EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES OF A POLYDISC

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Let $E$ be a subvariety of the unit polydisc 

$$U^N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N: |z_i| < 1, 1 \leq i \leq N\}$$

such that $E$ is the zero set of a holomorphic function $f$ on $U^N$, i.e., $E = Z(f)$ where $Z(f) = \{z \in U^N: f(z) = 0\}$. This amounts to saying that $E$ is a subvariety of pure dimension $N - 1$. In [2] Walter Rudin proved that if $E$ is bounded away from the torus $T^N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N: |z_i| = 1, 1 \leq i \leq N\}$, then there is a bounded holomorphic function $F$ on $U^N$ such that $E = Z(F)$.

Call such a subvariety $E$, that is, a pure $N - 1$ dimensional subvariety of $U^N$ bounded from $\Gamma^N$, a Rudin variety. We are interested in the following question: When is it possible to extend every bounded holomorphic function on a Rudin variety $E$ to one on $U^N$?

Examples show this is not always possible. We will say that a pure $N - 1$ dimensional subvariety $E$ of $U^N$ is a special Rudin variety if there exists an annular domain $Q^N = \{(z_1, \ldots, z_N) \in \mathbb{C}^N: r < |z_i| < 1, 1 \leq i \leq N\}$ for some $r(0 < r < 1)$ and a $\delta > 0$ such that

(i) $E \cap Q^N = \emptyset$ and

(ii) if $1 \leq k \leq N$ and $(z', \alpha, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $(z', \beta, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $\alpha \neq \beta$, then $|\alpha - \beta| \geq \delta$.

Obviously (i) implies that a special Rudin variety is a Rudin variety. We have the

**Theorem.** If $E$ is a special Rudin variety in $U^N$, then there exists a bounded linear transformation $T: H^\infty(E) \to H^\infty(U^N)$ (where $H^\infty$ is the corresponding Banach space of bounded holomorphic functions under sup norm) which extends each bounded holomorphic function on $E$ to one on $U^N$.

**Remark.** The proof of the theorem is a modification of the proof in [2] of Rudin’s theorem: the changes reflecting the fact that we are dealing with an additive problem while Rudin’s was of a multiplicative nature. I am further indebted to Professor Rudin for some comments (on a preliminary version of this paper) which led to improvement in the hypothesis of the theorem.

The following lemma is well-known and easy to prove.

**Lemma 1.** If $0 < r < 1$ and $Q = \{\lambda \in \mathbb{C}: r < |\lambda| < 1\}$ and 

$$h(\lambda) = \sum_{n=0}^\infty a_n \lambda^n, h_1(\lambda) = \sum_{n=1}^\infty a_n \lambda^n$$

are analytic in $Q$, then 

$$h_1(\lambda) = \frac{1}{1 - \lambda} (h(\lambda) - a_0).$$
for $\lambda \in Q$, then

$$|| h_i ||_q \leq K || h ||_q$$

where $K (> 1)$ is a constant depending only on $r$.

If $h$ is holomorphic on $Q^r = \{ (z_1, \cdots, z_N) : r < |z_i| < 1, 1 \leq i \leq N \}$ then $h$ has a Laurent expansion

\begin{equation}
(1) \quad h(z_1, z_2, \cdots, z_N) = \sum a(n_1, n_2, \cdots, n_N)z_1^{n_1}z_2^{n_2} \cdots z_N^{n_N}.
\end{equation}

Following [2], we define $\pi_j h_j, 1 \leq j \leq N$, to be the holomorphic function on $Q^r$ whose Laurent series is obtained by deleting in (1) all terms in which $n_i \neq 0$. Lemma 1 implies

**Lemma 2.** $|| \pi_j h ||_{q^N} \leq K || h ||_{q^N}$.

**Proof of the theorem.** Since $E$ is a subvariety of $U^N$ of pure dimension $N - 1$, there exists by [1, p. 251] a function $f$ holomorphic on $U^N$ such that at each point of $U^N$ the germ of $f$ generates the ideal of germs of holomorphic functions which vanish on the germ of $E$ at the given point. In particular, $E = Z(f)$. We will show that $\partial f/\partial z_k \neq 0$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leq k \leq N$. We give the proof for $k = 1$, the other cases are identical. Let $(\alpha, \alpha') \in (U \times Q^{N-1}) \cap E$. Now $f$ is regular in the first coordinate [1, p. 13] at $(\alpha, \alpha')$ since otherwise $f(\zeta, \zeta')$ vanishes in a neighborhood of $\alpha$ and hence for $|\zeta| < 1$ and so $E = Z(f) \supseteq (|\zeta| < 1)$, contradicting (i) in the definition of a special Rudin variety. Thus we can apply the Weierstrass preparation theorem and write in some neighborhood of $(\alpha, \alpha')$, $f = \Omega p$ where $\Omega$ is invertible and $p$ is a Weierstrass polynomial. Factor $p$ into primes:

