ON THE DECOMPOSITION OF INFINITELY DIVISIBLE CHARACTERISTIC FUNCTIONS WITH CONTINUOUS POISSON SPECTRUM. II

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Let \( f \) be an infinitely divisible characteristic function whose spectral functions are absolutely continuous functions with almost everywhere continuous derivatives. Some necessary conditions that \( f \) belong to the class \( I_0 \) of the infinitely divisible characteristic functions without indecomposable factors have been obtained by Cramér, Shimizu and the author and a sufficient condition that \( f \) belong to \( I_0 \) has been given by Ostrovskiy. In the present work, we prove that the condition of Ostrovskiy is not only a sufficient, but also a necessary condition that \( f \) belong to \( I_0 \).

Let \( f \) be the function of the variable \( t \) defined by

\[
\log f(t) = \int_{-\infty}^{0} \left[ e^{itu} - 1 - itu(1 + u^2)^{-1}\right] \varphi(u) \, du \\
+ \int_{0}^{\infty} \left[ e^{itu} - 1 - itu(1 + u^2)^{-1}\right] \psi(u) \, du
\]

where \( \log \) means the branch of logarithm defined by continuity from \( \log f(0) = 0 \) and where \( \varphi \) and \( \psi \) are almost everywhere nonnegative and continuous functions which are defined respectively on \( ]-\infty, 0[ \) and \( ]0, +\infty[ \) and satisfy the condition

\[
\int_{-\varepsilon}^{0} u^2 \varphi(u) \, du + \int_{0}^{\varepsilon} u^2 \psi(u) \, du < +\infty
\]

for any \( \varepsilon > 0 \). If we let

\[
M(x) = \int_{-\infty}^{x} \varphi(u) \, du \quad x < 0,
\]

\[
N(x) = -\int_{x}^{+\infty} \psi(u) \, du \quad x > 0,
\]

then we see that the conditions of the Lévy representation theorem ([4], Th. 5.5.2) are satisfied, so that \( f \) is an infinitely divisible characteristic function. In [3], we have proved the following result.

If the two following conditions are satisfied:

(a) \( \varphi(u) \geq k \) a.e. for \( -c(1 + 2^{-n}) < u < -c \),
(b) \( \psi(u) \geq k \) a.e. for \( d < u < d(1 + 2^{-n}) \),

where \( k, c \) and \( d \) are positive constants and \( n \) is a positive integer,
then the function $f$ defined by (1) has an indecomposable factor.

The following theorem completes this result.

**Theorem 1.** If

$$ψ(u) \geq k \text{ a.e. for } c < u < c(1 + 2^{-n}) \text{ and } d < u < d(1+2^{-n})$$

where $n$ is a positive integer and $k$, $c$ and $d \geq 2c$ are positive constants, then the function $f$ defined by (1) has an indecomposable factor.

This theorem is an immediate consequence of the

**Lemma.** Let $f$ be the characteristic function defined by

$$\log f(t) = \int_0^\infty (e^{itu} - 1 - itu(1 + u^2)^{-1})\alpha(u)du$$

where

$$\alpha(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ 0 & \text{otherwise} \end{cases}$$

c being a positive constant, $\lambda = 1 + 2^{-n}$ ($n$ positive integer) and $r \geq 2\lambda$. Then $f$ has an indecomposable factor.

**Proof.** Let $\beta$ be the function defined by

$$\beta(u) = \begin{cases} c & \text{if } 1 < u < \lambda \text{ or } r < u < r\lambda \\ -c\varepsilon & \text{if } \gamma < u < \delta \\ 0 & \text{otherwise} \end{cases}$$

$(2 < \gamma < \delta < 2\lambda)$ and $\alpha_m$ and $\beta_m$ be the functions defined by

$$\alpha_i(x) = \alpha(x); \quad \alpha_m(x) = \int_{-\infty}^\infty \alpha_{m-1}(x - t)\alpha_1(t)dt$$

$$\beta_i(x) = \beta(x); \quad \beta_m(x) = \int_{-\infty}^\infty \beta_{m-1}(x - t)\beta_1(t)dt .$$

We prove easily by induction that

$$\beta_m(x) = \alpha_m(x) \geq 0 \quad \text{if } x \notin [A_m, B_m]$$

where $A_m$ and $B_m$ are defined by

$$A_m = m + 2^{-n} \quad B_m = mr\lambda - 2^{-n} .$$

We prove now that
Indeed, if $\varepsilon < 1$, we have

\[
|\alpha_m(x)| \leq e^m(r\lambda - 1)^{m-1} \\
|\beta_m(x)| \leq e^m(r\lambda - 1)^{m-1}
\]

and from these formulae and from

\[
\alpha_m(x) - \beta_m(x) = \int_{-\infty}^{+\infty} [\alpha_{m-1}(x-t)\alpha_i(t) - \beta_{m-1}(x-t)\beta_i(t)]dt
\]

it follows by induction that

\[
|\alpha_m(x) - \beta_m(x)| \leq \varepsilon(2c)^m(r\lambda - 1)^{m-1}
\]

and this implies (3).

