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# EXTENDING BOUNDED HOLOMORPHIC FUNCTIONS FROM CERTAIN SUBVARIETIES OF A POLYDISC

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**Let $E$ be a subvariety of the unit polydisc**

$$U^N = \{(z_1, \cdots, z_N) \in C^N: |z_i| < 1, 1 \leqq i \leqq N\}$$

**such that $E$ is the zero set of a holomorphic function $f$ on $U^N$, i.e., $E = Z(f)$ where $Z(f) = \{z \in U^N: f(z) = 0\}$. This amounts to saying that $E$ is a subvariety of pure dimension $N-1$. In [2] Walter Rudin proved that if $E$ is bounded away from the torus $T^N = \{(z_1, \cdots, z_N) \in C^N: |z_i| = 1, 1 \leqq i \leqq N\}$, then there is a bounded holomorphic function $F$ on $U^N$ such that $E = Z(F)$. Call such a subvariety $E$, that is, a pure $N-1$ dimensional subvariety of $U^N$ bounded from $T^N$, a *Rudin variety*. We are interested in the following question: When is it possible to extend every bounded holomorphic function on a Rudin variety $E$ to one on $U^N$? Examples show this is not always possible. We will say that a pure $N-1$ dimensional subvariety $E$ of $U^N$ is a *special Rudin variety* if there exists an annular domain $Q^N = \{(z_1, \cdots, z_N) \in C^N: r < |z_i| < 1, 1 \leqq i \leqq N\}$ for some $r (0 < r < 1)$ and a $\delta > 0$ such that**

**(i) $E \cap Q^N = \emptyset$ and**

**(ii) if $1 \leqq k \leqq N$ and $(z', \alpha, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $(z', \beta, z'') \in (Q^{k-1} \times U \times Q^{N-k}) \cap E$ and $\alpha \neq \beta$, then $|\alpha - \beta| \geqq \delta$. Obviously (i) implies that a special Rudin variety is a Rudin variety. We have the**

**THEOREM. If $E$ is a special Rudin variety in $U^N$, then there exists a bounded linear transformation $T: H^\infty(E) \to H^\infty(U^N)$ (where $H^\infty$ is the corresponding Banach space of bounded holomorphic functions under sup norm) which extends each bounded holomorphic function on $E$ to one on $U^N$.**

REMARK. The proof of the theorem is a modification of the proof in [2] of Rudin's theorem: the changes reflecting the fact that we are dealing with an additive problem while Rudin's was of a multiplicative nature. I am further indebted to Professor Rudin for some comments (on a preliminary version of this paper) which led to improvement in the hypothesis of the theorem.

The following lemma is well-known and easy to prove.

LEMMA 1. *If $0 < r < 1$ and $Q = \{\lambda \in C: r < |\lambda| < 1\}$ and*

$$h(\lambda) = \sum_{-\infty}^{\infty} a_n \lambda^n,\ h_1(\lambda) = \sum_{-\infty}^{-1} a_n \lambda^n$$

*for* $\lambda \in Q$, *then*

$$\|h_1\|_Q \leqq K\|h\|_Q$$

*where* $K$ $(>1)$ *is a constant depending only on* $r$.

If $h$ is holomorphic on $Q^N = \{(z_1, \cdots, z_N): r < |z_i| < 1, 1 \leqq i \leqq N\}$ then $h$ has a Laurent expansion

$$(1) \qquad h(z_1, z_2, \cdots, z_N) = \sum a(n_1, n_2, \cdots, n_N) z_1^{n_1} z_2^{n_2} \cdots z_N^{n_N} .$$

Following [2], we define $\pi_j h$, $1 \leqq j \leqq N$, to be the holomorphic function on $Q^N$ whose Laurent series is obtained by deleting in (1) all terms in which $n_j \geqq 0$. Lemma 1 implies

LEMMA 2. $\|\pi_j h\|_{Q^N} \leqq K\|h\|_{Q^N}$.

