TRANSLATION KERNELS ON DISCRETE ABELIAN GROUPS

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Let $G$ be a compact Abelian group with discrete countable dual group $\Gamma = \hat{G}$ and let $f \in L^1(G)$ with Fourier transform $F = \hat{f}$. If $V$ is a finite subset of $\Gamma$ we consider the operator $F_V$ on $L^2(V)$:

$$(F_V \varphi)(\tau) = \sum_{\tau \in V} F(\tau - \tau) \varphi(\tau) \quad \varphi \in L^2(V), \, \tau \in V.$$ 

Then if $\{V_n\}$ is any suitably restricted sequence of finite subsets of $\Gamma$ we show that

$$\lim_{n \to \infty} \| F_{V_n} \| = \lim_{n \to \infty} \max_{|\varphi| = 1} |(F_{V_n} \varphi, \varphi)| = \| f \|_\infty$$

where $|F_V|$ is the operator norm of $F_V$ on $L^2(V)$ and $(F_V \varphi, \varphi)$ denotes the inner product of $F_V \varphi$ and $\varphi$ (over $V$).

This result is then translated into a statement concerning a special class of infinite matrices which generalize the classical Toeplitz matrices. We then apply these results in evaluating the norm of a special type of linear operator.

In [1] the author considered the asymptotic distribution of eigenvalues and characteristic numbers of certain sequences of operators $\{F_{V_n}\}$ over a locally compact group $\Gamma$ associated with sequences $\{V_n\}$ of Borel sets of $\Gamma$ of finite nonzero measure satisfying

$$(*) \quad \lim_{n \to \infty} |\gamma V_n \Delta V_n| / |V_n| = 0 \quad \text{for all } \gamma \in \Gamma,$$

where $|\cdot|$ is left Haar measure on $\Gamma$. We write $\{V_n\} \in W_r$, and say $\{V_n\}$ has the weak ratio property in case $(*)$ is satisfied (see [2]). In this paper we are considering countable Abelian $\Gamma$ and a more general family $T_\Gamma$ of sequences $\{V_n\}$ than those in $W_r$ (and hence in general the asymptotic distribution of the characteristic numbers of $\{F_{V_n}\}$ does not exist, [2]) but still restricted enough to guarantee an asymptotic formula for the maximal characteristic number of $F_{V_n}$ as $n \to \infty$.

1. The basic theorem. $\Gamma$ denotes an arbitrary countably infinite discrete Abelian group equipped with the counting measure.

DEFINITION 1. A sequence $\{V_n\}$ of finite nonempty subsets of $\Gamma$ has the translation property, written $\{V_n\} \in T_\Gamma$, if and only if to every finite subset $\Gamma_0 \subseteq \Gamma$ there corresponds an $n_0 = n_0(\Gamma_0)$ such that for $n \geq n_0$ there exists a $\tau_n = \tau_n(\Gamma_0) \in \Gamma$ with the property that $\tau_n + \Gamma_0 \subseteq V_n$. 

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PROPOSITION 1. (i) \( \{V_n\} \in T_r \) and \( V_n \subseteq V_n^* \) for \( n \in N^+ \) implies \( \{V_n^*\} \in T_r \). (ii) \( W_r \) is properly contained in \( T_r \).

Proof. (i) is immediate from the definition. To prove (ii), first assume \( \{V_n\} \in W_r \) and fix any finite nonempty subset \( \Gamma_0 \subseteq \Gamma \). Set \( C = \Gamma_0 \cup \{0\} \). We then readily conclude (see [2] where many properties of \( W_r \) are established)

\[
\lim_{n \to \infty} \frac{|C + V_n|}{|V_n|} = 1.
\]

But \( C + V_n = \bigcup_{\tau \in \Gamma_0} (\tau + \Gamma_0) \cup V_n \). Hence if \( \tau + \Gamma_0 \not\subseteq V_n \) for all \( \tau \in V_n \), we have \( |(\tau + \Gamma_0) \sim V_n| \geq 1 \) for all \( \tau \in V_n \) and consequently

\[
|C + V_n| \geq |V_n| + \frac{|V_n|}{|\Gamma_0|}.
\]

since we may choose \( |V_n| \) elements \( \tau + \gamma, \tau \in (\tau + \Gamma_0) \sim V_n \), where \( \tau \in V_n \) and \( \gamma, \tau \in \Gamma_0 \), and no element is duplicated more than \( |\Gamma_0| \) times.

But for sufficiently large \( n \) (2) violates (1) and therefore there is a \( \tau_n \in V_n \) for which \( \tau_n + \Gamma_0 \subseteq V_n \). Hence \( W_r \subseteq T_r \).

We now show inclusion is proper. For let \( \{V_n\} \in W_r \) (\( \neq \emptyset \), by [2]); we shall construct a sequence \( V_n^* \supseteq V_n \) such that \( \{V_n^*\} \in W_r \), which completes the proof of (ii) upon appealing to (i). Fix any \( \gamma \in \Gamma \sim \{0\} \). We inductively construct a sequence \( v_i^{(n)}, \cdots, v_{v_n(\gamma)}^{(n)} \) as follows: \( v_i^{(n)} \in V_n + \{0, \pm \gamma\} \), and

\[
v_i^{(n)} \in (V_n \cup \{v_1^{(n)}, \cdots, v_{v_n(\gamma)}^{(n)}\}) + \{0, \pm \gamma\} \quad (2 \leq k \leq |V_n|).
\]

We set \( V_n^* = V_n \cup \{v_1^{(n)}, \cdots, v_{|V_n(\gamma)|}^{(n)}\} \) and verify that

\[
\left| (\gamma + V_n^*) \cap V_n^* \right| \leq \frac{1}{2} \quad (n \in N^+)
\]

implying \( \{V_n^*\} \in W_r \). For

\[
(\gamma + V_n^*) \cap V_n^*
\]

\[
= ((\gamma + V_n) \cap V_n^*) \cup \{(\gamma + v_1^{(n)}), \cdots, \gamma + v_{v_n(\gamma)}^{(n)}\} \cap (V_n \cup \{v_1^{(n)}, \cdots, v_{v_n(\gamma)}^{(n)}\})
\]

\[
= ((\gamma + V_n) \cap V_n^*)
\]

since the second term in the union is empty by (I). Hence

\[
| (\gamma + V_n^*) \cap V_n^* | \leq | \gamma + V_n | = | V_n |, \quad \text{and therefore for } n \in N^+
\]

\[
\left| (\gamma + V_n^*) \cap V_n^* \right| = \left| (\gamma + V_n^*) \cap V_n^* \right| \leq \frac{|V_n|}{2} = \frac{1}{2}.
\]

