

Pacific Journal of Mathematics

**ON THE DECOMPOSITION OF INFINITELY DIVISIBLE
CHARACTERISTIC FUNCTIONS WITH CONTINUOUS
POISSON SPECTRUM. II**

ROGER CUPPENS

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Let f be an infinitely divisible characteristic function whose spectral functions are absolutely continuous functions with almost everywhere continuous derivatives. Some necessary conditions that f belong to the class I_0 of the infinitely divisible characteristic functions without indecomposable factors have been obtained by Cramér, Shimizu and the author and a sufficient condition that f belong to I_0 has been given by Ostrovskiy. In the present work, we prove that the condition of Ostrovskiy is not only a sufficient, but also a necessary condition that f belong to I_0 .

Let f be the function of the variable t defined by

$$(1) \quad \log f(t) = \int_{-\infty}^0 [e^{itu} - 1 - itu(1 + u^2)^{-1}] \varphi(u) du \\
 + \int_0^{\infty} [e^{itu} - 1 - itu(1 + u^2)^{-1}] \psi(u) du$$

where \log means the branch of logarithm defined by continuity from $\log f(0) = 0$ and where φ and ψ are almost everywhere nonnegative and continuous functions which are defined respectively on $]-\infty, 0[$ and $]0, +\infty[$ and satisfy the condition

$$\int_{-\varepsilon}^0 u^2 \varphi(u) du + \int_0^{\varepsilon} u^2 \psi(u) du < +\infty$$

for any $\varepsilon > 0$. If we let

$$M(x) = \int_{-\infty}^x \varphi(u) du \quad x < 0, \\
 N(x) = - \int_x^{+\infty} \psi(u) du \quad x > 0,$$

then we see that the conditions of the Lévy representation theorem ([4], Th. 5.5.2) are satisfied, so that f is an infinitely divisible characteristic function. In [3], we have proved the following result.

If the two following conditions are satisfied:

- (a) $\varphi(u) \geq k$ a.e. for $-c(1 + 2^{-n}) < u < -c$,
- (b) $\psi(u) \geq k$ a.e. for $d < u < d(1 + 2^{-n})$,

where k, c and d are positive constants and n is a positive integer,

then the function f defined by (1) has an indecomposable factor.
 The following theorem completes this result.

THEOREM 1. *If*

$$\psi(u) \geq k \text{ a.e. for } c < u < c(1 + 2^{-n}) \text{ and } d < u < d(1 + 2^{-n})$$

where n is a positive integer and k, c and $d \geq 2c$ are positive constants, then the function f defined by (1) has an indecomposable factor.

This theorem is an immediate consequence of the

LEMMA. *Let f be the characteristic function defined by*

$$\log f(t) = \int_0^\infty (e^{itu} - 1 - itu(1 + u^2)^{-1})\alpha(u)du$$

where

$$\alpha(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ 0 & \text{otherwise} \end{cases}$$

c being a positive constant, $\lambda = 1 + 2^{-n}$ (n positive integer) and $r \geq 2\lambda$. Then f has an indecomposable factor.

Proof. Let β be the function defined by

$$\beta(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ -c\varepsilon & \text{if } \gamma < u < \delta \\ 0 & \text{otherwise} \end{cases}$$

($2 < \gamma < \delta < 2\lambda$) and α_m and β_m be the functions defined by

$$\alpha_1(x) = \alpha(x); \alpha_m(x) = \int_{-\infty}^\infty \alpha_{m-1}(x-t)\alpha_1(t)dt$$

$$\beta_1(x) = \beta(x); \beta_m(x) = \int_{-\infty}^\infty \beta_{m-1}(x-t)\beta_1(t)dt.$$

We prove easily by induction that

$$(2) \quad \beta_m(x) = \alpha_m(x) \geq 0 \quad \text{if } x \in [A_m, B_m]$$

where A_m and B_m are defined by

$$A_m = m + 2^{-n}$$

$$B_m = mr\lambda - 2^{-n}.$$

We prove now that

$$(3) \quad \lim_{\varepsilon \rightarrow 0} \sup_{A_m \leq x \leq B_m} |\alpha_m(x) - \beta_m(x)| = 0 .$$

Indeed, if $\varepsilon < 1$, we have

$$\begin{aligned} |\alpha_m(x)| &\leq c^m(r\lambda - 1)^{m-1} \\ |\beta_m(x)| &\leq c^m(r\lambda - 1)^{m-1} \end{aligned}$$

and from these formulae and from

$$\begin{aligned} \alpha_m(x) - \beta_m(x) &= \int_{-\infty}^{\infty} [\alpha_{m-1}(x-t)\alpha_1(t) - \beta_{m-1}(x-t)\beta_1(t)]dt \\ &= \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)[\alpha_1(t) - \beta_1(t)]dt - \int_{-\infty}^{+\infty} [\beta_{m-1}(x-t) - \alpha_{m-1}(x-t)]\beta_1(t)dt \end{aligned}$$

it follows by induction that

$$|\alpha_m(x) - \beta_m(x)| \leq \varepsilon(2c)^m(r\lambda - 1)^{m-1}$$

and this implies (3).