*Proof of the theorem.* Since $E$ is a subvariety of $U^N$ of pure dimension $N-1$, there exists by [1, p. 251] a function $f$ holomorphic on $U^N$ such that at each point of $U^N$ the germ of $f$ generates the ideal of germs of holomorphic functions which vanish on the germ of $E$ at the given point. In particular, $E = Z(f)$. We will show that $\partial f/\partial z_k \neq 0$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leqq k \leqq N$. We give the proof for $k = 1$, the other cases are identical. Let $(\alpha, \alpha') \in (U \times Q^{N-1}) \cap E$. Now $f$ is regular in the first coordinate [1, p. 13] at $(\alpha, \alpha')$ since otherwise $f(\zeta, \alpha')$ vanishes in a neighborhood of $\alpha$ and hence for $|\zeta| < 1$ and so $E = Z(f) \supseteqq \{(\zeta, \alpha'): |\zeta| < 1\}$, contradicting (i) in the definition of a special Rudin variety. Thus we can apply the Weierstrass preparation theorem and write in some neighborhood of $(\alpha, \alpha')$, $f = \Omega p$ where $\Omega$ is invertible and $p$ is a Weierstrass polynomial. Factor $p$ into primes: $p = p_1^{e_1} \cdots p_t^{e_t}$ where $p$ and the $p_i$'s are of the form

$$(\zeta - \alpha)^n + a_{n-1}(\zeta')(\zeta - \alpha)^{n-1} + \cdots + a_0(\zeta')$$

for $(\zeta, \zeta')$ near $(\alpha, \alpha')$ with $a_j(\alpha') = 0$. Now the degree of each $p_i$ must be equal to 1 since otherwise there would exist $\zeta'_n \to \alpha'$ with $\zeta'_n$ off the discriminant locus of some $p_i$ and so there would exist $\alpha_n \neq \beta_n$ near $\alpha$ with $p_i(\alpha_n, \zeta'_n) = 0 = p_i(\beta_n, \zeta'_n)$ and thus $(\alpha_n, \zeta'_n)$ and $(\beta_n, \zeta'_n)$ are in $(U \times Q^{N-1}) \cap E$, but $\zeta'_n \to \alpha'$ implies $\alpha_n \to \alpha$ and $\beta_n \to \alpha$ and so $|\alpha_n - \beta_n| \to 0$, contradicting (ii). A similar argument also using (ii) shows that there cannot be more than one $p_i$ and so $f = \Omega p_1^{e_1}$ near $(\alpha, \alpha')$. Finally, since the germ of $f$ generates the ideal of $E$ at $(\alpha, \alpha')$, $e_1$ must be equal to 1. Thus $f(\zeta, \zeta') = \Omega(\zeta, \zeta')(\zeta - \alpha + a_0(\zeta'))$ and $\partial f/\partial \zeta(\alpha, \alpha') = \Omega(\alpha, \alpha') \neq 0$ as required.

Now by Theorem 1 of [2] applied to $E = Z(f)$ there is a bounded holomorphic function $F$ on $U^N$ such that $E = Z(F)$. Examination of the construction in [2] shows that $1/F$ is bounded on $Q^N$ since $F = f_1 e^{g-g_1}$ on $Q^N$ and $1/f_1$ and $|\operatorname{Re}(g - g_1)|$ are bounded on $Q^N$. We will show that there is an $\varepsilon > 0$ such that $|\partial F/\partial z_k| > \varepsilon$ on $(Q^{k-1} \times U \times Q^{N-k}) \cap E$ for $1 \leqq k \leqq N$. We do this for $k = 1$, the finitely many other cases are identical. From [2], $F = fe^g$ for some $g$ and so $\partial f/\partial z_1 \neq 0$ on $(U \times Q^{N-1}) \cap E$ implies $\partial F/\partial z_1 \neq 0$ there. Now for $z' \in Q^{N-1}$