We now prove a result, of independent interest, which is critical in the proof of Theorem 1.
PROPOSITION 2. Let $G$ be a compact Abelian group (with measure normalized to one), let $f \in L^1(G)$, and let $\rho$ be any positive number. Then

$$\|f\|_\infty = \sup_\omega \left| \int_G \omega(x) |\hat{f}(x)| dx \right|$$

where $\omega$ ranges over all trigonometric polynomials on $G$ satisfying $\|\omega\|_\rho \leq 1$.

Proof. Recall that a trigonometric polynomial is a finite linear combination of characters on $G$. Clearly

$$\left| \int_G \omega(x) |\hat{f}(x)| dx \right| \leq \|f\|_\infty \int_G \omega(x) |\hat{f}(x)| dx = \|f\|_\infty \|\omega\|_\rho \leq \|f\|_\infty .$$

We divide the proof of the converse inequality into two cases:

To prove the converse inequality, we first consider the case $\|g\|_\infty < +\infty$. Fix any $\delta > 0$ (until the conclusion of the argument). Let $S = S(\delta)$ be a measurable subset of the complex plane of diameter less than $\delta$ and such that

$$E = f^{-1}(S), \quad \|\chi_E f\|_\infty = \|f\|_\infty ,$$

where $\chi_E$ denotes the characteristic function of $E$. Hence for $s \in S$ and $x \in E$ we have

$$|s| - |(\chi_E f)(x)| \leq |s - (\chi_E f)(x)| < \delta ,$$

and consequently also

$$|s| - \|f\|_\infty = |s| - \|\chi_E f\|_\infty \leq \delta .$$

Therefore, if $g = \chi_E/|E|$ then

$$0 \leq \|f\|_\infty - \left| \int_E f g dx \right| \leq \|f\|_\infty - |s| + |s| - \left| \int_E f g dx \right|$$

$$\leq \|f\|_\infty - |s| + \left| s - \int_E f g dx \right|$$

(3) $$= \|f\|_\infty - |s| + \left| \frac{1}{|E|} \int_E (s - \chi_E f) dx \right| \leq 2\delta .$$

We next wish to approximate $g$ by a continuous function $h$, and at this point the estimate is rather delicate because this is also needed later in the case $\|f\|_\infty = +\infty$ and consequently we must avoid $\|f\|_\infty$ as a factor in the error of estimation. Now since $f \in L^1(G)$, to every $\varepsilon > 0$ there corresponds an $\eta = \eta(\varepsilon)$ such that for all measurable subsets $T$ of $G$ of measure at most $\eta$
\[ \int_{\mathcal{T}} |f| \, dx < \varepsilon . \]

We now choose \( \gamma = \gamma(\delta) \) satisfying

\begin{align*}
(4) & \quad (i) \quad \gamma < \delta \cdot |E|, \quad (ii) \quad \int_{\mathcal{T}} |f| \, dx < \delta \cdot |E| \quad \text{if} \quad |T| < \gamma .
\end{align*}

Furthermore, since Haar measure is regular, we may find an open set \( E^+ \) and a closed set \( E^- \) such that

\begin{align*}
(4') \quad E^- \subseteq E \subseteq E^+, \quad |E^+ \sim E^-| < \gamma .
\end{align*}

Finally (by Urysohn’s Lemma, since \( G \) is a normal topological space) there exists a continuous \( h_0: G \rightarrow [0,1] \) such that \( h_0 \cdot |E^-| = 1 \) and \( h_0 \cdot |G \sim E^+| = 0 \). Our candidate for \( h \) is then defined to be the nonnegative function \( h = h_0|E| \). Let us now estimate \( \int_{\mathcal{O}} fgdx - \int_{\mathcal{O}} fhdx : \)

\[ \left| \int_{\mathcal{O}} fgdx - \int_{\mathcal{O}} fhdx \right| \leq \int_{\mathcal{O}} |f| \cdot |g - h| \, dx \]

\[ = \left( \int_{E^-} + \int_{E^+ \sim E^-} + \int_{E^- \sim E^+} \right) |f| \cdot |g - h| \, dx = \int_{E^+ \sim E^-} |f| \cdot |g - h| \, dx \]

\[ \leq \max |g - h| \int_{E^+ \sim E^-} |f| \, dx \leq \frac{1}{|E^+|} \cdot \delta \cdot |E^-| = \delta \]

by (4), (4’) and the definitions of \( g \) and \( h \). Also, we have

\[ \int_{E^-} hdx \leq \int_{\mathcal{O}} hdx = \int_{E^+} hdx \leq \| h \|_{\infty} \cdot |E^+| , \]

implying the estimate

\[ \frac{|E^-|}{|E|} \leq \| h \|_{1} \leq \frac{|E^+|}{|E|} \leq 1 + \delta \quad \text{by virtue of (4) and (4’).} \]

Lastly, to any \( \alpha > 0 \) we may correspond a trigonometric polynomial \( \omega_{\alpha} \) satisfying \( \| h^{1/\rho} - \omega_{\alpha} \|_{\infty} < \alpha \), and consequently \( \| h^{1/\rho} - \omega_{\alpha} \|_{\infty} < \alpha \)

since \( h^{1/\rho} \geq 0 \). Thus by choosing \( \alpha_0 = \alpha_0(\delta) \) sufficiently small we may conclude

\[ \| h - \omega_{\alpha_0} \|_{1} \leq \| h - \omega_{\alpha_0} \|_{\infty} < \delta . \]

Also,

\[ \| \omega_{\alpha_0} \|_{1} \leq \| h - \omega_{\alpha_0} \|_{1} + \| h \|_{1} \leq \delta + (1 + \delta) = 1 + 2\delta . \]

