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WILLIAM RICHARD EMERSON

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WILLIAM R. EMERSON

**Let $G$ be a compact Abelian group with discrete countable dual group $\Gamma = \hat{G}$ and let $f \in L^1(G)$ with Fourier transform $F = \hat{f}$. If $V$ is a finite subset of $\Gamma$ we consider the operator $F_V$ on $L^2(V)$:**

$$(F_V\varphi)(\gamma) = \sum_{\tau \in V} F(\gamma - \tau)\varphi(\tau) \qquad \varphi \in L^2(V),\ \gamma \in V .$$

**Then if $\{V_n\}$ is any suitably restricted sequence of finite subsets of $\Gamma$ we show that**

$$\lim_{n\to\infty} | F_{V_n} | = \lim_{n\to\infty} \{\max_{||\varphi||_2=1} | (F_{V_n}\varphi, \varphi) |\} = || f ||_\infty$$

**where $| F_V |$ is the operator norm of $F_V$ on $L^2(V)$ and $(F_V\varphi,\varphi)$ denotes the inner product of $F_V\varphi$ and $\varphi$ (over $V$).**

**This result is then translated into a statement concerning a special class of infinite matrices which generalize the classical Toeplitz matrices. We then apply these results in evaluating the norm of a special type of linear operator.**

In [1] the author considered the asymptotic distribution of eigenvalues and characteristic numbers of certain sequences of operators $\{F_{V_n}\}$ over a locally compact group $\Gamma$ associated with sequences $\{V_n\}$ of Borel sets of $\Gamma$ of finite nonzero measure satisfying

$$(*) \qquad \lim_{n\to\infty} | \gamma V_n \triangle V_n |/| V_n | = 0 \qquad \text{for all } \gamma \in \Gamma ,$$

where $| \ |$ is left Haar measure on $\Gamma$. We write $\{V_n\} \in W_\Gamma$, and say $\{V_n\}$ has the weak ratio property in case $(*)$ is satisfied (see [2]). In this paper we are considering countable Abelian $\Gamma$ and a more general family $T_\Gamma$ of sequences $\{V_n\}$ than those in $W_\Gamma$ (and hence in general the asymptotic distribution of the characteristic numbers of $\{F_{V_n}\}$ does not exist, [2]) but still restricted enough to guarantee an asymptotic formula for the *maximal* characteristic number of $F_{V_n}$ as $n \to \infty$.

## 1. The basic theorem.

$\Gamma$ denotes an arbitrary countably infinite discrete Abelian group equipped with the counting measure.

DEFINITION 1. A sequence $\{V_n\}$ of finite nonempty subsets of $\Gamma$ has *the translation property*, written $\{V_n\} \in T_\Gamma$, if and only if to every finite subset $\Gamma_0 \subseteqq \Gamma$ there corresponds an $n_0 = n_0(\Gamma_0)$ such that for $n \geqq n_0$ there exists a $\tau_n = \tau_n(\Gamma_0) \in \Gamma$ with the property that $\tau_n + \Gamma_0 \subseteqq V_n$.

PROPOSITION 1. *(i) $\{V_n\} \in T_\Gamma$ and $V_n \subseteqq V_n^*$ for $n \in N^+$ implies $\{V_n^*\} \in T_\Gamma$. (ii) $W_\Gamma$ is properly contained in $T_\Gamma$.*

*Proof.* (i) is immediate from the definition. To prove (ii), first assume $\{V_n\} \in W_\Gamma$ and fix any finite nonempty subset $\Gamma_0 \subseteqq \Gamma$. Set $C = \Gamma_0 \cup \{0\}$. We then readily conclude (see [2] where many properties of $W_\Gamma$ are established)

$$\lim_{n\to\infty} \frac{|C + V_n|}{|V_n|} = 1 . \tag{1}$$

But $C + V_n = \bigcup_{\tau \in V_n}(\tau + \Gamma_0) \cup V_n$. Hence if $\tau + \Gamma_0 \not\subseteqq V_n$ for all $\tau \in V_n$, we have $|(\tau + \Gamma_0) \sim V_n| \geqq 1$ for all $\tau \in V_n$ and consequently

$$|C + V_n| \geqq |V_n| + \frac{|V_n|}{|\Gamma_0|} \tag{2}$$

since we may choose $|V_n|$ elements $\tau + \gamma_\tau \in (\tau + \Gamma_0) \sim V_n$, where $\tau \in V_n$ and $\gamma_\tau \in \Gamma_0$, and no element is duplicated more than $|\Gamma_0|$ times. But for sufficiently large $n$ (2) violates (1) and therefore there is a $\tau_n \in V_n$ for which $\tau_n + \Gamma_0 \subseteqq V_n$. Hence $W_\Gamma \subseteqq T_\Gamma$.

We now show inclusion is proper. For let $\{V_n\} \in W_\Gamma$ ($\neq \varnothing$, by [2]); we shall construct a sequence $V_n^* \supseteqq V_n$ such that $\{V_n^*\} \notin W_\Gamma$, which completes the proof of (ii) upon appealing to (i). Fix any $\gamma \in \Gamma \sim \{0\}$. We inductively construct a sequence $\nu_1^{(n)}, \cdots, \nu_{|V_n|}^{(n)}$ as follows: $\nu_1^{(n)} \notin V_n + \{0, \pm\gamma\}$, and

$$\nu_k^{(n)} \notin (V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{k-1}^{(n)}\}) + \{0, \pm\gamma\} \quad (2 \leqq k \leqq |V_n|) . \tag{I}$$

We set $V_n^* = V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{|V|}^{(n)}\}$ and verify that

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} \leqq \frac{1}{2} \qquad (n \in N^+)$$

implying $\{V_n^*\} \notin W_\Gamma$. For

$$\begin{aligned}&(\gamma + V_n^*) \cap V_n^* \\ &= ((\gamma + V_n) \cap V_n^*) \cup (\{\gamma + \nu_1^{(n)}, \cdots, \gamma + \nu_{|V_n|}^{(n)}\} \cap (V_n \cup \{\nu_1^{(n)}, \cdots, \nu_{|V_n|}^{(n)}\})) \\ &= ((\gamma + V_n) \cap V_n^*)\end{aligned}$$

since the second term in the union is empty by (I). Hence $|(\gamma + V_n^*) \cap V_n^*| \leqq |\gamma + V_n| = |V_n|$, and therefore for $n \in N^+$

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} = \frac{|(\gamma + V_n^*) \cap V_n^*|}{2|V_n|} \leqq \frac{|V_n|}{2|V_n|} = \frac{1}{2} .$$

We now prove a result, of independent interest, which is critical in the proof of Theorem 1.