$$z' \longrightarrow \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\partial F/\partial z_1(\zeta, z')}{F(\zeta, z')} d\zeta$$

is a continuous integer-valued function and so is a constant $m_1$ giving the number of zeros for $F(\cdot, z')$ in $U$. Since these zeros are the points of $(U \times Q^{N-1}) \cap E$ and $\partial F/\partial z_1 \neq 0$ there, it follows that the $m_1$ zeros $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$ are distinct simple zeros. By (ii) then, $|\alpha_i(z') - \alpha_j(z')| \geqq \delta$ for $i \neq j$. Write $F(\cdot, z') = BH$, where $B$ is the Blaschke product with zeros at $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$. Now since $1/F$ is bounded on $Q^N$ $1/H$ is bounded on $U$. But on $E$, $\partial F/\partial z_1 = \partial B/\partial z_1 \cdot H$ and since

$$|\alpha_i(z') - \alpha_j(z')| \geqq \delta, \partial B/\partial z_1$$

is bounded from zero on $E$ by some constant depending on $\delta$, and as $H$ is also bounded from zero independently of $z'$, it follows that $\partial F/\partial z_1$ is bounded from zero on $(U \times Q^{N-1}) \cap E$.

Let $d = \operatorname{dist}(E, Q^N)$ which we may assume is positive by increasing $r$ if need be. Let $g$ be a bounded holomorphic function on $E$. We shall extend $g$ to a bounded function on $U^N$. By the general Oka-Cartan theory [1], there is a holomorphic extension $G$ of $g$ to $U^N$; $G$ need not be bounded. Since $F \neq 0$ on $Q^N$, we may define a function $h_1$ on $U \times Q^{N-1}$ as follows: Let $(z_1, z') \in U \times Q^{N-1}$. Choose a circle $\Gamma$ about 0 lying in $Q$ and enclosing $z_1$ with positive orientation and set

$$h_1(z_1, z') = \frac{1}{2\pi i} \int_\Gamma \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

$h_1$ is clearly independent of the choice of $\Gamma$ and holomorphic on $U \times Q^{N-1}$. We claim that $G/F - h_1$ is bounded on $Q^N$. Let $(z_1, z') \in Q^N$ where $z_1 \in Q$, $z' \in Q^{N-1}$. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_{m_1}$ be small circles about $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$, the zeros of $F(\cdot, z')$. Then the Cauchy integral formula reads

$$(G/F)(z_1, z') = \frac{1}{2\pi i} \int_{\Gamma - \Gamma_1 - \cdots \Gamma_{m_1}} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

Therefore

$$(G/F - h_1)(z_1, z') = - \sum_1^{m_1} \frac{1}{2\pi i} \int_{\Gamma_k} \frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1} d\zeta .$$

Clearly for $r_k$ = radius of $\Gamma_k$,

$$\frac{1}{2\pi i}\int_{\Gamma_k}\frac{G(\zeta, z')/F(\zeta, z')}{\zeta - z_1}\,d\zeta$$
$$= \frac{1}{2\pi i}\int_{|\zeta-\alpha_k(z')|=r_k}\frac{G(\zeta, z')}{\zeta - z_1}\,\frac{\zeta - \alpha_k(z')}{F(\zeta, z') - F(\alpha_k(z'), z')}\,\frac{d\zeta}{\zeta - \alpha_k(z')}$$
$$\longrightarrow \frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1)\dfrac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')} \qquad \text{as } r_k \to 0\ .$$

So letting the radii of the $\Gamma_k$ go to zero we get

$$(G/F - h_1)(z_1, z') = -\sum_{k=1}^{m_1}\frac{g(\alpha_k(z'), z')}{(\alpha_k(z') - z_1)\dfrac{\partial F}{\partial \zeta_1}(\alpha_k(z'), z')}\ .$$

