon x_k \in W$ for each $k \in \sigma$; for n large enough, $|f_k(y_n) - a_k| < \varepsilon$ for all $k \in \sigma$ and so for such n, $\sum_{k\in\sigma}[f_k(y_n) - a_k] \cdot x_k \in W + \cdots + W \subset V$). Thus if U is a neighborhood of zero and V a barrel such that $V + V \subset U$ we have for each $\sigma \in \Phi$, an integer N_σ such that $\sum_{k\in\sigma} a_k x_k - S_\sigma(y_{N_\sigma}) \in V$. V_b is a neighborhood of zero (see Theorem 1.18) and since $\{y_n\}$ is bounded there is a scalar $s > 1$ such that $y_n \in sV_b$ for all n. Hence $S_\sigma(y_n) \in sV$ for all $\sigma \in \Phi$ and $n \in \omega$. Then for each $\sigma \in \Phi$ we have $\sum_{k\in\sigma} a_k x_k \in S_\sigma(y_{N_\sigma}) + V \subset sV + V \subset sU$, proving that $\{\sum_{k\in\sigma} a_k x_k\}_{\sigma\in\Phi}$ is bounded. Since $\{x_k, f_k\}$ is boundedly complete there is a $y \in E$ such that $a_k = f_k(y)$ for each $k \in \omega$, and recalling the definition of a_k we have $\lim_{n\to\infty} f_k(y_n - y) = 0$ for each $k \in \omega$.

We will prove that y is the weak sequential limit of $\{y_n\}$. Suppose it is not. Then $u_n = y_n - y$ does not tend weakly to θ and so there is an $h \in E^*$, an $\varepsilon > 0$, and a subsequence $\{z_m\}$ of $\{u_n\}$ such that

(i) $h(z_m) > \varepsilon$ for each $m \in \omega$.

In accordance with Variation 1.20 let B be the equicontinuous subset of E^* corresponding to $\{h\}$. By the result of the first paragraph of proof $\lim_{m\to\infty} S_\sigma(z_m) = \theta$. Therefore for the neighborhood of zero $U = \varepsilon/5[B \cup \{h\}]_0$ there is a positive integer m_0 such that $R_{m_0}(z_1) \in U$ and then there is a $n_1 > m_0$ such that $S_{m_0}(z_{n_1}) \in U$. By induction there are increasing sequences $\{m_k\}_{k=0}^{\infty}$ and $\{n_k\}_{k=1}^{\infty}$ such that

(ii) $|g(R_{m_k}(z_{n_k}))| \leqq \varepsilon/5$ and $|g(S_{m_k}(z_{n_{k+1}}))| \leqq \varepsilon/5$ for all $g \in B \cup \{h\}$ and $k \in \omega$.

For each positive integer k define $w_k = \sum_{k=m_{k-1}+1}^{m_k} f_k(z_{n_k})x_k$. Since $z_{n_k} - w_k = S_{m_{k-1}}(z_{n_k}) + R_{m_k}(z_{n_k})$ we have, using (ii),

(iii) $|g(w_k) - g(z_{n_k})| \leqq 2\varepsilon/5$ for all $g \in B \cup \{h\}$ and $k \in \omega$.

Now using (i) and (iii) we have $|h(w_k)| \geqq |h(z_{n_k})| - |h(z_{n_k}) - h(w_k)| > \varepsilon - 2\varepsilon/5$. Thus

(iv) $|h(w_k)| > 3\varepsilon/5$ for each $k \in \omega$.

Let F be the subspace of l^1 of finitely nonzero sequences. If $t = \{t_k\} \in F$ then define $T(t) = \sum_{k=1}^{\infty} t_k w_k$. For $t \neq 0$ pick the scalars ε_k so that $\varepsilon_k t_k h(w_k) = |t_k h(w_k)|$; then, $|h(\sum_{k=1}^{\infty} \varepsilon_k t_k w_k)| = \sum_{k=1}^{\infty} |t_k| \cdot |h(w_k)| > 3\varepsilon/5 \sum_{k=1}^{\infty} |t_k|$. By the generalized Gurevic Theorem there must be a $g_t \in B$ such that $|g_t(\sum_{k=1}^{\infty} t_k w_k)| > 3\varepsilon/5 \cdot \sum_{k=1}^{\infty} |t_k|$.

(v) For each $t \in F$ there is a $g_t \in B$ such that $|g_t(T(t))| \geqq 3\varepsilon/5 \, ||t||$.

