

Pacific Journal of Mathematics

**EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM
CERTAIN SUBVARIETIES OF A POLYDISC**

HERBERT JAMES ALEXANDER

EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES OF A POLYDISC

HERBERT ALEXANDER

Let E be a subvariety of the unit polydisc

$$U^N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : |z_i| < 1, 1 \leq i \leq N\}$$

such that E is the zero set of a holomorphic function f on U^N , i.e., $E = Z(f)$ where $Z(f) = \{z \in U^N : f(z) = 0\}$. This amounts to saying that E is a subvariety of pure dimension $N - 1$. In [2] Walter Rudin proved that if E is bounded away from the torus $T^N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : |z_i| = 1, 1 \leq i \leq N\}$, then there is a bounded holomorphic function F on U^N such that $E = Z(F)$. Call such a subvariety E , that is, a pure $N - 1$ dimensional subvariety of U^N bounded from T^N , a *Rudin variety*. We are interested in the following question: When is it possible to extend every bounded holomorphic function on a Rudin variety E to one on U^N ? Examples show this is not always possible. We will say that a pure $N - 1$ dimensional subvariety E of U^N is a *special Rudin variety* if there exists an annular domain $Q^N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : r < |z_i| < 1, 1 \leq i \leq N\}$ for some $r(0 < r < 1)$ and a $\delta > 0$ such that

(i) $E \cap Q^N = \emptyset$ and

(ii) if $1 \leq k \leq N$ and $(z', \alpha, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $(z', \beta, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $\alpha \neq \beta$, then $|\alpha - \beta| \geq \delta$. Obviously (i) implies that a special Rudin variety is a Rudin variety. We have the

THEOREM. If E is a special Rudin variety in U^N , then there exists a bounded linear transformation $T: H^\infty(E) \rightarrow H^\infty(U^N)$ (where H^∞ is the corresponding Banach space of bounded holomorphic functions under sup norm) which extends each bounded holomorphic function on E to one on U^N .

REMARK. The proof of the theorem is a modification of the proof in [2] of Rudin's theorem: the changes reflecting the fact that we are dealing with an additive problem while Rudin's was of a multiplicative nature. I am further indebted to Professor Rudin for some comments (on a preliminary version of this paper) which led to improvement in the hypothesis of the theorem.

The following lemma is well-known and easy to prove.

LEMMA 1. If $0 < r < 1$ and $Q = \{\lambda \in \mathbb{C} : r < |\lambda| < 1\}$ and

$$h(\lambda) = \sum_{-\infty}^{\infty} a_n \lambda^n, \quad h_1(\lambda) = \sum_{-\infty}^{-1} a_n \lambda^n$$

for $\lambda \in Q$, then

$$\|h_1\|_Q \leq K \|h\|_Q$$

where $K (> 1)$ is a constant depending only on r .

If h is holomorphic on $Q^N = \{(z_1, \dots, z_N) : r < |z_i| < 1, 1 \leq i \leq N\}$ then h has a Laurent expansion

$$(1) \quad h(z_1, z_2, \dots, z_N) = \sum a(n_1, n_2, \dots, n_N) z_1^{n_1} z_2^{n_2} \dots z_N^{n_N}.$$

Following [2], we define $\pi_j h, 1 \leq j \leq N$, to be the holomorphic function on Q^N whose Laurent series is obtained by deleting in (1) all terms in which $n_j \geq 0$. Lemma 1 implies

LEMMA 2. $\|\pi_j h\|_{Q^N} \leq K \|h\|_{Q^N}.$

Proof of the theorem. Since E is a subvariety of U^N of pure dimension $N - 1$, there exists by [1, p. 251] a function f holomorphic on U^N such that at each point of U^N the germ of f generates the ideal of germs of holomorphic functions which vanish on the germ of E at the given point. In particular, $E = Z(f)$. We will show that $\partial f / \partial z_k \neq 0$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leq k \leq N$. We give the proof for $k = 1$, the other cases are identical. Let $(\alpha, \alpha') \in (U \times Q^{N-1}) \cap E$. Now f is regular in the first coordinate [1, p. 13] at (α, α') since otherwise $f(\zeta, \alpha')$ vanishes in a neighborhood of α and hence for $|\zeta| < 1$ and so $E = Z(f) \cong \{(\zeta, \alpha') : |\zeta| < 1\}$, contradicting (i) in the definition of a special Rudin variety. Thus we can apply the Weierstrass preparation theorem and write in some neighborhood of $(\alpha, \alpha'), f = \Omega p$ where Ω is invertible and p is a Weierstrass polynomial. Factor p into primes: $p = p_i^{e_i} \dots p_t^{e_t}$ where p and the p_i 's are of the form