PROPOSITION 2. *Let $G$ be a compact Abelian group (with measure normalized to one), let $f \in L^1(G)$, and let $\rho$ be any positive number. Then*

$$\|f\|_\infty = \sup_\omega \left| \int_G |\omega(x)|^\rho f(x)dx \right|$$

*where $\omega$ ranges over all trigonometric polynomials on $G$ satisfying $\|\omega\|_\rho \leqq 1$.*

*Proof.* Recall that a trigonometric polynomial is a finite linear combination of characters on $G$. Clearly

$$\left| \int_G |\omega(x)|^\rho f(x)dx \right| \leqq \|f\|_\infty \int_G |\omega(x)|^\rho dx = \|f\|_\infty \|\omega\|_\rho^\rho \leqq \|f\|_\infty .$$

We divide the proof of the converse inequality into two cases:

To prove the converse inequality, we first consider the case $\|f\|_\infty < +\infty$. Fix any $\delta > 0$ (until the conclusion of the argument). Let $S = S(\delta)$ be a measurable subset of the complex plane of diameter less than $\delta$ and such that

$$E = f^{-1}(S), \ \|\chi_E f\|_\infty = \|f\|_\infty ,$$

where $\chi_E$ denotes the characteristic function of $E$. Hence for $s \in S$ and $x \in E$ we have

$$||s| - |(\chi_E f)(x)|| \leqq |s - (\chi_E f)(x)| < \delta ,$$

and consequently also

$$||s| - \|f\|_\infty| = ||s| - \|\chi_E f\|_\infty| \leqq \delta .$$

Therefore, if $g = \chi_E/|E|$ then

$$\begin{aligned} 0 &\leqq \|f\|_\infty - \left| \int_G fgdx \right| \leqq |\|f\|_\infty - |s|| + \left| |s| - \left| \int_G fgdx \right| \right| \\ &\leqq |\|f\|_\infty - |s|| + \left| s - \int_G fgdx \right| \\ &= |\|f\|_\infty - |s|| + \left| \frac{1}{|E|} \int_E (s - \chi_E f)dx \right| \leqq 2\delta . \end{aligned} \tag{3}$$

We next wish to approximate $g$ by a *continuous* function $h$, and at this point the estimate is rather delicate because this is also needed later in the case $\|f\|_\infty = +\infty$ and consequently we must avoid $\|f\|_\infty$ as a factor in the error of estimation. Now since $f \in L^1(G)$, to every $\varepsilon > 0$ there corresponds an $\eta = \eta(\varepsilon)$ such that for all measurable subsets $T$ of $G$ of measure at most $\eta$

$$\int_T |f|\,dx < \varepsilon\,.$$

We now choose $\gamma = \gamma(\delta)$ satisfying

$$(4)\quad \text{(i)}\quad \gamma < \delta\,|E|, \quad \text{(ii)}\quad \int_T |f|\,dx < \delta\,|E| \ \text{ if } \ |T| < \gamma\,.$$

Furthermore, since Haar measure is regular, we may find an open set $E^+$ and a closed set $E^-$ such that

$$(4')\qquad E^- \subseteqq E \subseteqq E^+\,, \qquad |E^+ \sim E^-| < \gamma\,.$$

Finally (by Urysohn's Lemma, since $G$ is a normal topological space) there exists a continuous $h_0\colon G \to [0, 1]$ such that $h_0 \,|\, E^- \equiv 1$ and $h_0 \,|\, G \sim E^+ \equiv 0$. Our candidate for $h$ is then defined to be the nonnegative function $h = h_0/|E|$. Let us now estimate $\int_G fg\,dx - \int_G fh\,dx$ :

$$\left| \int_G fg\,dx - \int_G fh\,dx \right| \leqq \int_G |f|\,|g - h|\,dx$$

$$(5)\quad = \left( \int_{E^-} + \int_{E^+ \sim E^-} + \int_{G \sim E^+} \right) |f|\,|g - h|\,dx = \int_{E^+ \sim E^-} |f|\,|g - h|\,dx$$

$$\leqq \max|g - h| \int_{E^+ \sim E^-} |f|\,dx \leqq \frac{1}{|E|} \cdot \delta\,|E| = \delta$$

by (4), (4′) and the definitions of $g$ and $h$. Also, we have

$$\int_{E^-} h\,dx \leqq \int_G h\,dx = \int_{E^+} h\,dx \leqq \|h\|_\infty\,|E^+|\,,$$

implying the estimate

$$(6)\qquad \frac{|E^-|}{|E|} \leqq \|h\|_1 \leqq \frac{|E^+|}{|E|} \leqq 1 + \delta \ \text{ by virtue of (4) and (4′)}\,.$$

Lastly, to any $\alpha > 0$ we may correspond a trigonometric polynomial $\omega_\alpha$ satisfying $\|h^{1/p} - \omega_\alpha\|_\infty < \alpha$, and consequently $\|h^{1/p} - |\omega_\alpha|\,\|_\infty < \alpha$ since $h^{1/p} \geqq 0$. Thus by choosing $\alpha_0 = \alpha_0(\delta)$ sufficiently small we may conclude

$$(7)\qquad \|h - |\omega_{\alpha_0}|^p\|_1 \leqq \|h - |\omega_{\alpha_0}|^p\|_\infty < \delta\,.$$

Also,

$$\|\omega_{\alpha_0}^p\|_1 \leqq \|h - |\omega_{\alpha_0}|^p\|_1 + \|h\|_1 \leqq \delta + (1 + \delta) = 1 + 2\delta\,.$$

We now let

$$\omega = \omega_{\alpha_0}/(1 + 2\delta)^{1/p}, \ \text{ implying } \ \|\omega\|_p \leqq 1\,.$$

Finally,

$$
\begin{aligned}
\left|\int_G |\omega|^p f dx\right| &= \frac{1}{1+2\delta}\left|\int_G |\omega_{\alpha_0}|^p f dx\right| \\
(8) \qquad &\geqq \frac{1}{1+2\delta}\left(\left|\int_G fgdx\right| - \left|\int_G fgdx - \int_G fhdx\right| - \left|\int_G hfdx - \int_G |\omega_{\alpha_0}|^p fdx\right|\right) \\
&\geqq \frac{1}{1+2\delta}((\|f\|_\infty - 2\delta) - \delta - \delta\|f\|_1) .
\end{aligned}
$$

By (3), (5), and (7). Our assertion follows upon letting $\delta \to 0$.