Let now $S(\alpha_m)$ be the spectrum of α_m . From the definition of α_m , it follows easily that

$$S(\alpha_m) = \bigcup_{j=0}^m [j + (m-j)r, (j + (m-j)r)\lambda] .$$

This implies that $S(\alpha_m)$ is all the interval $[m, mr\lambda]$ if

$$m > K = [(r-1)(2^n + 1)]$$

(here $[x]$ means the integer part of x) and therefore

$$(4) \quad \inf_{A_m \leq x \leq B_m} \alpha_m(x) > 0 \quad m = K + 2, K + 3, \dots .$$

From (3) and (4), it follows that

$$(5) \quad \beta_m(x) \geq 0 \quad m = K + 2, K + 3, \dots, 2K + 3$$

if ε is small enough. But, from the definition of β_m , we have for $k < m$

$$\beta_m(x) = \int_{-\infty}^{\infty} \beta_{m-k}(x-t)\beta_k(t)dt$$

so that, from (5)

$$(6) \quad \beta_m(x) \geq 0 \quad m \geq K + 2$$

if ε is small enough.

We consider now β_m for $m \leq K + 1$. β_m can be negative only on intervals of the kind

$$I = [j + kr + l\gamma, (j + kr)\lambda + l\delta]$$

where j and k are nonnegative integers and l a positive integer satisfying

$$j + k + l = m$$

and on I we have

$$|\beta_m(x)| \leq \varepsilon e^m (r\lambda - 1)^{m-1}.$$

But we have

$$j + 2l + kr < j + kr + l\gamma < (j + kr)\lambda + l\delta < (j + 2l + kr)\lambda$$

so that α_{m+l} is positive on I . Therefore, using (3), we have

$$\sum_{\substack{1 \leq j \leq k+1 \\ j \neq m-l}} \frac{\beta_j(x)}{j!} + \frac{\beta_{m+l}(x)}{(m+l)!} > 0$$

for $x \in I$ if ε is small enough. This implies that

$$\sum_{j=1}^{2K+2} \frac{\beta_j(x)}{j!} \geq 0$$

for any x and therefore from (6)

$$(7) \quad \sum_{j=1}^{\infty} \frac{\beta_j(x)}{j!} \geq 0$$

for any x if $\varepsilon > 0$ is small enough.

Let now g be the function defined by

$$\log g(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu(1 + u^2)^{-1})\beta(u)du.$$

Then

$$g(t) = \int_{-\infty}^{\infty} e^{itx} dG(x)$$

where G is the function

$$G(x) = e^{-\lambda} \left\{ \chi(x + \eta) + \int_{-\infty}^x \left[\sum_{n=1}^{\infty} \frac{\beta_n(y + \eta)}{n!} \right] dy \right\}.$$

Here χ is the degenerate distribution and λ and η are defined by

$$\begin{aligned} \lambda &= \int_{-\infty}^{\infty} \beta(u)du \\ \eta &= \int_{-\infty}^{\infty} u(1 + u^2)^{-1}\beta(u)du. \end{aligned}$$

From (7), it follows that g is a characteristic function if ε is small enough. Since g is not infinitely divisible, from the Khintchine's theorem ([4], Th. 6.2.2), g has an indecomposable factor and since g divides f , the lemma is proved.

As consequences of the Theorem 1, we obtain the following results which are respectively the results of Cramér [1] and Shimizu [6] quoted in the introduction.

COROLLARY 1. *If in an interval $[0, r]$ ($r > 0$), $\psi(u) \geq c > 0$ almost everywhere, then the function f defined by (1) has an indecomposable factor.*

COROLLARY 2. *If in an interval $[r, s]$ ($s > 2r > 0$), $\psi(u) \geq c > 0$ almost everywhere, then the function f defined by (1) has an indecomposable factor.*

The characterization announced in the introduction is the following.

THEOREM 2. *A necessary and sufficient condition that the function f defined by (1) has no indecomposable factor is the existence of an $r > 0$ such that one of the two following conditions is satisfied:*

- (a) $\varphi(u) \equiv 0$ a.e.; $\psi(u) = 0$ a.e. if $u \in [r, 2r]$;
- (b) $\psi(u) \equiv 0$ a.e.; $\varphi(u) = 0$ a.e. if $u \in [-2r, -r]$.

Proof. The sufficiency is a consequence of the Theorem 1 of Ostrovskiy [4] (see also [2], Th. 8.2), while the necessity follows immediately from the preceding theorem and from the Theorem 1 of [3] stated above.

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Pacific Journal of Mathematics

Vol. 29, No. 3

July, 1969

Herbert James Alexander, <i>Extending bounded holomorphic functions from certain subvarieties of a polydisc</i>	485
Edward T. Cline, <i>On an embedding property of generalized Carter subgroups</i>	491
Roger Cuppens, <i>On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum. II</i>	521
William Richard Emerson, <i>Translation kernels on discrete Abelian groups</i>	527
Robert William Gilmer, Jr., <i>Power series rings over a Krull domain</i>	543
Julien O. Hennefeld, <i>The Arens products and an imbedding theorem</i>	551
James Secord Howland, <i>Embedded eigenvalues and virtual poles</i>	565
Bruce Ansgar Jensen, <i>Infinite semigroups whose non-trivial homomorphs are all isomorphic</i>	583
Michael Joseph Kascic, Jr., <i>Polynomials in linear relations. II</i>	593
J. Gopala Krishna, <i>Maximum term of a power series in one and several complex variables</i>	609
Renu Chakravarti Laskar, <i>Eigenvalues of the adjacency matrix of cubic lattice graphs</i>	623
Thomas Anthony Mc Cullough, <i>Rational approximation on certain plane sets</i>	631
T. S. Motzkin and Ernst Gabor Straus, <i>Divisors of polynomials and power series with positive coefficients</i>	641
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix.</i>	653
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix. II</i>	663
Donald Steven Passman, <i>Exceptional $3/2$-transitive permutation groups</i>	669
Grigorios Tsagas, <i>A special deformation of the metric with no negative sectional curvature of a Riemannian space</i>	715
Joseph Zaks, <i>Trivially extending decompositions of E^n</i>	727