Since $(\alpha_k(z'), z') \in (U \times Q^{N-1}) \cap E$, recalling the significance of $d$ and $\varepsilon$ we get

$$\| G/F - h_1 \|_{Q^N} \leqq \frac{m_1 \| g \|_E}{d\varepsilon}\ .$$

In the same way for each $i, 1 < i \leqq N$ we have an integer $m_i$ and a function $h_i$ holomorphic on $Q^{i-1} \times U \times Q^{N-i}$ such that

$$\| G/F - h_i \|_{Q^N} \leqq \frac{m_i \| g \|_E}{d\varepsilon}\ .$$

Now let $m = \max\{m_i\colon 1 \leqq i \leqq N\}$ and let $A = m/d\varepsilon$. Subtracting in the above, we get $\| h_1 - h_i \|_{Q^N} \leqq 2A \| g \|_E$. Now following [2] closely, set $h = (1 - \pi_1)(1 - \pi_2) \cdots (1 - \pi_N)h_1$. Since $\pi_i h = 0$, $h$ extends (uniquely) to a holomorphic function on $U^N$. Since $h_j$ is holomorphic on

$$Q^{j-1} \times U \times Q^{N-j},\ \pi_j h_j = 0$$

and so $\pi_j h_1 = \pi_j(h_1 - h_j)$ and therefore by Lemma 2,

$$\| \pi_j h_1 \|_{Q^N} = \| \pi_j(h_1 - h_j \|_{Q^N} \leqq K \| h_1 - h_j \|_{Q^N} \leqq 2KA \| g \|_E\ .$$

Now, since $h - h_1 = -\sum \pi_i h_1 + \sum \pi_i \pi_j h_1 - + \cdots$ and since we get by induction and by use of Lemma 2 that $\| \pi_{i_1}\pi_{i_2} \cdots \pi_{i_S} h_1 \|_{Q^N} \leqq 2K^S A \| g \|_E$, *it* follows that $\| h - h_1 \|_{Q^N} \leqq BA \| g \|_E$ where $B$ depends only on $K$. Now consider $\bar{G} = G - Fh$. $\bar{G}$ is holomorphic on $U^N$ and extends $g$ since $G$ does. On $Q^N, \bar{G} = F(G/F - h_1) + F(h_1 - h)$. Therefore $\| \bar{G} \|_{Q^N} \leqq \| F \|_{U^N} A \| g \|_E + \| F \|_{U^N} BA \| g \|_E$. Thus $\bar{G}$ is bounded on $U^N$ and $\| \bar{G} \|_{U^N} \leqq \gamma \| g \|_E$ where $\gamma = A(1 + B) \| F \|_{U^N}$ is independent of $g$.

Next we show that $\bar{G}$ does not depend on the choice of $G$ made at the beginning of the construction. Suppose $\tilde{G}$ were another (not necessarily bounded) extension of $g$ to $U^N$. As above we get

$$\tilde{h}_1 = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{G}/F}{\zeta - z_1}\,d\zeta\,.$$

But then on $U \times Q^{N-1}$

$$(2)\qquad h_1 - \tilde{h}_1 = \frac{1}{2\pi i}\int \frac{(G-\tilde{G})/F}{\zeta - z_1}\,d\zeta\,.$$

Since for $z' \in Q^{N-1}$, $(G - \tilde{G})(\cdot, z')$ vanishes at $\alpha_1(z'), \cdots, \alpha_{m_1}(z')$ and since $F(\cdot, z')$ has simple zeros and only at these points, $(G - \tilde{G})/F(\cdot, z')$ is holomorphic on $U$ and the right hand side of (2) equals $(G - \tilde{G})/F$ and so on $U \times Q^{N-1}$

$$(3)\qquad h_1 - \tilde{h}_1 = (G - \tilde{G})/F\,.$$

Since the left hand side of (3) is holomorphic on $U \times Q^{N-1}$, so is the right and consequently $(G-\tilde{G})/F = (1-\pi_1)((G-\tilde{G})/F)$ on $Q^N$. In the same way we see that for each $j$, $(G-\tilde{G})/F = (1-\pi_j)((G-\tilde{G})/F)$ on $Q^N$. Therefore on $Q^N$ we have

$$(G-\tilde{G})/F = \prod_{j=1}^{N}(1-\pi_j)(G-\tilde{G})/F = \prod_{j=1}^{N}(1-\pi_j)(h_1-\tilde{h}_1) = h - \tilde{h}\,.$$