In case $\|f\|_\infty = +\infty$, we let $S_n$ be a measurable subset of the complex plane of diameter less than $\delta$ and such that $E_n = f^{-1}(S_n)$, $\|\chi_{E_n} f\|_\infty > n$. Equations (3) – (8) still hold with $\|f\|_\infty$ replaced by $\|\chi_{E_n} f\|_\infty > n$ wherever it occurs, and we readily construct trigonometric polynomials $\omega_n$ with $\|\omega_n\|_p \leqq 1$ and such that $\int_G |\omega_n|^p f dx$ is unbounded as $n \to +\infty$.

We now are ready to prove the basic theorem.

THEOREM 1. *Let $G$ be a compact group (with measure normalized to one), let $f \in L^1(G)$, and let $F = \hat{f} \in L^\infty(\Gamma)$, the Fourier Transform of $f$. Furthermore, let $\{V_n\} \in T_\Gamma$ and let $F_{V_n}$ be the Hilbert-Schmidt operator on $L^2(V_n)$:*

$$
(F_{V_n}\psi)(\gamma) = \int_{V_n} F(\gamma - \tau)\psi(\tau)d\tau = \sum_{\tau \in V_n} F(\gamma - \tau)\psi(\tau) \qquad (\psi \in L^2(V_n),\ \gamma \in V_n) .
$$

*Let $(F_{V_n}\psi, \psi)_{V_n}$ denote the inner product of $F_{V_n}\psi$ and $\psi$* over $V_n$, *and let $|F_{V_n}|$ denote the maximal characteristic number of $F_{V_n}$ as an operator on the Hilbert space $L^2(V_n)$. Then*

( i ) $$\lim_{n\to\infty} \max_{\|\psi\|_2 = 1} |(F_{V_n}\psi, \psi)_{V_n}| = \|f\|_\infty .$$

( ii ) $$\lim_{n\to\infty} |F_{V_n}| = \|f\|_\infty .$$

*Proof.* (i) By definition,

$$
\begin{aligned}
(F_{V_n}\psi, \psi)_{V_n} &= \sum_{\gamma, \tau \in V_n} F(\gamma - \tau)\psi(\tau)\overline{\psi(\gamma)} \\
&= \sum_{\gamma, \tau \in V_n}\left[\int_G \overline{(\gamma - \tau, x)} f(x)dx\right]\psi(\tau)\overline{\psi(\gamma)} \\
&= \int_G\left[\sum_{\gamma, \tau \in V_n} (\tau, x)\psi(\tau)\overline{(\gamma, x)\psi(\gamma)}\right] f(x)dx \\
&= \int_G \left|\sum_{\tau \in V_n} (\tau, x)\psi(\tau)\right|^2 f(x)dx .
\end{aligned}
$$

Note that

$$\omega_\psi(x) = \sum_{\tau \in V_n} (\tau, x)\psi(\tau)$$

is a trigonometric polynomial on $G$, and $\psi \to \omega_\psi$ is an isometry of $L^2(V_n)$ into $L^2(G)$ since $\|\omega_\psi\|_2^2 = \sum_{\tau \in V_n} |\psi(\tau)|^2 = \|\psi\|_2^2$. Therefore

$$(\dagger) \qquad \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \max_{\|\omega\|_2=1} \left| \int_G \left|\omega(x)\right|^2 f(x)dx \right|$$

where $\omega$ ranges over linear combinations of characters on $G$ generated by elements in $V_n$. Hence by Proposition 2 ($\rho = 2$),

$$\lim_{n\to\infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| \leqq \|f\|_\infty .$$

On the other hand, let $\omega$ be *any* trigonometric polynomial on $G$, say

$$\omega(x) = \sum_{1\leqq i\leqq k} (\gamma_i, x)c_i \qquad (c_i \in \mathbb{C},\ \gamma_i \in \Gamma) .$$

Let $\Gamma_0 = \{\gamma_1, \cdots, \gamma_k\}$, a finite subset of $\Gamma$. Now since $\{V_n\} \in T_\Gamma$ there exists an $n_0$ such that for $n \geqq n_0$ there exists $\tau_n \in \Gamma$ such that $\tau_n + \Gamma_0 \subseteqq V_n$. Hence for $n \geqq n_0$,

$$\omega_n(x) = (\tau_n, x)\omega(x) = \sum_{1\leqq i\leqq k} (\tau_n + \gamma_i, x)c_i$$

is a linear combination of characters on $G$ generated by elements of $V_n$. Since $|\omega(x)| = |\omega_n(x)|$ for all $x \in G$, the proof of (i) is completed by again applying Proposition 2 with $\rho = 2$.

(ii) Recall that $|F_{V_n}|$ is the norm of $F_{V_n}$ considered as an operator on $L^2(V_n)$, i.e.,

$$|F_{V_n}| = \max_{\|\psi\|_2=1} \|F_{V_n}\psi\|_2 .$$

but by the Cauchy-Schwarz Inequality, for $\|\psi\|_2 = 1$

$$|(F_{V_n}\psi, \psi)_{V_n}| \leqq \|F_{V_n}\psi\|_2 \|\psi\|_2 = \|F_{V_n}\psi\|_2 \leqq |F_{V_n}|$$

and therefore by (i),

$$\varliminf_{n\to\infty} |F_{V_n}| \geqq \varliminf_{n\to\infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \|f\|_\infty .$$

Thus, if $\|f\|_\infty = +\infty$ nothing remains to be proved. If $\|f\|_\infty < +\infty$ we have $f \in L^1(G) \cap L^\infty(G)$, and therefore by [3], p. 445, $|F_{V_n}| \leqq \|f\|_\infty$ for all $n \in N^+$. Hence $\varlimsup_{n\to\infty} |F_{V_n}| \leqq \|f\|_\infty$, and consequently $\lim_{n\to\infty} |F_{V_n}| = \|f\|_\infty$ in this case as well.