$$(\zeta - \alpha)^n + a_{n-1}(\zeta')(\zeta - \alpha)^{n-1} + \dots + a_0(\zeta')$$

for (ζ, ζ') near (α, α') with $a_j(\alpha') = 0$. Now the degree of each p_i must be equal to 1 since otherwise there would exist $\zeta'_n \rightarrow \alpha'$ with ζ'_n off the discriminant locus of some p_i and so there would exist $\alpha_n \neq \beta_n$ near α with $p_i(\alpha_n, \zeta'_n) = 0 = p_i(\beta_n, \zeta'_n)$ and thus (α_n, ζ'_n) and (β_n, ζ'_n) are in $(U \times Q^{N-1}) \cap E$, but $\zeta'_n \rightarrow \alpha'$ implies $\alpha_n \rightarrow \alpha$ and $\beta_n \rightarrow \alpha$ and so $|\alpha_n - \beta_n| \rightarrow 0$, contradicting (ii). A similar argument also using (ii) shows that there cannot be more than one p_i and so $f = \Omega p_i^{e_i}$ near (α, α') . Finally, since the germ of f generates the ideal of E at (α, α') , e_i must be equal to 1. Thus $f(\zeta, \zeta') = \Omega(\zeta, \zeta')(\zeta - \alpha + a_0(\zeta'))$ and $\partial f / \partial \zeta(\alpha, \alpha') = \Omega(\alpha, \alpha') \neq 0$ as required.

Now by Theorem 1 of [2] applied to $E = Z(f)$ there is a bounded holomorphic function F on U^N such that $E = Z(F)$. Examination of the

construction in [2] shows that $1/F$ is bounded on Q^N since $F = f_1 e^{g-g_1}$ on Q^N and $1/f_1$ and $|\operatorname{Re}(g - g_1)|$ are bounded on Q^N . We will show that there is an $\varepsilon > 0$ such that $|\partial F/\partial z_k| > \varepsilon$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leq k \leq N$. We do this for $k = 1$, the finitely many other cases are identical. From [2], $F = fe^g$ for some g and so $\partial f/\partial z_1 \neq 0$ on $(U \times Q^{N-1}) \cap E$ implies $\partial F/\partial z_1 \neq 0$ there. Now for $z' \in Q^{N-1}$

$$z' \rightarrow \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\partial F/\partial z_1(\zeta, z')}{F(\zeta, z')} d\zeta$$

is a continuous integer-valued function and so is a constant m_1 giving the number of zeros for $F(\cdot, z')$ in U . Since these zeros are the points of $(U \times Q^{N-1}) \cap E$ and $\partial F/\partial z_1 \neq 0$ there, it follows that the m_1 zeros $\alpha_1(z'), \dots, \alpha_{m_1}(z')$ are distinct simple zeros. By (ii) then, $|\alpha_i(z') - \alpha_j(z')| \geq \delta$ for $i \neq j$. Write $F(\cdot, z') = BH$, where B is the Blaschke product with zeros at $\alpha_1(z'), \dots, \alpha_{m_1}(z')$. Now since $1/F$ is bounded on Q^N $1/H$ is bounded on U . But on E , $\partial F/\partial z_1 = \partial B/\partial z_1 \cdot H$ and since

$$|\alpha_i(z') - \alpha_j(z')| \geq \delta, \partial B/\partial z_1$$

is bounded from zero on E by some constant depending on δ , and as H is also bounded from zero independently of z' , it follows that $\partial F/\partial z_1$ is bounded from zero on $(U \times Q^{N-1}) \cap E$.

