

# Pacific Journal of Mathematics

**ON THE DECOMPOSITION OF INFINITELY DIVISIBLE  
CHARACTERISTIC FUNCTIONS WITH CONTINUOUS  
POISSON SPECTRUM. II**

ROGER CUPPENS

# ON THE DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH CONTINUOUS POISSON SPECTRUM, II

ROGER CUPPENS

Let  $f$  be an infinitely divisible characteristic function whose spectral functions are absolutely continuous functions with almost everywhere continuous derivatives. Some necessary conditions that  $f$  belong to the class  $I_0$  of the infinitely divisible characteristic functions without indecomposable factors have been obtained by Cramér, Shimizu and the author and a sufficient condition that  $f$  belong to  $I_0$  has been given by Ostrovskiy. In the present work, we prove that the condition of Ostrovskiy is not only a sufficient, but also a necessary condition that  $f$  belong to  $I_0$ .

Let  $f$  be the function of the variable  $t$  defined by

$$(1) \quad \log f(t) = \int_{-\infty}^0 [e^{itu} - 1 - itu(1 + u^2)^{-1}] \varphi(u) du + \int_0^{\infty} [e^{itu} - 1 - itu(1 + u^2)^{-1}] \psi(u) du$$

where  $\log$  means the branch of logarithm defined by continuity from  $\log f(0) = 0$  and where  $\varphi$  and  $\psi$  are almost everywhere nonnegative and continuous functions which are defined respectively on  $] -\infty, 0[$  and  $] 0, +\infty[$  and satisfy the condition

$$\int_{-\varepsilon}^0 u^2 \varphi(u) du + \int_0^{\varepsilon} u^2 \psi(u) du < +\infty$$

for any  $\varepsilon > 0$ . If we let

$$M(x) = \int_{-\infty}^x \varphi(u) du \quad x < 0, \\ N(x) = - \int_x^{+\infty} \psi(u) du \quad x > 0,$$

then we see that the conditions of the Lévy representation theorem ([4], Th. 5.5.2) are satisfied, so that  $f$  is an infinitely divisible characteristic function. In [3], we have proved the following result.

If the two following conditions are satisfied:

- (a)  $\varphi(u) \geq k$  a.e. for  $-c(1 + 2^{-n}) < u < -c$ ,
- (b)  $\psi(u) \geq k$  a.e. for  $d < u < d(1 + 2^{-n})$ ,

where  $k, c$  and  $d$  are positive constants and  $n$  is a positive integer,

then the function  $f$  defined by (1) has an indecomposable factor.  
The following theorem completes this result.

**THEOREM 1.** *If*

$$\psi(u) \geq k \text{ a.e. for } c < u < c(1 + 2^{-n}) \text{ and } d < u < d(1 + 2^{-n})$$

where  $n$  is a positive integer and  $k$ ,  $c$  and  $d \geq 2c$  are positive constants, then the function  $f$  defined by (1) has an indecomposable factor.

This theorem is an immediate consequence of the

**LEMMA.** *Let  $f$  be the characteristic function defined by*

$$\log f(t) = \int_0^{\infty} (e^{itu} - 1 - itu(1 + u^2)^{-1})\alpha(u)du$$

where

$$\alpha(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ 0 & \text{otherwise} \end{cases}$$

$c$  being a positive constant,  $\lambda = 1 + 2^{-n}$  ( $n$  positive integer) and  $r \geq 2\lambda$ . Then  $f$  has an indecomposable factor.