$p = p_1^{\alpha_1} \cdots p_i^{\alpha_i}$

where $p_i$ and the $p_i$'s are of the form

$$(\zeta - \alpha)^n + a_{n-1}(\zeta')(\zeta - \alpha)^{n-1} + \cdots + a_0(\zeta')$$

for $(\zeta, \zeta')$ near $(\alpha, \alpha')$ with $a_j(\alpha') = 0$. Now the degree of each $p_i$ must be equal to 1 since otherwise there would exist $\zeta_n \rightarrow \alpha'$ with $\zeta'_n$ off the discriminant locus of some $p_i$ and so there would exist $\alpha_n \neq \beta_n$ near $\alpha$ with $p_i(\alpha_n, \zeta'_n) = 0 = p_i(\beta_n, \zeta'_n)$ and thus $(\alpha_n, \zeta'_n)$ and $(\beta_n, \zeta'_n)$ are in $(U \times Q^{N-1}) \cap E$, but $\zeta'_n \rightarrow \alpha'$ implies $\alpha_n \rightarrow \alpha$ and $\beta_n \rightarrow \alpha$ and so $|\alpha_n - \beta_n| \rightarrow 0$, contradicting (ii). A similar argument also using (ii) shows that there cannot be more than one $p_i$ near $(\alpha, \alpha')$. Finally, since the germ of $f$ generates the ideal of $E$ at $(\alpha, \alpha')$, $e_i$ must be equal to 1. Thus $f(\zeta, \zeta') = \Omega(\zeta, \zeta')(\zeta - \alpha + a_0(\zeta'))$ and $\partial f/\partial \zeta(\alpha, \alpha') = \Omega(\alpha, \alpha') \neq 0$ as required.

Now by Theorem 1 of [2] applied to $E = Z(f)$ there is a bounded holomorphic function $F$ on $U^N$ such that $E = Z(F)$. Examination of the
construction in [2] shows that $1/F$ is bounded on $Q^N$ since $F = f_e^{e^{-\sigma_1}}$ on $Q^N$ and $1/f_i$ and $|\text{Re} (g - g_i)|$ are bounded on $Q^N$. We will show that there is an $\varepsilon > 0$ such that $|\partial F/\partial z_k| > \varepsilon$ on $(Q^{N-k} \times U \times Q^{N-k}) \cap E$ for $1 \leq k \leq N$. We do this for $k = 1$, the finitely many other cases are identical. From [2], $F = fe^g$ for some $g$ and so $\partial f/\partial z_i \neq 0$ on $(U \times Q^{N-k}) \cap E$ implies $\partial F/\partial z_i \neq 0$ there. Now for $z' \in Q^{N-1}$

$$z' \mapsto \frac{1}{2\pi i} \int_{|z| = r} \frac{\partial F/\partial z_1(\zeta, z')}{F(\zeta, z')} \, d\zeta$$

is a continuous integer-valued function and so is a constant $m_1$ giving the number of zeros for $F(\cdot, z')$ in $U$. Since these zeros are the points of $(U \times Q^{N-1}) \cap E$ and $\partial F/\partial z_i \neq 0$ there, it follows that the $m_i$ zeros $\alpha_i(z'), \cdots, \alpha_m(z')$ are distinct simple zeros. By (ii) then, $|\alpha_i(z') - \alpha_j(z')| \geq \delta$ for $i \neq j$. Write $F(\cdot, z') = BH$, where $B$ is the Blaschke product with zeros at $\alpha_i(z'), \cdots, \alpha_m(z')$. Now since $1/F$ is bounded on $Q^N 1/H$ is bounded on $U$. But on $E$, $\partial F/\partial z_i = \partial B/\partial z_i$, $H$ and since

$$|\alpha_i(z') - \alpha_j(z')| \geq \delta, \partial B/\partial z_i$$

is bounded from zero on $E$ by some constant depending on $\delta$, and as $H$ is also bounded from zero independently of $z'$, it follows that $\partial F/\partial z_i$ is bounded from zero on $(U \times Q^{N-1}) \cap E$.