Let now $S(\alpha_m)$ be the spectrum of $\alpha_m$. From the definition of $\alpha_m$, it follows easily that

\[
S(\alpha_m) = \bigcup_{j=0}^{m} [j + (m - j)r, (j + (m - j)r)\lambda] .
\]

This implies that $S(\alpha_m)$ is all the interval $[m, mr\lambda]$ if

\[
m > K = [(r - 1)(2^n + 1)]
\]

(here $[x]$ means the integer part of $x$) and therefore

\[
(4) \inf_{A_m \leq x \leq B_m} \alpha_m(x) > 0 \quad m = K + 2, K + 3, \cdots .
\]

From (3) and (4), it follows that

\[
(5) \beta_m(x) \geq 0 \quad m = K + 2, K + 3, \cdots, 2K + 3
\]

if $\varepsilon$ is small enough. But, from the definition of $\beta_m$, we have for $k < m$

\[
\beta_m(x) = \int_{-\infty}^{+\infty} \beta_{m-k}(x-t)\beta_k(t)dt
\]

so that, from (5)

\[
(6) \beta_m(x) \geq 0 \quad m \geq K + 2
\]

if $\varepsilon$ is small enough.

We consider now $\beta_m$ for $m \leq K + 1$. $\beta_m$ can be negative only on intervals of the kind
\[ I = [j + kr + l\gamma, (j + kr)\lambda + l\delta] \]

where \( j \) and \( k \) are nonnegative integers and \( l \) a positive integer satisfying

\[ j + k + l = m \]

and on \( I \) we have

\[ |\beta_m(x)| \leq \varepsilon c^m (r\lambda - 1)^{m-1}. \]

But we have

\[ j + 2l + kr < j + kr + l\gamma < (j + kr)\lambda + l\delta < (j + 2l + kr)\lambda \]

so that \( \alpha_{m+1} \) is positive on \( I \). Therefore, using (3), we have

\[ \sum_{1 \leq j \leq K + 1} \frac{\beta_j(x)}{j!} + \frac{\beta_{m+1}(x)}{(m + b)!} > 0 \]

for \( x \in I \) if \( \varepsilon \) is small enough. This implies that

\[ \sum_{j=1}^{2K+2} \frac{\beta_j(x)}{j!} \geq 0 \]

for any \( x \) and therefore from (6)

(7) \[ \sum_{j=1}^{\infty} \frac{\beta_j(x)}{j!} \geq 0 \]

for any \( x \) if \( \varepsilon > 0 \) is small enough.

Let now \( g \) be the function defined by

\[ \log g(t) = \int_{-\infty}^{\infty} (e^{it\lambda} - 1 - it\lambda(1 + \lambda^{-1}))(\beta(u)du) . \]

Then

\[ g(t) = \int_{-\infty}^{\infty} e^{it\lambda} dG(x) \]

where \( G \) is the function

\[ G(x) = e^{-t} \left\{ \chi(x + \eta) + \int_{-\infty}^{x} \left[ \sum_{n=1}^{\infty} \frac{\beta_n(y + \eta)}{n!} \right] dy \right\} . \]

Here \( \chi \) is the degenerate distribution and \( \lambda \) and \( \eta \) are defined by

\[ \lambda = \int_{-\infty}^{\infty} \beta(u)du \]
\[ \eta = \int_{-\infty}^{\infty} u(1 + u^2)^{-1} \beta(u)du . \]
From (7), it follows that \( g \) is a characteristic function if \( \varepsilon \) is small enough. Since \( g \) is not infinitely divisible, from the Khintchine's theorem ([4], Th. 6.2.2), \( g \) has an indecomposable factor and since \( g \) divides \( f \), the lemma is proved.

As consequences of the Theorem 1, we obtain the following results which are respectively the results of Cramér [1] and Shimizu [6] quoted in the introduction.

**Corollary 1.** If in an interval \([0, r]\) \((r > 0)\), \( \psi(u) \geq c > 0 \) almost everywhere, then the function \( f \) defined by (1) has an indecomposable factor.

**Corollary 2.** If in an interval \([r, s]\) \((s > 2r > 0)\), \( \psi(u) \geq c > 0 \) almost everywhere, then the function \( f \) defined by (1) has an indecomposable factor.

The characterization announced in the introduction is the following.

**Theorem 2.** A necessary and sufficient condition that the function \( f \) defined by (1) has no indecomposable factor is the existence of an \( r > 0 \) such that one of the two following conditions is satisfied:

(a) \( \varphi(u) = 0 \) a.e.; \( \psi(u) = 0 \) a.e. if \( u \notin [r, 2r] \);

(b) \( \psi(u) = 0 \) a.e.; \( \varphi(u) = 0 \) a.e. if \( u \notin [-2r, -r] \).

**Proof.** The sufficiency is a consequence of the Theorem 1 of Ostrovskiy [4] (see also [2], Th. 8.2), while the necessity follows immediately from the preceding theorem and from the Theorem 1 of [3] stated above.

**References**


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