Define $H(t) = \sum_{k=1}^{\infty} t_k z_{n_k}$ for each $t \in F$. From (iii) we get $|g(H(t) - T(t))| \leqq 2\varepsilon/5 \, ||t||$ for all $g \in B$; hence

$$|g(H(t))| \geqq |g(T(t))| - \frac{2\varepsilon}{5} ||t||.$$

Now using (v) we have

(vi) for $t \in F$ there is a $g_t \in B$ such that $|g_t(H(t))| \geqq \varepsilon/5 \, ||t||$.

If Q is the Minkowski functional of B_0 then $|g(x)| \leqq Q(x)$ for each $x \in E$ and all $g \in B$. Thus in view of (vi) we have $|Q(H(t))| \geqq \varepsilon/5 \, ||t||$ for every $t \in F$. It now follows from Lemmas 2.2 and 2.3 that H is a topological isomorphism from F into E.

Let $\overline{H(F)}$ be the closure of $H(F)$ in the completion $\hat{E}$ of E. Then $\overline{H(F)}$ is topologically isomorphic to l^1, and hence is weakly ω-complete. Since the weakly Cauchy sequence $\{z_{n_k}\}$ is contained in $H(F)$, it is weakly convergent to an element z of $\hat{E}$. Therefore $\{u_n\}$ is weakly convergent to z and we have for each $k \in \omega$, $f_k(z) = \lim_{n \to \infty} f_k(u_n) = 0$ by the final result of the first paragraph of proof. This implies that $z = \theta$ and hence $\{u_n\}$ does converge weakly to θ, contradicting the original assumption that it does not.

(2) $\rightarrow$ (3). Lemma 2.10.

(3) $\rightarrow$ (1). If $\{x_k\}$ is not boundedly complete then there is a sequence $\{a_k\}$ such that $D = \{\sum_{k=1}^{\infty} a_k x_k\}_{n=1}^{\infty}$ is bounded but the series does not converge and hence is not Cauchy since E is ω-complete. Thus there is an equicontinuous subset A of E^* and increasing sequences $\{m_k\}$ and $\{n_k\}$ with $m_{k+1} > n_k$ such that $y_k \equiv \sum_{j=m_k}^{n_k} a_j x_j \notin A_0$ for all $k \in \omega$. Hence, for each positive integer k there is a $g_k \in A$ such that

(i) $|g_k(y_k)| > 1$.

Let F be the subspace of c_0 of finitely nonzero sequences. Define $T: F \rightarrow [y_k]$ as follows: $T(t) = \sum_{k \in \sigma} t_k y_k$ where σ is the set of subscripts on which t is not 0. We shall show that T is continuous. Since $\{x_k\}$ is unconditional and D is bounded, the set of unordered partial sums $\{\sum_{k \in \sigma} a_k x_k : \sigma \in \Phi\}$ is bounded (if V is a barrel then V_b is also a neighborhood of zero and hence there is a scalar s such that $\sum_{k=1}^{n} a_k x_k \in s V_b$ for all $n \in \omega$; it follows that $\sum_{k \in \sigma} a_k x_k \in sV$ for all

$\sigma \in \Phi$). Hence if B_0 is a neighborhood of zero (B is equicontinuous) there is a number M such that $|f\sum_{k\in\sigma}a_kx_k| < M$ for all $f\in B$ and $\sigma\in\Phi$. Let $\rho = \bigcup_{k\in\sigma}\{j: m_k \leqq j \leqq n_k\}$; then $|\sum_{k\in\sigma}f(y_k)| = |f\sum_{k\in\sigma}y_k| = |f\sum_{j\in\rho}a_jx_j| < M$. Therefore $\sum_{k\in\sigma}|f(y_k)| \leqq 4M$ for all $f\in B$ and $\sigma\in\Phi$. Now suppose $t\in F$ and $\|t\| < 1/4M$. If $f\in B$ then $|f(T(t))| = |\sum_{k\in\sigma}t_kf(y_k)| \leqq \sum_{k\in\sigma}|t_k|\cdot|f(y_k)| \leqq \|t\|\cdot\sum_{k\in\sigma}|f(y_k)| < 1$. Therefore $T(t)\in B_0$, and we have proved that T is continuous.