We now let

\[ \omega = \omega_{\alpha}(1 + 2\delta)^{1/\rho} , \quad \text{implying} \quad \| \omega \|_{\rho} \leq 1 . \]
Finally,
\[
\left| \int_g |\omega |^p f dx \right| = \frac{1}{1 + 2\delta} \left| \int_g |\omega_{\alpha_0}|^p f dx \right|
\]
(8) \[\geq \frac{1}{1 + 2\delta} \left( \left| \int_g f g dx \right| - \left| \int_g f h dx \right| - \left| \int_g h f dx - \int_g |\omega_{\alpha_0}|^p f dx \right| \right)\]
\[
\geq \frac{1}{1 + 2\delta} (||f||_\infty - 2\delta) - \delta - \delta ||f||_1 .
\]

By (3), (5), and (7). Our assertion follows upon letting \(\delta \to 0\).

In case \(||f||_\infty = +\infty\), we let \(S_n\) be a measurable subset of the complex plane of diameter less than \(\delta\) and such that \(E_n = f^{-1}(S_n)\), \(||\chi_{E_n}f||_\infty > n\). Equations (3) – (8) still hold with \(||f||_\infty\) replaced by \(||\chi_{E_n}f||_\infty > n\) wherever it occurs, and we readily construct trigonometric polynomials \(\omega_n\) with \(||\omega_n||_p \leq 1\) and such that \(\int_g |\omega_n|^p f dx\) is unbounded as \(n \to +\infty\).

We now are ready to prove the basic theorem.

**Theorem 1.** Let \(G\) be a compact group (with measure normalized to one), let \(f \in L^1(G)\), and let \(\hat{f} \in L^\infty(I)\), the Fourier Transform of \(f\). Furthermore, let \(\{V_n\} \in T_r\) and let \(F_{V_n}\) be the Hilbert-Schmidt operator on \(L^2(V_n)\):
\[
(F_{V_n}\psi)(\gamma) = \int_{V_n} F(\gamma - \tau) \psi(\tau) d\tau = \sum_{\tau \in V_n} F(\gamma - \tau) \psi(\tau)
\]
\((\psi \in L^2(V_n), \gamma \in V_n)\).

Let \((F_{V_n}\psi, \psi)_{V_n}\) denote the inner product of \(F_{V_n}\psi\) and \(\psi\) over \(V_n\), and let \(|F_{V_n}|\) denote the maximal characteristic number of \(F_{V_n}\) as an operator on the Hilbert space \(L^2(V_n)\). Then

(i) \[\lim_{n \to \infty} \max_{||\psi||_2 = 1} |(F_{V_n}\psi, \psi)_{V_n}| = ||f||_\infty .\]
(ii) \[\lim_{n \to \infty} |F_{V_n}| = ||f||_\infty .\]

**Proof.** (i) By definition,
\[
(F_{V_n}\psi, \psi)_{V_n} = \sum_{\tau, \gamma \in V_n} F(\gamma - \tau) \psi(\tau) \overline{\psi(\gamma)}
\]
\[
= \sum_{\tau, \gamma \in V_n} \left[ \left( \gamma - \tau, x \right) f(x) dx \right] \psi(\tau) \overline{\psi(\gamma)}
\]
\[
= \int_{\tau, \gamma \in V_n} \left( \gamma, x \right) \psi(\tau) \psi(\gamma) f(x) dx
\]
\[
= \int_{\tau, x} \left[ \sum_{\gamma \in V_n} \left( \tau, x \right) \psi(\gamma) \psi(\gamma) \right] f(x) dx .
\]
Note that
\[ \omega_\varphi(x) = \sum_{\tau \in V_n} (\tau, x) \psi(\tau) \]

is a trigonometric polynomial on \( G \), and \( \psi \rightarrow \omega_\varphi \) is an isometry of \( L^2(V_n) \) into \( L^2(G) \) since \( ||\omega_\varphi||_2^2 = \sum_{\tau \in V_n} |\psi(\tau)|^2 = ||\psi||_2^2 \). Therefore

\[
\max_{||\psi||_2 = 1} |(F_{V_n} \psi, \psi)_{V_n}| = \max_{||\omega||_2 = 1} \left| \int_\sigma \omega(x) f(x) dx \right|
\]

where \( \omega \) ranges over linear combinations of characters on \( G \) generated by elements in \( V_n \). Hence by Proposition 2 \((\rho = 2)\),

\[
\lim_{n \to \infty} \max_{||\psi||_2 = 1} |(F_{V_n} \psi, \psi)_{V_n}| \leq ||f||_\infty .
\]

On the other hand, let \( \omega \) be any trigonometric polynomial on \( G \), say

\[ \omega(x) = \sum_{i \leq k} (\gamma_i, x) e_i \quad (e_i \in \mathcal{C}, \gamma_i \in \Gamma) . \]

Let \( \Gamma_0 = \{\gamma_1, \cdots, \gamma_k\} \), a finite subset of \( \Gamma \). Now since \( \{V_n\} \in T_r \) there exists an \( n_0 \) such that for \( n \geq n_0 \) there exists \( \tau_n \in \Gamma \) such that \( \tau_n + \Gamma_0 \subseteq V_n \). Hence for \( n \geq n_0 \),

\[ \omega_n(x) = (\tau_n, x) \omega(x) = \sum_{i \leq k} (\tau_n + \gamma_i, x) e_i \]

is a linear combination of characters on \( G \) generated by elements of \( V_n \). Since \( |\omega(x)| = |\omega_n(x)| \) for all \( x \in G \), the proof of (i) is completed by again applying Proposition 2 with \( \rho = 2 \).