We now conversely prove that the hypothesis $\{V_n\} \in T_\Gamma$ is in fact necessary for the conclusion of Theorem 1. More precisely,

THEOREM 1′. *Using the notation of Theorem* 1, *if* $\{V_n\}$ *is any sequence of finite subsets of* $\Gamma$ *for which conclusion* (i) *holds for all trigonometric polynomials* $f$ *on* $G$, *then* $\{V_n\} \in T_\Gamma$.

*Proof.* Assume $\{V_n\} \notin T_\Gamma$, i.e., there exists a finite subset $\Gamma_0$ of $\Gamma$ such that no translate of $\Gamma_0$ lies in $V_m$ for an appropriate subsequence $m \to \infty$. We then assert that

$$f(x) = \frac{1}{|\Gamma_0|} \sum_{\tau \in \Gamma_0} (\tau, x) \qquad (\|f\|_\infty = f(0) = 1)$$

is a trigonometric polynomial for which (i) fails. More precisely we show for all these $m$:

$$\operatorname*{Max}_{\|\psi\|_2=1} |(F_{V_m}\psi, \psi)_{V_m}| \leqq \left(1 - \frac{1}{2|\Gamma_0|}\right)\|f\|_\infty .$$

Recalling relation (†) of the proof of Theorem 1. We have:

$$(\dagger) \qquad \operatorname*{Max}_{\|\psi\|_2=1} |(F_{V_m}\psi, \psi)_{V_m}| = \operatorname*{Max}_{\|\omega\|_2=1} \left| \int_G |\omega(x)|^2 f(x)dx \right|$$

where $\omega$ ranges over all linear combinations of characters on $G$ generated by elements in $V_m$.

However, any such $\omega$ is of the form

$$\omega(x) = \sum_{\tau \in V_m} (\tau, x)a_\tau$$

where

$$\sum_{\tau \in V_m} |a_\tau|^2 = \|\omega\|_2^2 \leqq 1 ,$$

implying

$$|\omega(x)|^2 = \sum_{\tau_1, \tau_2 \in V_m} (\tau_1 - \tau_2, x)a_{\tau_1}\bar{a}_{\tau_2} ,$$

and finally

$$\int_G |\omega(x)|^2 f(x)dx = \frac{1}{|\Gamma_0|} \sum_{\substack{\tau_1, \tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} a_{\tau_1}\bar{a}_{\tau_2} .$$

Consequently,

$$\begin{aligned}\left|\int_G |\omega(x)|^2 f(x)dx\right| &\leqq \frac{1}{|\Gamma_0|}\sum_{\substack{\tau_1,\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} |a_{\tau_1}|\,|a_{\tau_2}| \leqq \frac{1}{2|\Gamma_0|}\sum_{\substack{\tau_1,\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} (|a_{\tau_1}|^2+|a_{\tau_2}|^2)\\
&= \frac{1}{2|\Gamma_0|}\left(\sum_{\tau_1\in V_m}|a_{\tau_1}|^2\left(\sum_{\substack{\tau_2\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} 1\right)+\sum_{\tau_2\in V_m}|a_{\tau_2}|^2\left(\sum_{\substack{\tau_1\in V_m\\ \tau_2-\tau_1\in\Gamma_0}} 1\right)\right)\\
&= \frac{1}{2|\Gamma_0|}\left(\sum_{\tau_1\in V_m}|a_{\tau_1}|^2\,|(V_m-\tau_1)\cap\Gamma_0|\right.\\
&\qquad\left.+\sum_{\tau_2\in V_m}|a_{\tau_2}|^2\,|(\tau_2-V_m)\cap\Gamma_0|\right)\\
&\leqq \frac{1}{2|\Gamma_0|}\left\{(|\Gamma_0|-1)\sum_{\tau_1\in V_m}|a_{\tau_1}|^2+|\Gamma_0|\sum_{\tau_2\in V_m}|a_{\tau_2}|^2\right\}\\
&= \left(1-\frac{1}{2|\Gamma_0|}\right)\sum_{\tau\in V_m}|a_\tau|^2 \leqq \left(1-\frac{1}{2|\Gamma_0|}\right)\\
&= \left(1-\frac{1}{2|\Gamma_0|}\right)||f||_\infty\end{aligned}$$

since no translate $V_m-\tau_1$ contains $\Gamma_0$ by hypothesis. Our assertion now readily follows.

## 2. A class of doubly-infinite matrices.

We now translate the theorem of the preceding section into a statement concerning a class of doubly-infinite complex matrices $M=(\alpha_{i,j})_{i,j=1}^\infty$ whose entries $\alpha_{i,j}$ are determined by a "group law".

DEFINITION 2. Let $M=(\alpha_{i,j})_{i,j=1}^\infty$ be a matrix with complex entries. We then write

$$M\sim(\Gamma,\Lambda,F)$$

if and only if

(i) $\Gamma$ is a countable Abelian group.

(ii) $\Lambda$ is a subset of $\Gamma$.

(iii) $F\colon\Gamma\to\mathbb{C}$.

(iv) There exists an ordering of $\Lambda=\{\lambda_1,\cdots,\lambda_n,\cdots\}$ such that for all $i,j\in N^+$,

$$\alpha_{i,j}=F(\lambda_i-\lambda_j)\ .$$

REMARK. For *any* $M=(\alpha_{i,j})_{i,j=1}^\infty$ with complex entries we may take $\Gamma$ to be $Q^\times$, the multiplicative group of rational numbers, and $\Lambda$ to be $P=\{p_n\colon n\in N^+\}$, the set of all positive integral primes, upon defining $F$ by

$$F(r)=\begin{cases}\alpha_{i,j} & \text{if } r=p_i/p_j\\ 0 & \text{otherwise}\ .\end{cases}$$

We then have $M \sim (Q^{\times}, P, F)$.