*Proof.* Let  $\beta$  be the function defined by

$$\beta(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ -c\varepsilon & \text{if } \gamma < u < \delta \\ 0 & \text{otherwise} \end{cases}$$

( $2 < \gamma < \delta < 2\lambda$ ) and  $\alpha_m$  and  $\beta_m$  be the functions defined by

$$\alpha_1(x) = \alpha(x); \alpha_m(x) = \int_{-\infty}^{\infty} \alpha_{m-1}(x-t)\alpha_1(t)dt$$

$$\beta_1(x) = \beta(x); \beta_m(x) = \int_{-\infty}^{\infty} \beta_{m-1}(x-t)\beta_1(t)dt.$$

We prove easily by induction that

$$(2) \quad \beta_m(x) = \alpha_m(x) \geq 0 \quad \text{if } x \notin [A_m, B_m]$$

where  $A_m$  and  $B_m$  are defined by

$$\begin{aligned} A_m &= m + 2^{-n} \\ B_m &= mr\lambda - 2^{-n}. \end{aligned}$$

We prove now that

$$(3) \quad \lim_{\varepsilon \rightarrow 0} \sup_{A_m \leq x \leq B_m} |\alpha_m(x) - \beta_m(x)| = 0 .$$

Indeed, if  $\varepsilon < 1$ , we have

$$\begin{aligned} |\alpha_m(x)| &\leq c^m(r\lambda - 1)^{m-1} \\ |\beta_m(x)| &\leq c^m(r\lambda - 1)^{m-1} \end{aligned}$$

and from these formulae and from

$$\begin{aligned} \alpha_m(x) - \beta_m(x) &= \int_{-\infty}^{\infty} [\alpha_{m-1}(x-t)\alpha_1(t) - \beta_{m-1}(x-t)\beta_1(t)]dt \\ &= \int_{-\infty}^{+\infty} \alpha_{m-1}(x-t)[\alpha_1(t) - \beta_1(t)]dt - \int_{-\infty}^{+\infty} [\beta_{m-1}(x-t) - \alpha_{m-1}(x-t)]\beta_1(t)dt \end{aligned}$$

it follows by induction that

$$|\alpha_m(x) - \beta_m(x)| \leq \varepsilon(2c)^m(r\lambda - 1)^{m-1}$$

and this implies (3).

Let now  $S(\alpha_m)$  be the spectrum of  $\alpha_m$ . From the definition of  $\alpha_m$ , it follows easily that

$$S(\alpha_m) = \bigcup_{j=0}^m [j + (m-j)r, (j + (m-j)r)\lambda] .$$

This implies that  $S(\alpha_m)$  is all the interval  $[m, mr\lambda]$  if

$$m > K = [(r-1)(2^n + 1)]$$

(here  $[x]$  means the integer part of  $x$ ) and therefore

$$(4) \quad \inf_{A_m \leq x \leq B_m} \alpha_m(x) > 0 \quad m = K + 2, K + 3, \dots .$$

From (3) and (4), it follows that

$$(5) \quad \beta_m(x) \geq 0 \quad m = K + 2, K + 3, \dots, 2K + 3$$

if  $\varepsilon$  is small enough. But, from the definition of  $\beta_m$ , we have for  $k < m$

$$\beta_m(x) = \int_{-\infty}^{\infty} \beta_{m-k}(x-t)\beta_k(t)dt$$

so that, from (5)

$$(6) \quad \beta_m(x) \geq 0 \quad m \geq K + 2$$

if  $\varepsilon$  is small enough.

We consider now  $\beta_m$  for  $m \leq K + 1$ .  $\beta_m$  can be negative only on intervals of the kind

$$I = [j + kr + l\gamma, (j + kr)\lambda + l\delta]$$

where  $j$  and  $k$  are nonnegative integers and  $l$  a positive integer satisfying

$$j + k + l = m$$

and on  $I$  we have

$$|\beta_m(x)| \leq \varepsilon c^m (r\lambda - 1)^{m-1}.$$

But we have

$$j + 2l + kr < j + kr + l\gamma < (j + kr)\lambda + l\delta < (j + 2l + kr)\lambda$$

so that  $\alpha_{m+l}$  is positive on  $I$ . Therefore, using (3), we have

$$\sum_{\substack{1 \leq j \leq k+1 \\ j \neq m+l}} \frac{\beta_j(x)}{j!} + \frac{\beta_{m+l}(x)}{(m+l)!} > 0$$

for  $x \in I$  if  $\varepsilon$  is small enough. This implies that

$$\sum_{j=1}^{2K+2} \frac{\beta_j(x)}{j!} \geq 0$$

for any  $x$  and therefore from (6)

$$(7) \quad \sum_{j=1}^{\infty} \frac{\beta_j(x)}{j!} \geq 0$$

for any  $x$  if  $\varepsilon > 0$  is small enough.