(ii) Recall that \( |F_{V_n}| \) is the norm of \( F_{V_n} \) considered as an operator on \( L^2(V_n) \), i.e.,

\[ |F_{V_n}| = \max_{||\psi||_2 = 1} ||F_{V_n} \psi||_2 . \]

but by the Cauchy-Schwarz Inequality, for \( ||\psi||_2 = 1 \)

\[ |(F_{V_n} \psi, \psi)_{V_n}| \leq ||F_{V_n} \psi||_2 ||\psi||_2 = ||F_{V_n} \psi||_2 \leq |F_{V_n}| \]

and therefore by (i),

\[ \lim_{n \to \infty} |F_{V_n}| \geq \lim_{n \to \infty} \max_{||\psi||_2 = 1} |(F_{V_n} \psi, \psi)_{V_n}| = ||f||_\infty . \]

Thus, if \( ||f||_\infty = +\infty \) nothing remains to be proved. If \( ||f||_\infty < +\infty \) we have \( f \in L^1(G) \cap L^\omega(G) \), and therefore by \([3], p. 445\), \( |F_{V_n}| \leq ||f||_\infty \) for all \( n \in N^+ \). Hence \( \lim_{n \to \infty} |F_{V_n}| \leq ||f||_\infty \), and consequently \( \lim_{n \to \infty} |F_{V_n}| = ||f||_\infty \) in this case as well.
We now conversely prove that the hypothesis \( \{ V_n \} \in T_r \) is in fact necessary for the conclusion of Theorem 1. More precisely,

**Theorem 1'**. Using the notation of Theorem 1, if \( \{ V_n \} \) is any sequence of finite subsets of \( \Gamma \) for which conclusion (i) holds for all trigonometric polynomials \( f \) on \( G \), then \( \{ V_n \} \in T_r \).

**Proof.** Assume \( \{ V_n \} \in T_r \), i.e., there exists a finite subset \( \Gamma_o \) of \( \Gamma \) such that no translate of \( \Gamma_o \) lies in \( V_n \) for an appropriate subsequence \( n \to \infty \). We then assert that

\[
f(x) = \frac{1}{|\Gamma_o|} \sum_{\tau \in \Gamma_o} (\tau, x)
\]

is a trigonometric polynomial for which (i) fails. More precisely we show for all these \( n \):

\[
\max_{||\varphi||=1} |(F_{V_n} \varphi, \varphi)_{V_n}| \leq \left(1 - \frac{1}{2|\Gamma_o|}\right)||f||_\infty .
\]

Recalling relation \((\dagger)\) of the proof of Theorem 1. We have:

\[
(\dagger) \quad \max_{||\varphi||=1} |(F_{V_n} \varphi, \varphi)_{V_n}| = \max_{||\omega||=1} \left| \int_G \omega(x) |\varphi(x)|^2 dx \right|
\]

where \( \omega \) ranges over all linear combinations of characters on \( G \) generated by elements in \( V_n \).

However, any such \( \omega \) is of the form

\[
\omega(x) = \sum_{\tau \in V_n} (\tau, x) a_\tau
\]

where

\[
\sum_{\tau \in V_n} |a_\tau|^2 = ||\omega||_2^2 \leq 1 ,
\]

implying

\[
|\omega(x)|^2 = \sum_{\tau_1, \tau_2 \in V_n} (\tau_1 - \tau_2, x) a_{\tau_1} \overline{a}_{\tau_2} ,
\]

and finally

\[
\int_G |\omega(x)|^2 f(x) dx = \frac{1}{|\Gamma_o|} \sum_{\tau \in \Gamma_o} a_\tau \overline{a}_{\tau}. \]

Consequently,
\[
\left| \int_{\Omega} \omega(x) |f(x)| dx \right| \leq \frac{1}{|\Gamma_0|} \sum_{\tau_1, \tau_2 \in \Gamma} |a_{\tau_1}| |a_{\tau_2}| \leq \frac{1}{2|\Gamma_0|} \sum_{\tau_1, \tau_2 \in \Gamma} (|a_{\tau_1}|^2 + |a_{\tau_2}|^2)
\]

\[
= \frac{1}{2|\Gamma_0|} \left( \sum_{\tau_1 \in \Gamma} |a_{\tau_1}|^2 \left( \sum_{\tau_2 \in \Gamma} 1 \right) + \sum_{\tau_2 \in \Gamma} |a_{\tau_2}|^2 \left( \sum_{\tau_1 \in \Gamma} 1 \right) \right)
\]

\[
= \frac{1}{2|\Gamma_0|} \left( \sum_{\tau_1 \in \Gamma} |a_{\tau_1}|^2 |(V_m - \tau) \cap \Gamma \right) + \sum_{\tau_2 \in \Gamma} |a_{\tau_2}|^2 |(\tau_2 - V_m) \cap \Gamma \right)\]

\[
\leq \frac{1}{2|\Gamma_0|} \left\{ (|\Gamma_0| - 1) \sum_{\tau_1 \in \Gamma} |a_{\tau_1}|^2 + |\Gamma_0| \sum_{\tau_2 \in \Gamma} |a_{\tau_2}|^2 \right\}
\]

\[
= \left( 1 - \frac{1}{2|\Gamma_0|} \right) \sum_{\tau_1 \in \Gamma} |a_{\tau_1}|^2 \leq \left( 1 - \frac{1}{2|\Gamma_0|} \right)
\]

\[
= \left( 1 - \frac{1}{2|\Gamma_0|} \right) \|f\|_\infty
\]

since no translate \( V_m - \tau_1 \) contains \( \Gamma_0 \) by hypothesis. Our assertion now readily follows.

2. A class of doubly-infinite matrices. We now translate the theorem of the preceding section into a statement concerning a class of doubly-infinite complex matrices \( M = (\alpha_{i,j})_{i,j=1}^{\infty} \) whose entries \( \alpha_{i,j} \) are determined by a "group law".

**DEFINITION 2.** Let \( M = (\alpha_{i,j})_{i,j=1}^{\infty} \) be a matrix with complex entries. We then write

\[
M \sim (\Gamma, \Lambda, F)
\]

if and only if

(i) \( \Gamma \) is a countable Abelian group.

(ii) \( \Lambda \) is a subset of \( \Gamma \).

(iii) \( F: \Gamma \to \mathcal{C} \).