Under suitable restrictions on $(\Gamma, \Lambda, F)$ we shall be able to compute the norm and quadratic norm of the matrix $M$, which are defined as follows:

DEFINITION 3. The norm of $M$, $|M|$, and the quadratic norm of $M$, $|M|_I$, are defined by

$$|M| = \sup_{\|X\|_2 \leqq 1} \|MX\|_2 , \qquad |M|_I = \sup_{\|X\|_2 \leqq 1} |(MX, X)| ,$$

where $X = (\{x_i\})$ ranges over elements of the complex Hilbert space $l^2$ with only *finitely* many $x_i \neq 0$, and $MX = (\{\sum_j \alpha_{i,j} x_j\})$.

LEMMA 1. *If $M$ induces a bounded operator on $l^2$, then*

$$\text{(i)} \quad |M| = \sup_{\|X\|_2 \leqq 1} \|MX\| , \qquad \text{(ii)} \quad |M|_I = \sup_{\|X\|_2 \leqq 1} |(MX, X)| ,$$

*where $X = (\{x_i\})$ ranges over all elements of $l^2$ (with $\|X\|_2 \leqq 1$). Hence in this case $|M|$ is the standard norm of $M$ considered as a bounded linear operator on $l^2$.*

*Proof.* For $x \in l^2$, let $X_n$ be the projection of $X$ on its first $n$ components (0 elsewhere). Since $M$ is bounded and consequently closed, $\lim_{n \to \infty} MX_n = MX$ and (i) follows since $X_n$ has at most $n$ nonzero components. Also

$$(MX, X) = (MX_n, X_n) + (M(X - X_n), X_n) + (MX, X - X_n) ,$$

and therefore

$$\begin{aligned} &|(MX, X) - (MX_n, X_n)| \\ &\leqq \|M(X - X_n)\|_2 \|X_n\|_2 + \|MX\|_2 \|X - X_n\|_2 \to 0 \end{aligned}$$

as $n \to \infty$, and (ii) clearly follows.

THEOREM 2. *Let $M \sim (\Gamma, \Lambda, F)$ where*

( i ) *$F \in A(\Gamma)$, i.e., $F = \hat{f}$ for some $f \in L^1(G)$.*

( ii ) *To each finite subset $\Gamma_0 \subseteqq \Gamma$ there corresponds a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteqq \Lambda$.*

*Then $|M| = |M|_I = \|f\|_\infty$.*

*Proof.* Assume $\Lambda = \{\lambda_1, \cdots, \lambda_n, \cdots\}$ as in Definition 2, and set $V_n = \{\lambda_1, \cdots, \lambda_n\}$. Then hypothesis (ii) clearly implies $\{V_n\} \in T_\Gamma$. The theorem will follow from the two inequalities

( i ) $|M| \leqq \|f\|_\infty$

( ii ) $\|f\|_\infty \leqq |M|_I$,

since $|M|_I \leqq |M|$ by the Cauchy-Schwarz inequality.

(i): If $||f||_\infty = +\infty$ there is nothing to prove. Otherwise $f \in L^1(G) \cap L^\infty(G)$, and therefore the operator $M'\colon L^2(\Gamma) \to L^2(\Gamma)$ defined by

$$(M'\varphi)(\gamma) = \sum_{\tau \in \Gamma} F(\gamma - \tau)\varphi(\tau) \qquad (\varphi \in L^2(\Gamma)\ ,\ \gamma \in \Gamma)$$

has norm $|M'| = ||f||_\infty$ by [3], § 3.2., p. 441. Hence if we only consider $\varphi$ with support in $\Lambda$ and restrict $\gamma$ to $\Lambda$, $M'$ restricts to an operator $M''\colon L^2(\Lambda) \to L^2(\Lambda)$ with $|M''| \leqq |M'|$. Now consider the isometry of $l^2$ onto $L^2(\Lambda)$ given by $X = (\{x_n\}) \rightsquigarrow \varphi_X$ where $\varphi(\lambda_n) = x_n$ for $n \in N^+$. Then for this $\varphi = \varphi_X$ and $\lambda_i \in \Lambda$,

$$(M''\varphi)(\lambda_i) = \sum_{\lambda \in \Lambda} F(\lambda_i - \lambda)\varphi(\lambda) = \sum_j F(\lambda_i - \lambda_j)\varphi(\lambda_j) = \sum_j \alpha_{i,j} x_j$$

which is the $i^{\text{th}}$ component of $MX$, and therefore $|M| = |M''| \leqq |M'| = ||f||_\infty$ (and thus $M$ induces a bounded linear operator if $||f||_\infty < +\infty$).

(ii): For $n \in N^+$, consider the isometry of $L^2(V_n)$ (which is none other than $n$-dimensional Euclidean space) into $l^2$ given by $\varphi \to X_\varphi$ where $X_\varphi = (\{x_j^\varphi\})$ and $x_j^\varphi = \varphi(\lambda_j)$ for $1 \leqq j \leqq n$ and 0 otherwise. Hence $X_\varphi$ has only finitely many nonzero components, and each $X \in l^2$ with only finitely many nonzero components is in the image of $L^2(V_n)$ under the above isometry for $n = n(X)$ sufficiently large. Now consider $F_{V_n}$ on $L^2(V_n)$:

$$\begin{aligned}(F_{V_n}\varphi, \varphi)_{V_n} &= \sum_{1 \leqq i \leqq n} \sum_{1 \leqq j \leqq n} F(\lambda_i - \lambda_j)\varphi(\lambda_j)\overline{\varphi(\lambda_i)} \\ &= \sum_{1 \leqq i, j \leqq n} \alpha_{i,j} x_j \bar{x}_i = (MX_\varphi, X_\varphi) \qquad (\text{in } l^2)\ .\end{aligned}$$

But by Theorem 1 (i) $\lim_{n\to\infty} \max_{||\varphi||_2 = 1} |(F_{V_n}\varphi, \varphi)_{V_n}| = ||f||_\infty$ and therefore $|M|_I = ||f||_\infty$ since

$$|M|_I = \sup_{\substack{||x||_2 \leqq 1 \\ x_i \neq 0 \text{ finitely}}} |(MX, X)| = \lim_{n\to\infty} \max_{||\varphi||_2 \leqq 1} |(F_{V_n}\varphi, \varphi)_{V_n}| = ||f||_\infty\ .$$

COROLLARY 1. (1) *Hypothesis* (i) *of Theorem* 2 *is satisfied if*

(i)′ $$\sum_{\gamma \in \Gamma} |F(\gamma)|^2 < +\infty\ .$$

(2) *Hypothesis* (ii) *of Theorem* 2 *is satisfied if*

(ii)′ $\Lambda + \Lambda \subseteqq \Lambda$ *and* (ii)″ $\Lambda$ *generates* $\Gamma$ .