Thus $G - Fh = \tilde{G} - F\tilde{h}$ on $Q^N$ and so on $U^N$. Since the extensions thus coincide, we have a well-defined map $T\colon H^\infty(E) \to H^\infty(U^N)$ such that $\|T(g)\|_{U^N} \leqq \gamma \|g\|_E$.

To see that $T$ is linear, let $g$ and $\tilde{g}$ be bounded holomorphic functions on $E$ and let $\lambda$ be a complex number. Let $G$ and $\tilde{G}$ respectively be arbitrary holomorphic extensions to $U^N$. Let $\tilde{\tilde{h}}_1, h_1, \tilde{h}_1$ and $\tilde{\tilde{h}}, h, \tilde{h}$ be the $h_1$ and the $h$ for $G + \lambda\tilde{G}$, $G$ and $\tilde{G}$ respectively. Then

$$\begin{aligned}\tilde{\tilde{h}}_1 &= \frac{1}{2\pi i}\int \frac{(G+\lambda\tilde{G})/F}{\zeta - z_1}\,d\zeta \\ &= \frac{1}{2\pi i}\int \frac{G/F}{\zeta - z_1}\,d\zeta + \lambda\cdot\frac{1}{2\pi i}\int \frac{\tilde{G}}{\zeta - z_1}\,d\zeta = h_1 + \lambda\tilde{h}_1\end{aligned}$$

and $\tilde{\tilde{h}} = \Pi(1-\pi_j)\tilde{h}_1 = [\Pi(1-\pi_j)](h_1 + \lambda\tilde{h}_1) = h + \lambda\tilde{h}$. Therefore

$$\begin{aligned}T(g + \lambda\tilde{g}) &= (G + \lambda\tilde{G}) - F(h + \lambda\tilde{h}) \\ &= (G - Fh) + \lambda(\tilde{G} - F\tilde{h}) = T(g) + \lambda T(\tilde{g})\,.\end{aligned}$$

EXAMPLE. Let E be the Rudin variety in $U^2$ given by $E = Z((z_2 - \frac{1}{2})(z_1z_2 - \frac{1}{2}))$. Then $E$ is the disjoint union of $Z(z_2 - \frac{1}{2})$ and $Z(z_1z_2 - \frac{1}{2})$. Let $g \in H^\infty(E)$ be given by

$$g \mid Z\Big((z_2 - \frac{1}{2}\Big) = 0 \quad\text{and}\quad g \mid Z\Big(z_1z_2 - \frac{1}{2}\Big) = 1\,.$$

Then $g$ admits no bounded holomorphic extension to $U^2$. For if $G$ were a bounded extension of $g$ to $U^2$ we would have for $z \in U$, $z$ near 1,

$$1 = G\left(z, \frac{1}{2z}\right) - G\left(z, \frac{1}{2}\right) = \frac{1}{2\pi i}\int_{|\zeta|=1} G(z, \zeta)\left(\frac{1}{\zeta - \frac{1}{2z}} - \frac{1}{\zeta - \frac{1}{2}}\right)d\zeta$$

$$= \left(\frac{1}{2z} - \frac{1}{2}\right)\frac{1}{2\pi i}\int_{|\zeta|=1} \frac{G(z, \zeta)}{\left(\zeta - \frac{1}{2z}\right)\left(\zeta - \frac{1}{2}\right)}\, d\zeta \,.$$

But as $z \to 1$, the integral is bounded and $(1/2z) - (1/2) \to 0$, a contradiction.

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