(iv) There exists an ordering of \( \Lambda = \{\lambda_1, \ldots, \lambda_n, \ldots\} \) such that for all \( i, j \in \mathbb{N}^+ \),

\[
\alpha_{i,j} = F(\lambda_i - \lambda_j).
\]

**REMARK.** For any \( M = (\alpha_{i,j})_{i,j=1}^{\infty} \) with complex entries we may take \( \Gamma \) to be \( \mathbb{Q}^* \), the multiplicative group of rational numbers, and \( \Lambda \) to be \( P = \{p_n: n \in \mathbb{N}^+\} \), the set of all positive integral primes, upon defining \( F \) by

\[
F(r) = \begin{cases} 
\alpha_{i,j} & \text{if } r = p_i/p_j \\
0 & \text{otherwise}
\end{cases}
\]
We then have $M \sim (Q^*, P, F)$.

Under suitable restrictions on $(\Gamma, A, F)$ we shall be able to compute the norm and quadratic norm of the matrix $M$, which are defined as follows:

**Definition 3.** The norm of $M$, $|M|$, and the quadratic norm of $M$, $|M|_I$, are defined by

$$|M| = \sup_{||X||_2 \leq 1} ||MX||_2, \quad |M|_I = \sup_{||X||_2 \leq 1} |(MX, X)|,$$

where $X = (\{x_i\})$ ranges over elements of the complex Hilbert space $l^2$ with only finitely many $x_i \neq 0$, and $M = (\{\sum_i \alpha_i x_i\})$.

**Lemma 1.** If $M$ induces a bounded operator on $l^2$, then

(i) $|M| = \sup_{||X||_2 \leq 1} ||MX||$, 
(ii) $|M|_I = \sup_{||X||_2 \leq 1} |(MX, X)|$,

where $X = (\{x_i\})$ ranges over all elements of $l^2$ (with $||X||_2 \leq 1$). Hence in this case $|M|$ is the standard norm of $M$ considered as a bounded linear operator on $l^2$.

**Proof.** For $x \in l^2$, let $X_n$ be the projection of $X$ on its first $n$ components (0 elsewhere). Since $M$ is bounded and consequently closed, $\lim_{n \to \infty} MX_n = MX$ and (i) follows since $X_n$ has at most $n$ nonzero components. Also

$$(MX, X) = (MX_n, X_n) + (M(X - X_n), X_n) + (MX, X - X_n),$$

and therefore

$$|(MX, X) - (MX_n, X_n)| \leq ||M(X - X_n)||_2 ||X_n||_2 + ||MX||_2 ||X - X_n||_2 \to 0$$
as $n \to \infty$, and (ii) clearly follows.

**Theorem 2.** Let $M \sim (\Gamma, \Lambda, F)$ where

(i) $F \in A(\Gamma)$, i.e., $F = \hat{f}$ for some $f \in L^1(G)$.

(ii) To each finite subset $\Gamma_0 \subseteq \Gamma$ there corresponds a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteq \Lambda$.

Then $|M| = |M|_I = ||f||_{\infty}$.

**Proof.** Assume $\Lambda = \{\lambda_i, \cdots, \lambda_n, \cdots\}$ as in Definition 2, and set $V_n = \{\lambda_1, \cdots, \lambda_n\}$. Then hypothesis (ii) clearly implies $\{V_n\} \in T_f$. The theorem will follow from the two inequalities

(i) $|M| \leq ||f||_{\infty}$

(ii) $||f||_{\infty} \leq |M|_I$, 

where $\gamma(\Gamma_0)$ corresponds to $\{V_n\}$.
since $|M|^t \leq |M|$ by the Cauchy-Schwarz inequality.

(i): If $\|f\|_{\infty} = +\infty$ there is nothing to prove. Otherwise $f \in L^1(G) \cap L^\infty(G)$, and therefore the operator $M': L^1(\Gamma) \to L^1(\Gamma)$ defined by

$$(M' \varphi)(\gamma) = \sum_{\tau \in \Gamma} F(\gamma - \tau) \varphi(\tau) \quad (\varphi \in L^1(\Gamma), \ \gamma \in \Gamma)$$

has norm $|M'| = \|f\|_{\infty}$ by [3], § 3.2., p. 441. Hence if we only consider $\varphi$ with support in $A$ and restrict $\gamma$ to $A$, $M'$ restricts to an operator $M'': L^1(A) \to L^1(A)$ with $|M''| \leq |M'|$. Now consider the isometry of $l^1$ onto $L^1(A)$ given by $X = (\langle \varphi_n \rangle) \mapsto \varphi_x$ where $\varphi(\lambda_n) = \varphi_n$ for $n \in N^+$. Then for this $\varphi = \varphi_x$ and $\lambda_i \in A$,

$$(M'' \varphi)(\lambda_i) = \sum_{\tau \in \Gamma} F(\lambda_i - \lambda) \varphi(\lambda) = \sum_{\tau \in \Gamma} F(\lambda_i - \lambda_j) \varphi(\lambda_j) = \sum_{\tau \in \Gamma} \alpha_i, \varphi_j$$

which is the $i^{th}$ component of $MX$, and therefore $|M''| = \|f\|_{\infty}$ (and thus $M$ induces a bounded linear operator if $\|f\|_{\infty} < +\infty$).

(ii): For $n \in N^+$, consider the isometry of $L^2(V_\infty)$ (which is none other than $n$-dimensional Euclidean space) into $l^1$ given by $\varphi \to X_\varphi$ where $X_\varphi = (|x_i^j|)$ and $x_i^j = \varphi(\lambda_i)$ for $1 \leq j \leq n$ and 0 otherwise. Hence $X_\varphi$ has only finitely many nonzero components, and each $X \in l^1$ with only finitely many nonzero components is in the image of $L^2(V_\infty)$ under the above isometry for $n = n(X)$ sufficiently large. Now consider $F_{V_\infty}$ on $L^2(V_\infty)$:

$$(F_{V_\infty} \varphi, \varphi)_{V_\infty} = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} F(\lambda_i - \lambda_j) \varphi(\lambda_j) \overline{\varphi(\lambda_i)} = \sum_{1 \leq i \leq n} \alpha_i, x_i \overline{x_i} = (MX, X) \quad (\text{in } l^1).$$

But by Theorem 1 (i) $\lim_{n \to \infty} \max_{\|\psi\|_{L^1} = 1} |(F_{V_\infty} \varphi, \varphi)_{V_\infty}| = \|f\|_{\infty}$ and therefore $|M|^t = \|f\|_{\infty}$ since

$$|M|^t = \sup_{\|\psi\|_{L^1} = 1} |(MX, X)| = \lim_{n \to \infty} \max_{\|\psi\|_{L^1} = 1} |(F_{V_\infty} \varphi, \varphi)_{V_\infty}| = \|f\|_{\infty}.$$ 

**Corollary 1.** (1) Hypothesis (i) of Theorem 2 is satisfied if

(i)' $\sum_{\tau \in \Gamma} |F(\gamma)|^t < +\infty$.