Let $d = \text{dist}(E, Q^N)$ which we may assume is positive by increasing $r$ if need be. Let $g$ be a bounded holomorphic function on $E$. We shall extend $g$ to a bounded function on $U^N$. By the general Oka-Cartan theory [1], there is a holomorphic extension $G$ of $g$ to $U^N$; $G$ need not be bounded. Since $F \neq 0$ on $Q^N$, we may define a function $h_i$ on $U \times Q^{N-1}$ as follows: Let $(z_1, z') \in U \times Q^{N-1}$. Choose a circle $\Gamma$ about 0 lying in $Q$ and enclosing $z_1$ with positive orientation and set

$$h_i(z_1, z') = \frac{1}{2\pi i} \int_{|z| = r} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} \, d\zeta.$$ 

$h_i$ is clearly independent of the choice of $\Gamma$ and holomorphic on $U \times Q^{N-1}$. We claim that $G/F - h_i$ is bounded on $Q^N$. Let $(z_1, z') \in Q^N$ where $z_1 \in Q$, $z' \in Q^{N-1}$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_m$ be small circles about $\alpha_i(z'), \cdots, \alpha_m(z')$, the zeros of $F(\cdot, z')$. Then the Cauchy integral formula reads

$$(G/F)(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma_1 \cdots \Gamma_m} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} \, d\zeta.$$ 

Therefore

$$(G/F - h_i)(z_1, z') = -\sum_{1}^{m_1} \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} \, d\zeta.$$ 

Clearly for $r_k = \text{radius of } \Gamma_k$,
\[
\frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta = \frac{1}{2\pi i} \int_{\zeta - \alpha_\kappa(z') = r_k} \frac{G(\zeta, z')}{\zeta - z_1} \frac{\zeta - \alpha_\kappa(z')}{F(\zeta, z') - F(\alpha_\kappa(z'), z')} \frac{d\zeta}{\zeta - \alpha_\kappa(z')}
\]

So letting the radii of the \(\Gamma_k\) go to zero we get

\[
(G/F - h_i)(z_i, z') = -\sum_{k=1}^{m_i} \frac{g(\alpha_\kappa(z'), z')}{(\alpha_\kappa(z') - z_i) \frac{\partial F}{\partial z_i}(\alpha_\kappa(z'), z')}
\]

Since \((\alpha_\kappa(z'), z') \in (U \times Q^{N-1}) \cap E\), recalling the significance of \(d\) and \(\varepsilon\) we get

\[
\| G/F - h_i \|_{Q^N} \leq \frac{m_i \| g \|_E}{d\varepsilon}.
\]

In the same way for each \(i, 1 \leq i \leq N\) we have an integer \(m_i\) and a function \(h_i\) holomorphic on \(Q^{i-1} \times U \times Q^{N-i}\) such that

\[
\| G/F - h_i \|_{Q^N} \leq \frac{m_i \| g \|_E}{d\varepsilon}.
\]

Now let \(m = \max\{m_i; 1 \leq i \leq N\}\) and let \(A = m/d\varepsilon\). Subtracting in the above, we get \(\| h_i - h_i \|_{Q^N} \leq 2A \| g \|_E\). Now following [2] closely, set \(h = (1 - \pi_j)(1 - \pi_\kappa) \cdots (1 - \pi_N)h_{i_j}\). Since \(\pi_i h = 0\), \(h\) extends (uniquely) to a holomorphic function on \(U^N\). Since \(h_j\) is holomorphic on \(Q^{N-1} \times U \times Q^{N-j}\), \(\pi_j h_j = 0\)

and so \(\pi_j h_i = \pi_j(h_i - h_j)\) and therefore by Lemma 2,

\[
\| \pi_j h_i \|_{Q^N} = \| \pi_j(h_i - h_j) \|_{Q^N} \leq K \| h_i - h_j \|_{Q^N} \leq 2KA \| g \|_E.
\]

Now, since \(h - h_i = -\sum \pi_i h_i + \sum \pi_i \pi_j h_i - + \cdots\) and since we get by induction and by use of Lemma 2 that \(\| \pi_i \pi_j \cdots \pi_{i_j} h_i \|_{Q^N} \leq 2K^A \| g \|_E\), it follows that \(\| h - h_i \|_{Q^N} \leq BA \| g \|_E\) where \(B\) depends only on \(K\).

Now consider \(\bar{G} = G - F h_i\). \(\bar{G}\) is holomorphic on \(U^N\) and extends \(g\) since \(G\) does. On \(Q^N\), \(\bar{G} = F(\bar{G}/F - h_i) + F(h_i - h)\). Therefore \(\| \bar{G} \|_{Q^N} \leq \| F \|_{C^N A} \| g \|_E + \| F \|_{C^N B A} \| g \|_E\). Thus \(\bar{G}\) is bounded on \(U^N\) and \(\| \bar{G} \|_{C^N} \leq \gamma \| g \|_E\) where \(\gamma = A(1 + B) \| F \|_{C^N A}\) is independent of \(g\).