Let t be a nonzero member of F and pick j so that $|t_j| = \|t\|$. Now choose $\{\alpha_k\}$ so that $\alpha_kt_kg_j(y_k) = |t_kg_j(y_k)|$. Then $|g_j\sum_{k\in\sigma}\alpha_kt_ky_k| = |\sum_{k\in\sigma}\alpha_kt_kg_j(y_k)| = \sum_{k\in\sigma}|t_k|\cdot|g_j(y_k)| \geqq |t_j|\cdot|g_j(y_j)| > |t_j| = \|t\|$. If B is the equicontinuous set corresponding to A then by the Generalized Gurevic Theorem there must be an $h_t\in B$ such that

(ii) $|h_tT(t)| > \|t\|$.

If Q is the Minkowski functional of B_0 then $|h(x)| \leqq Q(x)$ for all $x\in E$ and $h\in B$. Therefore, $|Q(T(t))| \geqq \|t\|$ for each $t\in F$. By Lemma 2.3 T is one-to-one and open and so T is a topological isomorphism of F onto $[y_k]$. Thus their completions, c_0 and $\overline{[y_k]}$, are topologically isomorphic.

A converse to 2.9 can now be proved. If $\{x_k, f_k\}$ is an unconditional shrinking basis in a barrelled space E, then $\{f_k\}$ is a strongly unconditional, boundedly complete basis for E^* (2.1 and 1.5). Therefore if E^* is a barrelled space (in the strong topology) then by Theorem 2.11, E^* is weakly ω-complete. Thus we have proved the following.

THEOREM 2.12. *Let E be a barrelled space$_2$ with an adjoint E^* which is also barrelled, and suppose that E has an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E^* is weakly ω-complete.*

If the adjoint of a locally convex metrizable space is separable (in the strong topology) then E^* is a bound space and hence barrelled [10, p. 217, 22.17]. Thus as a corollary to 2.12 we have the following.

THEOREM 2.13. *Let E be a barrelled metrizable space with an unconditional basis $\{x_k, f_k\}$. If $\{x_k, f_k\}$ is shrinking, then E^* is weakly ω-complete.*

We include a summary of the results on shrinking bases.

COROLLARY 2.14. *Let E be a locally convex space$_2$ with an unconditional Schauder basis $\{x_k, f_k\}$. Consider the following six conditions.*

(1) *$\{f_k, \pi(x_k)\}$ is a Schauder basis for the dual space E^*.*

(2) *$\{f_k, \pi(x_k)\}$ is a boundedly complete Schauder basis for E^*.*

(3) $\{f_k, \pi(x_k)\}$ *is an unconditional Schauder basis for* E^*.

(4) E *is a non-*l^1 *space.*

(5) E^* *is separable.*

(6) E^* *is weakly* ω*-complete.*

Then (5) *implies* (4) *(even if* E *has no basis) and so each of* (4) *and* (5) *is a necessary condition for* (1). (1), (2), *and* (3) *are equivalent if* E *is barrelled* ((1) *and* (2) *are equivalent even if the basis* $\{x_k\}$ *is conditional). Each of* (4) *and* (5) *is sufficient for* (1) *if* E *is a* ω*-complete barrelled space.* (6) *implies* (1) *if* E *is a locally convex space such that weak* bounded sets are strongly bounded*; *and* (1) *implies* (6) *if* E *is both barrelled and metrizable. Thus all six statements are eqvivalent if* E *is a Fréchet space.*

Recall (Proposition 1.6) that a barrelled space$_2$ with a Schauder basis is reflexive if and only if the basis is both boundedly complete and shrinking. Also recall that a reflexive space is barrelled and its adjoint is reflexive and barrelled. Thus in view of 2.11 and 2.14 we have the following.

THEOREM 2.15. *Let* E *be a* ω*-complete barrelled space*$_2$ *with an unconditional Schauder basis. Consider the following conditions on* E *and* E^*.

(1) E *is weakly* ω*-complete.*

(2) E *is a non-*c_0 *space.*

(3) E *is a non-*l^1 *space.*

(4) E^* *is weakly* ω*-complete.*

(5) E^* *is a non-*c_0 *space.*

(6) E^* *is a non-*l^1 *space.*

If E *is reflexive then* (1), (2), *and* (3) *hold. If in addition* E^* *is* ω*-complete (as it will be if* E *is a bound space) then all six statements hold. Then conjunction of* (1) *and* (3) *imply that* E *is reflexive as does also the conjunction of* (2) *and* (3).

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