(2) Hypothesis (ii) of Theorem 2 is satisfied if

(ii)' $A + A \subseteq A$ and (ii)'' $A$ generates $\Gamma$.

**Proof.** (1), (i)' implies $f(x) = \sum_{\tau \in \Gamma} F(\gamma, x) \varphi(\gamma) \in L^1(G)$ and therefore also $f \in L^1(G)$ since $G$ is compact and hence of finite measure. Clearly $F = \hat{f} \in A(\Gamma)$. 


(2) First note that any element of $\Gamma$ is the difference of two elements in $\Lambda$, i.e., $\Gamma = \Lambda - \Lambda$. For by (ii)', if $y \in \Gamma$ we have $y = \lambda_{i_1} + \cdots + \lambda_{i_k} - \cdots - \lambda_{i_n}$ for some suitable finite sequence of integers $i_1, \cdots, i_n$ (if no terms with a plus sign occur we may take $k = 0$, if none with a minus sign occur take $k = n$). But by (ii)', $\gamma^+ = \lambda_{i_1} + \cdots + \lambda_{i_k} \in A$ if $k > 0$, $\gamma^- = \lambda_{i_{k+1}} + \cdots + \lambda_{i_n} \in A$ if $k < n$. If $k = 0$ we may write $\gamma = \lambda_i - (\lambda_1 + \cdots + \lambda_n + \lambda_i)$ and similarly $\gamma = (\lambda_1 + \cdots + \lambda_n + \lambda_i) - \lambda_i$ if $k = n$.

Now let $\Gamma_0 = \{\gamma_1, \cdots, \gamma_k\}$ be a nonempty finite subset of $\Gamma$. Then for appropriate $a_i, b_i \in N^+$ we have

$$\gamma_i = \lambda_{a_i} - \lambda_{b_i} \quad (1 \leq i \leq k).$$

Consequently, for $1 \leq i \leq k$,

$$\gamma_i = \lambda_{a_i} + \lambda_{b_1} + \cdots + \lambda_{b_i} + \cdots + \lambda_{b_k} - (\lambda_{b_1} + \cdots + \lambda_{b_k})$$

where (*) denotes deletion of a term. Hence if we set $\gamma = \lambda_{b_1} + \cdots + \lambda_{b_k}$ we have $\gamma + \Gamma_0 \subseteq A$ since

$$\lambda_{a_i} + \lambda_{b_1} + \cdots + \lambda_{b_i} + \cdots + \lambda_{b_k} \in A$$

by (ii)'.

We now apply Theorem 2 to completely solve the norm evaluation in the case $M \sim (\Gamma, A, F)$ where $F \geq 0$ and $A$ satisfies (ii). We make use of the following simple lemma:

**Lemma 2.** If $M = (\alpha_{i,j})_{i,j=1}$ and $M' = (\alpha'_{i,j})_{i,j=1}$ where

$$\alpha_{i,j} \geq 0$$

then

(i) $|M| = \sup_{||x||_2 \leq 1} ||MX||$, $|M|_I = \sup_{||x||_2 \leq 1} (MX, X)$

where $X = (\{x_i\})$ has only finitely many nonzero coordinates, all positive.

(ii) $|M'| \leq |M|$, $|M'|_I \leq |M|_I$.

**Proof.** For $X = (\{x_i\}) \in \ell^2$ we define $X^+ = (\{x_i\})$. Note $||X||_2 = ||X^+||_2$ and $X^+$ and $X$ have the same cardinality of nonzero coordinates. Also, $\alpha_{i,j} \geq 0$ clearly implies $||MX||_2 \leq ||MX^+||_2$ and $|(MX, X)| \leq (MX^+, X^+)$ and (i) readily follows. But $\alpha_{i,j} \geq \alpha'_{i,j} \geq 0$ also implies each component of $M'X^+$ is dominated by the corresponding component of $MX^+$ and hence (ii) follows from (i).

**Theorem 3.** Let $M \sim (\Gamma, A, F)$ where
\( F(\gamma) \geq 0 \) for all \( \gamma \in \Gamma \).

(ii) If \( \Gamma_0 \) is any finite subset of \( \Gamma \) there exists a \( \gamma = \gamma(\Gamma_0) \) such that \( \gamma + \Gamma_0 \subseteq A \). Then

\[
| M | = | M |_I = \sum_{\gamma \in \Gamma} F(\gamma) \quad (\text{possibly } + \infty).
\]

**Proof.** Since \( | M |_I \leq | M | \) it suffices to show that

(1) \( | M | \leq \sum_{\gamma \in \Gamma} F(\gamma) \),

(2) \( | M |_I = \sum_{\gamma \in \Gamma} F(\gamma) \).

\( \alpha_i = \sum_{\gamma \in \Gamma} F(\gamma) \)

If \( \sum_{\gamma \in \Gamma} F(\gamma) < +\infty \) there is nothing to prove since in this case the result is included in Theorem 2 because \( f(x) = \sum_{\gamma \in \Gamma} (\gamma(x) F(\gamma) \) is a continuous function on \( \Gamma \), \( F = \hat{f} \), and \( \| f \|_\infty = f(0) = \sum_{\gamma \in \Gamma} F(\gamma) \).