*Proof*. (1), (i)′ implies $f(x) = \sum_{\gamma \in \Gamma}(\gamma, x)F(\gamma) \in L^2(G)$ and therefore also $f \in L^1(G)$ since $G$ is compact and hence of finite measure. Clearly $F = \hat{f} \in A(\Gamma)$.

(2) First note that any element of $\Gamma$ is the difference of two elements in $\Lambda$, i.e., $\Gamma = \Lambda - \Lambda$. For by (ii)$''$, if $\gamma \in \Gamma$ we have $\gamma = \lambda_{i_1} + \cdots + \lambda_{i_k} - \cdots - \lambda_{i_n}$ for some suitable finite sequence of integers $i_1, \cdots, i_n$ (if no terms with a plus sign occur we may take $k = 0$, if none with a minus sign occur take $k = n$). But by (ii)$'$, $\gamma^+ = \lambda_{i_1} + \cdots + \lambda_{i_k} \in \Lambda$ if $k > 0$, $\gamma^- = \lambda_{i_{k+1}} + \cdots + \lambda_{i_n} \in \Lambda$ if $k < n$. If $k = 0$ we may write $\gamma = \lambda_1 - (\lambda_1 + \cdots + \lambda_n + \lambda_1)$ and similarly $\gamma = (\lambda_1 + \cdots + \lambda_n + \lambda_1) - \lambda_1$ if $k = n$.

Now let $\Gamma_0 = \{\gamma_1, \cdots, \lambda_k\}$ be a nonempty finite subset of $\Gamma$. Then for appropriate $a_i, b_i \in N^+$ we have

$$\gamma_i = \lambda_{a_i} - \lambda_{b_i} \qquad (1 \leqq i \leqq k) .$$

Consequently, for $1 \leqq i \leqq k$,

$$\gamma_i = \lambda_{a_i} + \lambda_{b_1} + \cdots + \widehat{\lambda}_{b_i} + \cdots + \lambda_{b_k} - (\lambda_{b_1} + \cdots + \lambda_{b_k})$$

where $(\hat{\ })$ denotes deletion of a term. Hence if we set $\gamma = \lambda_{b_1} + \cdots + \lambda_{b_k}$ we have $\gamma + \Gamma_0 \subseteqq \Lambda$ since

$$\lambda_{a_i} + \lambda_{b_1} + \cdots + \widehat{\lambda}_{b_i} + \cdots + \lambda_{b_k} \in \Lambda$$

by (ii)$'$.

We now apply Theorem 2 to completely solve the norm evaluation in the case $M \sim (\Gamma, \Lambda, F)$ where $F \geqq 0$ and $\Lambda$ satisfies (ii). We make use of the following simple lemma:

LEMMA 2. *If* $M = (\alpha_{i,j})_{i,j=1}^{\infty}$ *and* $M' = (\alpha'_{i,j})_{i,j=1}^{\infty}$ *where*

$$\alpha_{i,j} \geqq \alpha'_{i,j} \geqq 0 \qquad \textit{for all } i, j \in N^+$$

*then*

(i) $$|M| = \sup_{||x||_2 \leqq 1} ||MX|| , \quad |M|_I = \sup_{||x||_2 \leqq 1} (MX, X)$$

*where* $X = (\{x_i\})$ *has only finitely many nonzero coordinates, all positive.*

(ii) $$|M'| \leqq |M| , \quad |M'|_I \leqq |M|_I .$$

*Proof.* For $X = (\{x_i\}) \in l^2$ we define $X^+ = (\{|x_i|\})$. Note $||X||_2 = ||X^+||_2$ and $X^+$ and $X$ have the same cardinality of nonzero coordinates. Also, $\alpha_{i,j} \geqq 0$ clearly implies $||MX||_2 \leqq ||MX^+||_2$ and $|(MX, X)| \leqq (MX^+, X^+)$ and (i) readily follows. But $\alpha_{i,j} \geqq \alpha'_{i,j} \geqq 0$ also implies each component of $M'X^+$ is dominated by the corresponding component of $MX^+$ and hence (ii) follows from (i).

THEOREM 3. *Let* $M \sim (\Gamma, \Lambda, F)$ *where*

(i) *$F(\gamma) \geqq 0$ for all $\gamma \in \Gamma$.*

(ii) *If $\Gamma_0$ is any finite subset of $\Gamma$ there exists a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteqq \Lambda$. Then*

$$|M| = |M|_I = \sum_{\gamma \in \Gamma} F(\gamma) \qquad (\textit{possibly } +\infty) .$$

*Proof.* Since $|M|_I \leqq |M|$ it suffices to show that

$$(1) \quad |M| \leqq \sum_{\gamma \in \Gamma} F(\gamma) , \qquad (2) \quad |M|_I = \sum_{\gamma \in \Gamma} F(\gamma) .$$

If $\sum_{\gamma \in \Gamma} F(\gamma) < +\infty$ there is nothing to prove since in this case the result is included in Theorem 2 because $f(x) = \sum_{\gamma \in \Gamma}(\gamma, x)F(\gamma)$ is a continuous function on $G$, $F = \hat{f}$, and $\|f\|_\infty = f(0) = \sum_{\gamma \in \Gamma} F(\gamma)$.