Next we show that \(\bar{G}\) does not depend on the choice of \(G\) made at the beginning of the construction. Suppose \(\bar{G}\) were another (not necessarily bounded) extension of \(g\) to \(U^N\). As above we get
\[ \tilde{h}_i = \frac{1}{2\pi i} \int_{\Gamma} \frac{G/F}{\zeta - z_i} d\zeta. \]

But then on \( U \times Q^{N-1} \)

\[ (2) \quad h_i - \tilde{h}_i = \frac{1}{2\pi i} \int_{\Gamma} \frac{(G - \tilde{G})/F}{\zeta - z_i} d\zeta. \]

Since for \( z' \in Q^{N-1}, (G - \tilde{G})(\cdot, z') \) vanishes at \( \alpha_i(z'), \ldots, \alpha_m(z') \) and since \( F(\cdot, z') \) has simple zeros and only at these points, \( (G - \tilde{G})/F(\cdot, z') \) is holomorphic on \( U \) and the right hand side of (2) equals \( (G - \tilde{G})/F \) and so on \( U \times Q^{N-1} \)

\[ (3) \quad h_i - \tilde{h}_i = (G - \tilde{G})/F. \]

Since the left hand side of (3) is holomorphic on \( U \times Q^{N-1} \), so is the right and consequently \( (G - \tilde{G})/F = (1 - \pi) ((G - \tilde{G})/F) \) on \( Q^N \). In the same way we see that for each \( j, (G - \tilde{G})/F = (1 - \pi_j) ((G - \tilde{G})/F) \) on \( Q^N \). Therefore on \( Q^N \) we have

\[ (G - \tilde{G})/F = \prod_{j=1}^{N} (1 - \pi_j) ((G - \tilde{G})/F) = \prod_{j=1}^{N} (1 - \pi_j) (h_i - \tilde{h}_i) = h - \tilde{h}. \]

Thus \( G - Fh = \tilde{G} - F\tilde{h} \) on \( Q^N \) and so on \( U^N \). Since the extensions thus coincide, we have a well-defined map \( T: H^\infty(E) \rightarrow H^\infty(U^N) \) such that \( \|T(g)\|_{U^N} \leq \gamma \|g\|_E \).

To see that \( T \) is linear, let \( g \) and \( \tilde{g} \) be bounded holomorphic functions on \( E \) and let \( \lambda \) be a complex number. Let \( G \) and \( \tilde{G} \) respectively be arbitrary holomorphic extensions to \( U^N \). Let \( \tilde{h}_i, h_i, \tilde{h}_i \) and \( \tilde{h}, h, \tilde{h} \) be the \( h_i \) and the \( h \) for \( G + \lambda \tilde{G}, G \) and \( \tilde{G} \) respectively. Then

\[ \tilde{h}_i = \frac{1}{2\pi i} \int_{\Gamma} \frac{(G + \lambda \tilde{G})/F}{\zeta - z_i} d\zeta \]

\[ = \frac{1}{2\pi i} \int_{\Gamma} \frac{G/F}{\zeta - z_i} d\zeta + \lambda \cdot \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{G}}{\zeta - z_i} d\zeta = h_i + \lambda \tilde{h}_i \]

and \( \tilde{h} = \Pi (1 - \pi_j) \tilde{h}_i = [\Pi (1 - \pi_j)] (h_i + \lambda \tilde{h}_i) = h + \lambda \tilde{h} \). Therefore

\[ T(g + \lambda \tilde{g}) = (G + \lambda \tilde{G}) - F(h + \lambda \tilde{h}) \]

\[ = (G - Fh) + \lambda (\tilde{G} - F\tilde{h}) = T(g) + \lambda T(\tilde{g}). \]

**Example.** Let \( E \) be the Rudin variety in \( U^2 \) given by \( E = Z((z_1 - \frac{1}{2})(z_1 z_2 - \frac{1}{2})) \). Then \( E \) is the disjoint union of \( Z(z_2 - \frac{1}{2}) \) and \( Z(z_1 z_2 - \frac{1}{2}) \). Let \( g \in H^\infty(E) \) be given by

\[ g \mid Z(z_1 z_2 - \frac{1}{2}) = 0 \quad \text{and} \quad g \mid Z(z_2 - \frac{1}{2}) = 1 \.]
Then $g$ admits no bounded holomorphic extension to $U^2$. For if $G$ were a bounded extension of $g$ to $U^2$ we would have for $z \in U, z$ near 1,

$$1 = G(z, \frac{1}{2z}) - G(z, \frac{1}{2}) = \frac{1}{2\pi i} \int_{|\zeta|=1} G(z, \zeta) \left( \frac{1}{\zeta - \frac{1}{2z}} - \frac{1}{\zeta - \frac{1}{2}} \right) d\zeta$$

$$= \left( \frac{1}{2z} - \frac{1}{2} \right) \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{G(z, \zeta)}{\left( \zeta - \frac{1}{2z} \right) \left( \zeta - \frac{1}{2} \right)} d\zeta.$$ 

But as $z \to 1$, the integral is bounded and $(1/2z) - (1/2) \to 0$, a contradiction.

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