On the other hand, if \( \sum_{\gamma \in \Gamma} F(\gamma) = +\infty \) then \( F \) may not be in \( A(\Gamma) \) and hence we cannot apply Theorem 2 directly. Clearly (1) is true in this case and we need only verify (2). Let \( \Gamma^\omega = \{ \gamma_1, \cdots, \gamma_s \} \) be any finite subset of \( \Gamma \) and define

\[
M_{\Gamma^\omega} = (\alpha_{i,j})_{i,j=1}^n
\]

where

\[
\alpha_{i,j} = \begin{cases} F(\gamma_i) & \text{if } \lambda_i - \lambda_j = \gamma_\iota \in \Gamma^\omega \\ 0 & \text{otherwise} \end{cases}
\]

i.e., \( M_{\Gamma^\omega} \sim (\Gamma, A, F_{\Gamma^\omega}) \) where \( F_{\Gamma^\omega}(\gamma) = F(\gamma) I_{\Gamma^\omega}(\gamma) \). Since \( F \geq 0 \), \( \alpha_{i,j} \geq \alpha_{i,j}^{\omega} \geq 0 \) for \( i, j \in N^+ \), and Lemma 2 implies \( | M |_I \leq | M_{\Gamma^\omega} |_I \). But by Theorem 2

\[
| M_{\Gamma^\omega} |_I = \text{ess sup} \left| \sum_{\gamma \in \Gamma} (\gamma(x) F_{\Gamma^\omega}(\gamma) \right| = \sum_{\gamma \in \Gamma^\omega} F(\gamma)
\]

since \( \sum_{\gamma \in \Gamma} (\gamma(x) F_{\Gamma^\omega}(\gamma) \) is continuous and \( F_{\Gamma^\omega} \geq 0 \). This in turn implies

\[
| M | \leq \text{sup} \left( \sum_{\gamma \in \Gamma^\omega} F(\gamma) \right) = +\infty.
\]

**Corollary 2.** Under the hypothesis of Theorem 3,

\[
| M | = | M |_I = \text{sup} \left( \sum_{\gamma \in \Gamma^\omega} \alpha_{i,j} \right) = \text{sup} \left( \sum_{i \in N^+} \alpha_{i,j} \right).
\]

**Proof.** We prove only \( | M | = | M |_I = \text{sup} (\sum_{i \in N^+} \alpha_{i,j}) \) the proof of the other equality being similar. By Theorem 3, we need only verify \( \text{sup} (\sum_{i \in N^+} \alpha_{i,j}^\omega) = \sum_{\gamma \in \Gamma} F(\gamma) \). First

\[
\sum_{i \in N^+} \alpha_{i,j} = \sum_{j \in N^+} F(\lambda_i - \lambda_j) = \sum_{\gamma \in \Gamma^\omega} F(\gamma) \leq \sum_{\gamma \in \Gamma} F(\gamma).
\]
Let \( \Gamma' = \{ \gamma_i, \ldots, \gamma_n \} \) be any finite subset of \( \Gamma \) and let \( \Gamma_0 = \{0, -\gamma_i, \ldots, -\gamma_n \} \). Condition (ii) insures the existence of an \( \alpha \in \Gamma \) such that \( \alpha + \Gamma_0 \subseteq \Lambda \). In particular \( \alpha \in \Lambda \), say \( \alpha = \lambda_k(\alpha) \). But

\[
\Gamma' \subseteq -\Gamma_0 = \alpha - (\alpha + \Gamma_0) \subseteq \lambda_k(\alpha) - \Lambda,
\]

and thus for \( i = k(\alpha) \) we have

\[
\sum_{j \in N^+} \alpha_{i,j} = \sum_{\gamma \in \lambda_k(\alpha) - \Lambda} F(\gamma) \geq \sum_{\gamma \in \Gamma'} F(\gamma)
\]

since \( F \geq 0 \), and our assertion follows.

3. An application. In this section we apply the results of § 2 to evaluate the norm of a special type of linear operator.

**Definition 4.** Let \( T \) be the circle group, considered as the real numbers \( \mathbb{R} \mod 2\pi \), and let \( L^2 = L^2(T, dt) \) be the associated Hilbert function space with respect to normalized Lebesgue measure. Let \( \mathcal{M} \subseteq L^2 \) be the submanifold

\[
\mathcal{M} = \{ f \in L^2; \int_T f(t)dt = 0 \} .
\]

Furthermore, let \( Z' = \mathbb{Z} \sim \{0\} \) and for \( a = \{a_n\}_{n \in \mathbb{Z}'} \in L^1(Z') \) define \( H_a : \mathcal{M} \rightarrow \mathcal{M} \) by

\[
(H_a f)(t) = \sum_{n \in \mathbb{Z}'} a_n f(nt)
\]

(where equality of functions is to be taken in the \( L^2 \) sense).

We now show that the mapping \( a \mapsto H_a \) is a one-to-one bounded linear transformation from \( L^1(Z') \) into \( \mathcal{M}^* \), the dual space of \( \mathcal{M} \). For

\[
\|H_a f\|_2^2 = \left\| \sum_{n \in \mathbb{Z}'} a_n f(nt) \sum_{m \in \mathbb{Z}'} a_m f(mt) \right\|_1
\]

\[
= \left\| \sum_{m,n \in \mathbb{Z}'} a_n a_m f(nt) f(mt) \right\|_1 \leq \sum_{m,n \in \mathbb{Z}'} |a_n| |a_m| \| f(nt) f(mt) \|_1
\]

\[
\leq \left( \sum_{m,n \in \mathbb{Z}'} |a_n| |a_m| \| f(nt) \|_2 \| f(mt) \|_2 \right)^2 \| f \|_2^2 = \| a \|_1^2 \| f \|_2^2
\]

since \( \| f(nt) \|_2 = \| f(t) \|_2 \) for all \( n \in \mathbb{Z}' \). Therefore \( \| H_a \|_{op} \leq \| a \|_1 \).

Also, \( f \in \mathcal{M} \) implies \( H_a f \in \mathcal{M} \) since

\[
\int_T (H_a f)(t)dt = \sum_{n \in \mathbb{Z}'} a_n \int_T f(nt)dt = 0 .
\]

Therefore, since \( H_a \) is clearly linear, \( H_a \in \mathcal{M}^* \) and the mapping
\( a \mapsto H_a \) is bounded and linear from \( L^1(\mathbb{Z}') \) to \( \mathcal{M}^* \). Finally, the mapping is one-to-one since

\[
H_a(e^{it}) = \sum_{n \in \mathbb{Z}'} a_n e^{int} = 0 \quad \Rightarrow \quad a = 0.
\]

We now apply Corollary 1 to evaluate the norm of \( H_a \).