On the other hand, if $\sum_{\gamma \in \Gamma} F(\gamma) = +\infty$ then $F$ may not be in $A(\Gamma)$ and hence we cannot apply Theorem 2 directly. Clearly (1) is true in this case and we need only verify (2). Let $\Gamma' = \{\gamma_1, \cdots, \gamma_n\}$ be any finite subset of $\Gamma$ and define

$$M_{\Gamma'} = (\alpha_{i,j}^{\Gamma'})_{i,j=1}^{\infty}$$

where

$$\alpha_{i,j}^{\Gamma'} = \begin{cases} F(\gamma_\nu) & \text{if } \lambda_i - \lambda_j = \gamma_\nu \in \Gamma' \\ 0 & \text{otherwise} , \end{cases}$$

i.e., $M_{\Gamma'} \sim (\Gamma, \Lambda, F_{\Gamma'})$ where $F_{\Gamma'}(\gamma) = F(\gamma) I_{\Gamma'}(\gamma)$. Since $F \geqq 0$, $\alpha_{i,j} \geqq \alpha_{i,j}^{\Gamma'} \geqq 0$ for $i, j \in N^+$, and Lemma 2 implies $|M|_I \geqq |M_{\Gamma'}|_I$. But by Theorem 2

$$|M_{\Gamma'}|_I = \operatorname*{ess\,sup}_{x \in G} \left| \sum_{\gamma \in \Gamma} (\gamma, x) F_{\Gamma'}(\gamma) \right| = \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $\sum_{\gamma \in \Gamma}(\gamma, x) F_{\Gamma'}(\gamma)$ is continuous and $F_{\Gamma'} \geqq 0$. This in turn implies

$$|M|_I \geqq \sup_{|\Gamma'| < +\infty} \sum_{\gamma \in \Gamma'} F(\gamma) = +\infty .$$

COROLLARY 2. *Under the hypothesis of Theorem* 3.

$$|M| = |M|_I = \sup_{i \in N^+} \Big( \sum_{j \in N^+} \alpha_{i,j} \Big) = \sup_{j \in N^+} \Big( \sum_{i \in N^+} \alpha_{i,j} \Big) .$$

*Proof.* We prove only $|M| = |M|_I = \sup_{i \in N^+}(\sum_{j \in N^+} \alpha_{i,j})$ the proof of the other equality being similar. By Theorem 3. we need only verify $\sup_{i \in N^+}(\sum_{j \in N^+} \alpha_{i,j}) = \sum_{\gamma \in \Gamma} F(\gamma)$. First

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{j \in N^+} F(\lambda_i - \lambda_j) = \sum_{\gamma \in \lambda_i - \Lambda} F(\gamma) \leqq \sum_{\gamma \in \Gamma} F(\gamma) .$$

Let $\Gamma' = \{\gamma_1, \cdots, \gamma_n\}$ be any finite subset of $\Gamma$ and let $\Gamma_0 = \{0, -\gamma_1, \cdots, -\gamma_n\}$. Condition (ii) insures the existence of an $\alpha \in \Gamma$ such that $\alpha + \Gamma_0 \subseteqq \Lambda$. In particular $\alpha \in \Lambda$, say $\alpha = \lambda_{k(\alpha)}$. But

$$\Gamma' \subseteqq -\Gamma_0 = \alpha - (\alpha + \Gamma_0) \subseteqq \lambda_{k(\alpha)} - \Lambda ,$$

and thus for $i = k(\alpha)$ we have

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{\gamma \in \lambda_{k(\alpha)} - \Lambda} F(\gamma) \geqq \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $F \geqq 0$, and our assertion follows.

## 3. An application.

In this section we apply the results of § 2 to evaluate the norm of a special type of linear operator.

DEFINITION 4. Let $T$ be the circle group, considered as the real numbers $R^+ \bmod 2\pi$, and let $L^2 = L^2(T, dt)$ be the associated Hilbert function space with respect to normalized Lebesgue measure. Let $\mathscr{M} \subseteqq L^2$ be the submanifold

$$\mathscr{M} = \left\{f \in L^2 \colon \int_T f(t)dt = 0\right\} .$$

Furthermore, let $\boldsymbol{Z}' = \boldsymbol{Z} \sim \{0\}$ and for $a = \{a_n\}_{n \in \boldsymbol{Z}'} \in L^1(\boldsymbol{Z}')$ define $H_a \colon \mathscr{M} \to \mathscr{M}$ by

$$(H_a f)(t) = \sum_{n \in \boldsymbol{Z}'} a_n f(nt)$$

(where equality of functions is to be taken in the $L^2$ sense).

We now show that the mapping $a \rightsquigarrow H_a$ is a one-to-one bounded linear transformation from $L^1(\boldsymbol{Z}')$ into $\mathscr{M}^*$, the dual space of $\mathscr{M}$. For

$$\begin{aligned} \| H_a f \|_2^2 &= \left\| \sum_{n \in \boldsymbol{Z}'} a_n f(nt) \sum_{m \in \boldsymbol{Z}'} \overline{a_m f(mt)} \right\|_1 \\ &= \left\| \sum_{m,n \in \boldsymbol{Z}'} a_n \overline{a_m} f(nt) \overline{f(mt)} \right\|_1 \leqq \sum_{m,n \in \boldsymbol{Z}'} | a_n | \, | a_m | \, \| f(nt) f(mt) \|_1 \\ &\leqq \| \sum_{m,n \in \boldsymbol{Z}'} | a_n | \, | a_m | \, \| f(nt) \|_2 \, \| f(mt) \|_2 = \left( \sum_{n \in \boldsymbol{Z}'} | a_n | \right)^2 \| f \|_2^2 = \| a \|_1^2 \, \| f \|_2^2 \end{aligned}$$

since $\| f(nt) \|_2 = \| f(t) \|_2$ for all $n \in \boldsymbol{Z}'$. Therefore $\| H_a \|_{op} \leqq \| a \|_1$. Also, $f \in \mathscr{M}$ implies $H_a f \in \mathscr{M}$ since

$$\int_T (H_a f)(t)dt = \sum_{n \in \boldsymbol{Z}'} a_n \int_T f(nt)dt = 0 .$$

Therefore, since $H_a$ is clearly linear, $H_a \in \mathscr{M}^*$ and the mapping

$a \rightsquigarrow H_a$ is bounded and linear from $L^1(\boldsymbol{Z}')$ to $\mathscr{M}^*$. Finally, the mapping is one-to-one since

$$H_a(e^{it}) = \sum_{n \in \boldsymbol{Z}'} a_n e^{int} = 0 \Leftrightarrow a = 0 .$$

We now apply Corollary 1 to evaluate the norm of $H_a$.