Let now  $g$  be the function defined by

$$\log g(t) = \int_{-\infty}^{\infty} (e^{itu} - 1 - itu(1 + u^2)^{-1})\beta(u)du.$$

Then

$$g(t) = \int_{-\infty}^{\infty} e^{itx} dG(x)$$

where  $G$  is the function

$$G(x) = e^{-\lambda} \left\{ \chi(x + \eta) + \int_{-\infty}^x \left[ \sum_{n=1}^{\infty} \frac{\beta_n(y + \eta)}{n!} \right] dy \right\}.$$

Here  $\chi$  is the degenerate distribution and  $\lambda$  and  $\eta$  are defined by

$$\begin{aligned} \lambda &= \int_{-\infty}^{\infty} \beta(u)du \\ \eta &= \int_{-\infty}^{\infty} u(1 + u^2)^{-1}\beta(u)du. \end{aligned}$$

From (7), it follows that  $g$  is a characteristic function if  $\varepsilon$  is small enough. Since  $g$  is not infinitely divisible, from the Khintchine's theorem ([4], Th. 6.2.2),  $g$  has an indecomposable factor and since  $g$  divides  $f$ , the lemma is proved.

As consequences of the Theorem 1, we obtain the following results which are respectively the results of Cramér [1] and Shimizu [6] quoted in the introduction.

**COROLLARY 1.** *If in an interval  $[0, r]$  ( $r > 0$ ),  $\psi(u) \geq c > 0$  almost everywhere, then the function  $f$  defined by (1) has an indecomposable factor.*

**COROLLARY 2.** *If in an interval  $[r, s]$  ( $s > 2r > 0$ ),  $\psi(u) \geq c > 0$  almost everywhere, then the function  $f$  defined by (1) has an indecomposable factor.*

The characterization announced in the introduction is the following.

**THEOREM 2.** *A necessary and sufficient condition that the function  $f$  defined by (1) has no indecomposable factor is the existence of an  $r > 0$  such that one of the two following conditions is satisfied:*

- (a)  $\varphi(u) \equiv 0$  a.e.;  $\psi(u) = 0$  a.e. if  $u \in [r, 2r]$ ;
- (b)  $\psi(u) \equiv 0$  a.e.;  $\varphi(u) = 0$  a.e. if  $u \in [-2r, -r]$ .

*Proof.* The sufficiency is a consequence of the Theorem 1 of Ostrovskiy [4] (see also [2], Th. 8.2), while the necessity follows immediately from the preceding theorem and from the Theorem 1 of [3] stated above.

#### REFERENCES

1. H. Cramér, *On the factorization of certain probability distributions*, Arkiv för Mat. **1** (1949), 61-65.
2. R. Cuppens, *Décomposition des fonctions caractéristiques des vecteurs aléatoires*, Publ. Inst. Statist. Univ. Paris (1967), 63-153.
3. ———, *On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum* (to appear in Proc. Amer. Math. Soc.)
4. E. Lukacs, *Characteristic functions*, Charles Griffin and Co., Ltd, London, 1960.
5. I. V. Ostrovskiy, *On the decomposition of infinitely divisible laws without gaussian factor* (in Russian), Dokl. Akad. Nauk SSSR **161** (1965), 48-51.
6. R. Shimizu, *On the decomposition of infinitely divisible characteristic functions with a continuous Poisson spectrum*, Ann. Inst. Statist. Math. **16** (1964), 384-407.