**Theorem 4.** Let \( a = \{a_n\} \in L^1(\mathbb{Z}') \), and for \( r \in Q^* \) let

\[
F(r) = \sum_{m, n \in \mathbb{Z}'} a_m \bar{a}_n.
\]

Then

\[
\| H_a \|_{op} = \max \left| \sum_{r \in \mathbb{Q}^*} (r, x)F(r) \right|^{\frac{1}{2}}
\]

where \( \hat{Q}^* \) is the compact dual of the discrete group \( Q^* \).

**Proof.** Let \( f \in \mathcal{M} \), and let the Fourier expansion of \( f \) be

\[
f(t) = \sum_{m \in \mathbb{Z}'} b_m e^{imt}.
\]

Then

\[
(H_a f)(t) = \sum_{n \in \mathbb{Z}'^*} a_n f(nt) = \sum_{m \in \mathbb{Z}'} a_n \left[ \sum_{m \in \mathbb{Z}'} b_m e^{imnt} \right]
\]

\[
= \sum_{m, n \in \mathbb{Z}'} a_n b_n e^{imnt} = \sum_{r \in \mathbb{Q}^*} c_r e^{r_it},
\]

where \( c_r = \sum_{m, n \in \mathbb{Z}'} a_n b_m \), and \( L^2 \) convergence is the justification for the rearrangement of summation. Therefore

\[
\| H_a f \|^2 = \sum_{r \in \mathbb{Q}^*} |c_r|^2 = \sum_{r \in \mathbb{Q}^*} \left( \sum_{m, n \in \mathbb{Z}'} \bar{a}_m b_m \sum_{m', n' \in \mathbb{Z}'} a_{m'} b_{m'} \right)
\]

\[
= \sum_{m, n, m', n' \in \mathbb{Z}'} \bar{a}_m b_m a_{m'} = \sum_{m, n, m', n' \in \mathbb{Z}'} \left\{ \left( \sum_{n, n' \in \mathbb{Z}'} a_n a_{n'} \right) b_m b_{m'} \right\}
\]

where the manipulation of the quadruple sum is justified by absolute convergence:

\[
\sum_{m, n, m', n' \in \mathbb{Z}'} |\bar{a}_m b_m a_{m'}| = \sum_{n, n', \mathbb{Z}'} a_n \left| a_{n'} \right| \left( \sum_{m, m' \in \mathbb{Z}'} \left| b_m \right| \left| b_{m'} \right| \right)
\]

\[
\leq \sum_{n, n', \mathbb{Z}'} a_n \left| a_{n'} \right| \left( \sum_{m \in \mathbb{Z}'} \left| b_m \right|^2 \right) = \| a \|^2 \| f \|^2 < +\infty
\]

by Cauchy-Schwarz. Upon setting
\[ \alpha_{i,j} = \sum_{m/n=i/j} a_m \bar{a}_n = F\left( \frac{\bar{r}}{j} \right) \quad \text{for } i, j \in \mathbb{Z}' \]

and

\[ M = (\alpha_{i,j}) \quad \text{(order } \mathbb{Z}' = (1, -1, 2, -2, \cdots)) \, , \]

we obtain

\[ M \sim (\mathbb{Q}^\times, \mathbb{Z}', F') \, . \]

Also, upon identifying \( \mathcal{M} \) with \( \mathcal{L}^2 \) by \( f \mapsto X_f = (b_1, b_{-1}, b_2, b_{-2}, \cdots) \) we have

\[ \| H_a f \|_2 = (MX_f, X_f) \, . \]

But

\[
\sum_{r \in \mathbb{Q}^\times} |F(r)| = \sum_{i,j \in \mathbb{Z}'} \int_{[1, j]} |F\left( \frac{\bar{r}}{j} \right)| \leq \sum_{i,j \in \mathbb{Z}'} \sum_{m/n=i/j} |a_n| |a_m| \\
= \left( \sum_{n \in \mathbb{Z}'} |a_n|^2 \right)^{1/2} = \| a \|_1 < +\infty ,
\]

and hence

\[ f(x) = \sum_{r \in \mathbb{Q}^\times} (r, x)F(r) \]

is a continuous function on \( \hat{\mathbb{Q}}^\times \) with Fourier transform \( F \). The theorem follows upon applying Theorem 2 (Corollary 1 (i)) to \( M \sim (\mathbb{Q}, \mathbb{Z}', F') \, . \)

**COROLLARY 3.** If \( a = \{a_n\}_{n \in \mathbb{Z}'} \in L'(\mathbb{Z}') \) and \( a_n \geq 0 \) for all \( n \in \mathbb{Z}' \), then

\[ \| H_a \|_{op} = \| a \|_1 . \]

**Proof.** By Theorem 4,

\[ \| H_{ap} \|_{op} = \max_{x \in \hat{\mathbb{Q}}^\times} \left| \sum_{r \in \mathbb{Q}^\times} (r, x)F(r) \right|^{1/2} \leq \left( \sum_{r \in \mathbb{Q}^\times} |F(r)| \right)^{1/2} = \| a \|_1 \]

since

\[ a_n \geq 0, |F(r)| = \sum_{m/n=r} a_m a_n, \text{ and } \sum_{r \in \mathbb{Q}^\times} |F(r)| = \| a \|_1 \, . \]

But upon setting \( x = 0 \) we obtain
\[
\left| \sum_{r \in Q_x} (r, 0)F(r) \right|^\frac{1}{2} = \left| \sum_{r \in Q_x} F(r) \right|^\frac{1}{2} = \left( \sum_{r \in Q_x} |F(r)| \right)^\frac{1}{2} = \|a\|_1,
\]

and thus the proof is complete.

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Received April 14, 1967, and in revised form November 22, 1968.

**New York University**

**New York, New York**
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