Let $d = \operatorname{dist}(E, Q^N)$ which we may assume is positive by increasing r if need be. Let g be a bounded holomorphic function on E . We shall extend g to a bounded function on U^N . By the general Oka-Cartan theory [1], there is a holomorphic extension G of g to U^N ; G need not be bounded. Since $F \neq 0$ on Q^N , we may define a function h_1 on $U \times Q^{N-1}$ as follows: Let $(z_1, z') \in U \times Q^{N-1}$. Choose a circle Γ about 0 lying in Q and enclosing z_1 with positive orientation and set

$$h_1(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta.$$

h_1 is clearly independent of the choice of Γ and holomorphic on $U \times Q^{N-1}$. We claim that $G/F - h_1$ is bounded on Q^N . Let $(z_1, z') \in Q^N$ where $z_1 \in Q$, $z' \in Q^{N-1}$. Let $\Gamma_1, \Gamma_2, \dots, \Gamma_{m_1}$ be small circles about $\alpha_1(z'), \dots, \alpha_{m_1}(z')$, the zeros of $F(\cdot, z')$. Then the Cauchy integral formula reads

$$(G/F)(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma - \Gamma_1 - \dots - \Gamma_{m_1}} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta.$$

Therefore

$$(G/F - h_1)(z_1, z') = - \sum_1^{m_1} \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta.$$

Clearly for $r_k = \text{radius of } \Gamma_k$,

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta \\ = & \frac{1}{2\pi i} \int_{|\zeta - \alpha_k(z')| = r_k} \frac{G(\zeta, z')}{\zeta - z_1} \frac{\zeta - \alpha_k(z')}{F(\zeta, z') - F(\alpha_k(z'), z')} \frac{d\zeta}{\zeta - \alpha_k(z')} \\ \rightarrow & \frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1) \frac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')} \quad \text{as } r_k \rightarrow 0. \end{aligned}$$

So letting the radii of the Γ_k go to zero we get

$$(G/F - h_1)(z_1, z') = - \sum_{k=1}^{m_1} \frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1) \frac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')}.$$

Since $(\alpha_k(z'), z') \in (U \times Q^{N-1}) \cap E$, recalling the significance of d and ε we get

$$\|G/F - h_1\|_{Q^N} \leq \frac{m_1 \|g\|_E}{d\varepsilon}.$$

In the same way for each $i, 1 < i \leq N$ we have an integer m_i and a function h_i holomorphic on $Q^{i-1} \times U \times Q^{N-i}$ such that

$$\|G/F - h_i\|_{Q^N} \leq \frac{m_i \|g\|_E}{d\varepsilon}.$$

Now let $m = \max \{m_i : 1 \leq i \leq N\}$ and let $A = m/d\varepsilon$. Subtracting in the above, we get $\|h_1 - h_i\|_{Q^N} \leq 2A \|g\|_E$. Now following [2] closely, set $h = (1 - \pi_1)(1 - \pi_2) \cdots (1 - \pi_N)h_1$. Since $\pi_i h = 0, h$ extends (uniquely) to a holomorphic function on U^N . Since h_j is holomorphic on

$$Q^{j-1} \times U \times Q^{N-j}, \pi_j h_j = 0$$

and so $\pi_j h_1 = \pi_j(h_1 - h_j)$ and therefore by Lemma 2,

$$\|\pi_j h_1\|_{Q^N} = \|\pi_j(h_1 - h_j)\|_{Q^N} \leq K \|h_1 - h_j\|_{Q^N} \leq 2KA \|g\|_E.$$

Now, since $h - h_1 = - \sum \pi_i h_1 + \sum \pi_i \pi_j h_1 - + \cdots$ and since we get by induction and by use of Lemma 2 that $\|\pi_{i_1} \pi_{i_2} \cdots \pi_{i_s} h_1\|_{Q^N} \leq 2K^s A \|g\|_E$, it follows that $\|h - h_1\|_{Q^N} \leq BA \|g\|_E$ where B depends only on K . Now consider $\bar{G} = G - Fh$. \bar{G} is holomorphic on U^N and extends g since G does. On $Q^N, \bar{G} = F(G/F - h_1) + F(h_1 - h)$. Therefore $\|\bar{G}\|_{Q^N} \leq \|F\|_{U^N A} \|g\|_E + \|F\|_{U^N B A} \|g\|_E$. Thus \bar{G} is bounded on U^N and $\|\bar{G}\|_{U^N} \leq \gamma \|g\|_E$ where $\gamma = A(1 + B) \|F\|_{U^N}$ is independent of g .