THEOREM 4. *Let $a = \{a_n\}_{n \in \boldsymbol{Z}'} \in L^1(\boldsymbol{Z}')$, and for $r \in Q^\times$ let*

$$F(r) = \sum_{\substack{m/n=r \\ m,n \in \boldsymbol{Z}'}} a_m \bar{a}_n .$$

*Then*

$$\| H_a \|_{op} = \max_{x \in \hat{Q}^\times} \left| \sum_{r \in Q^\times} (r, x) F(r) \right|^{\frac{1}{2}}$$

*where $\hat{Q}^\times$ is the compact dual of the discrete group $Q^\times$.*

*Proof.* Let $f \in \mathscr{M}$, and let the Fourier expansion of $f$ be

$$f(t) = \sum_{m \in \boldsymbol{Z}'} b_m e^{imt} .$$

Then

$$(H_a f)(t) = \sum_{n \in \boldsymbol{Z}'} a_n f(nt) = \sum_{n \in \boldsymbol{Z}'} a_n \Big[ \sum_{m \in \boldsymbol{Z}'} b_m e^{imnt} \Big]$$
$$= \sum_{m,n \in \boldsymbol{Z}'} a_n b_m e^{imnt} = \sum_{p \in \boldsymbol{Z}'} c_p e^{ipt} ,$$

where $c_p = \sum_{mn=p \; m,n \in \boldsymbol{Z}'} a_n b_m$, and $L^2$ convergence is the justification for the rearrangement of summation. Therefore

$$\| H_a f \|_2^2 = \sum_{p \in \boldsymbol{Z}'} | c_p |^2 = \sum_{p \in \boldsymbol{Z}'} \Big( \sum_{\substack{mn=p \\ m,n \in \boldsymbol{Z}'}} \overline{a_n b_m} \sum_{\substack{m'n'=p \\ m',n' \in \boldsymbol{Z}'}} a_{n'} b_{m'} \Big)$$
$$= \sum_{\substack{m,n,m',n' \in \boldsymbol{Z}' \\ mn=m'n'}} \bar{a}_n \bar{b}_m a_{n'} b_{m'} = \sum_{m,m' \in \boldsymbol{Z}'} \Big\{ \Big( \sum_{\substack{n,n' \in \boldsymbol{Z}' \\ n'/n=m/m'}} a_{n'} \overline{a_n} \Big) b_{m'} \overline{b_{m'}} \Big\}$$

where the manipulation of the quadruple sum is justified by absolute convergence:

$$\sum_{\substack{m,n,m',n' \in \boldsymbol{Z}' \\ mn=m'n'}} | \bar{a}_n \bar{b}_m a_{n'} b_{m'} | = \sum_{n,n' \in \boldsymbol{Z}'} | a_n | \, | a_{n'} | \Big( \sum_{\substack{m,m' \in \boldsymbol{Z}' \\ n'/n=m/m'}} | b_m | (b_{m'}) \Big)$$
$$\leqq \sum_{n,n' \in \boldsymbol{Z}'} | a_n | \, | a_{n'} | \Big( \sum_{m \in \boldsymbol{Z}'} | b_m |^2 \Big) = \| a \|_1^2 \, \| f \|_2^2 < +\infty$$

by Cauchy-Schwarz. Upon setting

$$\alpha_{i,j} = \sum_{m/n=i/j} a_m \bar{a}_n = F\left(\frac{i}{j}\right) \qquad \text{for } i, j \in \boldsymbol{Z}'$$

and

$$M = (\alpha_{ij}) \quad (\text{order } \boldsymbol{Z}' = (1, -1, 2, -2, \cdots)) ,$$

we obtain

$$M \sim (Q^\times, \boldsymbol{Z}', F) .$$

Also, upon identifying $\mathscr{M}$ with $\mathscr{L}^2$ by $f \leftrightarrow X_f = (b_1, b_{-1}, b_2, b_{-2}, \cdots)$ we have

$$\| H_a f \|_2^2 = (MX_f, X_f) .$$

But

$$\sum_{r \in Q^\times} | F(r) | = \sum_{\substack{i,j \in \boldsymbol{Z}' \\ (i,j)=1 \\ i>0}} \left| F\left(\frac{i}{j}\right) \right| \leqq \sum_{\substack{i,j \in \boldsymbol{Z}' \\ (i,j)=1 \\ i>0}} \sum_{m/n=i/j} | a_n | \, | a_m |$$
$$= \left( \sum_{n \in \boldsymbol{Z}'} | a_n | \right)^2 = \| a \|_1^2 < +\infty ,$$

and hence

$$f(x) = \sum_{r \in Q^\times} (r, x) F(r)$$

is a continuous function on $\hat{Q}^\times$ with Fourier transform $F$. The theorem follows upon applying Theorem 2 (Corollary 1 (i)) to $M \sim (Q, \boldsymbol{Z}', F)$ .

COROLLARY 3. *If* $a = \{a_n\}_{n \in \boldsymbol{Z}'} \in L^1(\boldsymbol{Z}')$ *and* $a_n \geqq 0$ *for all* $n \in \boldsymbol{Z}'$, *then*

$$\| H_a \|_{op} = \| a \|_1 .$$

*Proof.* By Theorem 4,

$$\| H_{ap} \|_{op} = \max_{x \in \hat{Q}^\times} \left| \sum_{r \in Q^\times} (r, x) F(r) \right|^{\frac{1}{2}} \leqq \left( \sum_{r \in Q^\times} | F(r) | \right)^{\frac{1}{2}} = \| a \|_1$$

since

$$a_n \geqq 0, |F(r)| = \sum_{\substack{m/n=r \\ m,n \in \boldsymbol{Z}'}} a_n a_m, \text{ and } \sum_{r \in Q^\times} | F(r) | = \| a \|_1^2 .$$

But upon setting $x = 0$ we obtain

$$\left| \sum_{r \in Q^\times} (r, 0) F(r) \right|^{\frac{1}{2}} = \left| \sum_{r \in Q^\times} F(r) \right|^{\frac{1}{2}} = \left( \sum_{r \in Q^\times} | F(r) | \right)^{\frac{1}{2}} = \| a \|_1 \, ,$$

and thus the proof is complete.

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