Received May 27, 1968. This work was supported by the National Science Foundation under grant NSF-GP-6175.



# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

H. ROYDEN  
Stanford University  
Stanford, California

J. DUGUNDJI  
Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. R. PHELPS  
University of Washington  
Seattle, Washington 98105

RICHARD ARENS  
University of California  
Los Angeles, California 90024

## ASSOCIATE EDITORS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF BRITISH COLUMBIA  
CALIFORNIA INSTITUTE OF TECHNOLOGY  
UNIVERSITY OF CALIFORNIA  
MONTANA STATE UNIVERSITY  
UNIVERSITY OF NEVADA  
NEW MEXICO STATE UNIVERSITY  
OREGON STATE UNIVERSITY  
UNIVERSITY OF OREGON  
OSAKA UNIVERSITY  
UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY  
UNIVERSITY OF TOKYO  
UNIVERSITY OF UTAH  
WASHINGTON STATE UNIVERSITY  
UNIVERSITY OF WASHINGTON

\* \* \*

AMERICAN MATHEMATICAL SOCIETY  
CHEVRON RESEARCH CORPORATION  
TRW SYSTEMS  
NAVAL WEAPONS CENTER

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies.

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, double spaced with large margins. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. It should not contain references to the bibliography. Manuscripts, in duplicate if possible, may be sent to any one of the four editors. Please classify according to the scheme of Math. Rev. **36**, 1539-1546. All other communications to the editors should be addressed to the managing editor, Richard Arens, University of California, Los Angeles, California, 90024.

50 reprints are provided free for each article; additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is published monthly. Effective with Volume 16 the price per volume (3 numbers) is \$8.00; single issues, \$3.00. Special price for current issues to individual faculty members of supporting institutions and to individual members of the American Mathematical Society: \$4.00 per volume; single issues \$1.50. Back numbers are available.

Subscriptions, orders for back numbers, and changes of address should be sent to Pacific Journal of Mathematics, 103 Highland Boulevard, Berkeley, California, 94708.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.), 7-17, Fujimi 2-chome, Chiyoda-ku, Tokyo, Japan.



Herbert James Alexander, <i>Extending bounded holomorphic functions from certain subvarieties of a polydisc</i> .....	485
Edward T. Cline, <i>On an embedding property of generalized Carter subgroups</i> .....	491
Roger Cuppens, <i>On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum. II</i> .....	521
William Richard Emerson, <i>Translation kernels on discrete Abelian groups</i> .....	527
Robert William Gilmer, Jr., <i>Power series rings over a Krull domain</i> .....	543
Julien O. Hennefeld, <i>The Arens products and an imbedding theorem</i> .....	551
James Secord Howland, <i>Embedded eigenvalues and virtual poles</i> .....	565
Bruce Ansgar Jensen, <i>Infinite semigroups whose non-trivial homomorphs are all isomorphic</i> .....	583
Michael Joseph Kascic, Jr., <i>Polynomials in linear relations. II</i> .....	593
J. Gopala Krishna, <i>Maximum term of a power series in one and several complex variables</i> .....	609
Renu Chakravarti Laskar, <i>Eigenvalues of the adjacency matrix of cubic lattice graphs</i> .....	623
Thomas Anthony Mc Cullough, <i>Rational approximation on certain plane sets</i> .....	631
T. S. Motzkin and Ernst Gabor Straus, <i>Divisors of polynomials and power series with positive coefficients</i> .....	641
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix.</i> .....	653
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix. II</i> .....	663
Donald Steven Passman, <i>Exceptional 3/2-transitive permutation groups</i> .....	669
Grigorios Tsagas, <i>A special deformation of the metric with no negative sectional curvature of a Riemannian space</i> .....	715
Joseph Zaks, <i>Trivially extending decompositions of <math>E^n</math></i> .....	727