Next we show that \bar{G} does not depend on the choice of G made at the beginning of the construction. Suppose \tilde{G} were another (not necessarily bounded) extension of g to U^N . As above we get

$$\tilde{h}_1 = \frac{1}{2\pi i} \int_r \frac{\tilde{G}/F}{\zeta - z_1} d\zeta .$$

But then on $U \times Q^{N-1}$

$$(2) \quad h_1 - \tilde{h}_1 = \frac{1}{2\pi i} \int \frac{(G - \tilde{G})/F}{\zeta - z_1} d\zeta .$$

Since for $z' \in Q^{N-1}$, $(G - \tilde{G})(\cdot, z')$ vanishes at $\alpha_1(z'), \dots, \alpha_{m_1}(z')$ and since $F(\cdot, z')$ has simple zeros and only at these points, $(G - \tilde{G})/F(\cdot, z')$ is holomorphic on U and the right hand side of (2) equals $(G - \tilde{G})/F$ and so on $U \times Q^{N-1}$

$$(3) \quad h_1 - \tilde{h}_1 = (G - \tilde{G})/F .$$

Since the left hand side of (3) is holomorphic on $U \times Q^{N-1}$, so is the right and consequently $(G - \tilde{G})/F = (1 - \pi_1)((G - \tilde{G})/F)$ on Q^N . In the same way we see that for each j , $(G - \tilde{G})/F = (1 - \pi_j)((G - \tilde{G})/F)$ on Q^N . Therefore on Q^N we have

$$(G - \tilde{G})/F = \prod_{j=1}^N (1 - \pi_j)(G - \tilde{G})/F = \prod_{j=1}^N (1 - \pi_j)(h_1 - \tilde{h}_1) = h - \tilde{h} .$$

Thus $G - Fh = \tilde{G} - F\tilde{h}$ on Q^N and so on U^N . Since the extensions thus coincide, we have a well-defined map $T: H^\infty(E) \rightarrow H^\infty(U^N)$ such that $\|T(g)\|_{U^N} \leq \gamma \|g\|_E$.

To see that T is linear, let g and \tilde{g} be bounded holomorphic functions on E and let λ be a complex number. Let G and \tilde{G} respectively be arbitrary holomorphic extensions to U^N . Let $\tilde{h}_1, h_1, \tilde{h}_1$ and \tilde{h}, h, \tilde{h} be the h_1 and the h for $G + \lambda\tilde{G}, G$ and \tilde{G} respectively. Then

$$\begin{aligned} \tilde{\tilde{h}}_1 &= \frac{1}{2\pi i} \int \frac{(G + \lambda\tilde{G})/F}{\zeta - z_1} d\zeta \\ &= \frac{1}{2\pi i} \int \frac{G/F}{\zeta - z_1} d\zeta + \lambda \cdot \frac{1}{2\pi i} \int \frac{\tilde{G}}{\zeta - z_1} d\zeta = h_1 + \lambda\tilde{h}_1 \end{aligned}$$

and $\tilde{\tilde{h}} = \Pi(1 - \pi_j)\tilde{\tilde{h}}_1 = [\Pi(1 - \pi_j)](h_1 + \lambda\tilde{h}_1) = h + \lambda\tilde{h}$. Therefore

$$\begin{aligned} T(g + \lambda\tilde{g}) &= (G + \lambda\tilde{G}) - F(h + \lambda\tilde{h}) \\ &= (G - Fh) + \lambda(\tilde{G} - F\tilde{h}) = T(g) + \lambda T(\tilde{g}) . \end{aligned}$$

EXAMPLE. Let E be the Rudin variety in U^2 given by $E = Z((z_2 - \frac{1}{2})(z_1z_2 - \frac{1}{2}))$. Then E is the disjoint union of $Z(z_2 - \frac{1}{2})$ and $Z(z_1z_2 - \frac{1}{2})$. Let $g \in H^\infty(E)$ be given by

$$g|Z\left(z_2 - \frac{1}{2}\right) = 0 \quad \text{and} \quad g|Z\left(z_1z_2 - \frac{1}{2}\right) = 1 .$$

Then g admits no bounded holomorphic extension to U^2 . For if G were a bounded extension of g to U^2 we would have for $z \in U, z$ near 1,

$$\begin{aligned} 1 &= G\left(z, \frac{1}{2z}\right) - G\left(z, \frac{1}{2}\right) = \frac{1}{2\pi i} \int_{|\zeta|=1} G(z, \zeta) \left(\frac{1}{\zeta - \frac{1}{2z}} - \frac{1}{\zeta - \frac{1}{2}} \right) d\zeta \\ &= \left(\frac{1}{2z} - \frac{1}{2} \right) \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{G(z, \zeta)}{\left(\zeta - \frac{1}{2z} \right) \left(\zeta - \frac{1}{2} \right)} d\zeta. \end{aligned}$$

But as $z \rightarrow 1$, the integral is bounded and $(1/2z) - (1/2) \rightarrow 0$, a contradiction.

REFERENCES

1. Robert C. Gunning and Hugo Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Englewood Cliffs, New Jersey, 1965.
2. Walter Rudin, *Zero-sets in polydiscs*, Bull. Amer. Math. Soc. **73** (1967), 580-583.

Received January 8, 1968. The research for this paper was partially supported by the following contracts: NONR 222 (85) and NONR 3656 (08).

UNIVERSITY OF MICHIGAN

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

H. ROYDEN
Stanford University
Stanford, California

J. DUGUNDJI
Department of Mathematics
University of Southern California
Los Angeles, California 90007

R. R. PHELPS
University of Washington
Seattle, Washington 98105

RICHARD ARENS
University of California
Los Angeles, California 90024

ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA
CALIFORNIA INSTITUTE OF TECHNOLOGY
UNIVERSITY OF CALIFORNIA
MONTANA STATE UNIVERSITY
UNIVERSITY OF NEVADA
NEW MEXICO STATE UNIVERSITY
OREGON STATE UNIVERSITY
UNIVERSITY OF OREGON
OSAKA UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY
UNIVERSITY OF TOKYO
UNIVERSITY OF UTAH
WASHINGTON STATE UNIVERSITY
UNIVERSITY OF WASHINGTON
* * *
AMERICAN MATHEMATICAL SOCIETY
CHEVRON RESEARCH CORPORATION
TRW SYSTEMS
NAVAL WEAPONS CENTER

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.

Pacific Journal of Mathematics

Vol. 29, No. 3

July, 1969

| | |
|--|-----|
| Herbert James Alexander, <i>Extending bounded holomorphic functions from certain subvarieties of a polydisc</i> | 485 |
| Edward T. Cline, <i>On an embedding property of generalized Carter subgroups</i> | 491 |
| Roger Cuppens, <i>On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum. II</i> | 521 |
| William Richard Emerson, <i>Translation kernels on discrete Abelian groups</i> | 527 |
| Robert William Gilmer, Jr., <i>Power series rings over a Krull domain</i> | 543 |
| Julien O. Hennefeld, <i>The Arens products and an imbedding theorem</i> | 551 |
| James Secord Howland, <i>Embedded eigenvalues and virtual poles</i> | 565 |
| Bruce Ansgar Jensen, <i>Infinite semigroups whose non-trivial homomorphs are all isomorphic</i> | 583 |
| Michael Joseph Kascic, Jr., <i>Polynomials in linear relations. II</i> | 593 |
| J. Gopala Krishna, <i>Maximum term of a power series in one and several complex variables</i> | 609 |
| Renu Chakravarti Laskar, <i>Eigenvalues of the adjacency matrix of cubic lattice graphs</i> | 623 |
| Thomas Anthony Mc Cullough, <i>Rational approximation on certain plane sets</i> | 631 |
| T. S. Motzkin and Ernst Gabor Straus, <i>Divisors of polynomials and power series with positive coefficients</i> | 641 |
| Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix.</i> | 653 |
| Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix. II</i> | 663 |
| Donald Steven Passman, <i>Exceptional 3/2-transitive permutation groups</i> | 669 |
| Grigorios Tsagas, <i>A special deformation of the metric with no negative sectional curvature of a Riemannian space</i> | 715 |
| Joseph Zaks, <i>Trivially extending decompositions of E^n</i> | 727 |