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Let G be a compact Abelian group with discrete countable dual group $\Gamma = \hat{G}$ and let $f \in L^1(G)$ with Fourier transform $F = \hat{f}$. If V is a finite subset of Γ we consider the operator F_V on $L^2(V)$:

$$(F_V \varphi)(\gamma) = \sum_{\tau \in V} F(\gamma - \tau) \varphi(\tau) \quad \varphi \in L^2(V), \gamma \in V.$$

Then if $\{V_n\}$ is any suitably restricted sequence of finite subsets of Γ we show that

$$\lim_{n \rightarrow \infty} \|F_{V_n}\| = \lim_{n \rightarrow \infty} \{\max_{\|\varphi\|_2=1} |(F_{V_n} \varphi, \varphi)|\} = \|f\|_\infty$$

where $\|F_V\|$ is the operator norm of F_V on $L^2(V)$ and $(F_V \varphi, \varphi)$ denotes the inner product of $F_V \varphi$ and φ (over V).

This result is then translated into a statement concerning a special class of infinite matrices which generalize the classical Toeplitz matrices. We then apply these results in evaluating the norm of a special type of linear operator.

In [1] the author considered the asymptotic distribution of eigenvalues and characteristic numbers of certain sequences of operators $\{F_{V_n}\}$ over a locally compact group Γ associated with sequences $\{V_n\}$ of Borel sets of Γ of finite nonzero measure satisfying

$$(*) \quad \lim_{n \rightarrow \infty} |\gamma V_n \triangle V_n| / |V_n| = 0 \quad \text{for all } \gamma \in \Gamma,$$

where $|\cdot|$ is left Haar measure on Γ . We write $\{V_n\} \in W_r$, and say $\{V_n\}$ has the weak ratio property in case (*) is satisfied (see [2]). In this paper we are considering countable Abelian Γ and a more general family T_r of sequences $\{V_n\}$ than those in W_r (and hence in general the asymptotic distribution of the characteristic numbers of $\{F_{V_n}\}$ does not exist, [2]) but still restricted enough to guarantee an asymptotic formula for the *maximal* characteristic number of F_{V_n} as $n \rightarrow \infty$.

1. The basic theorem. Γ denotes an arbitrary countably infinite discrete Abelian group equipped with the counting measure.

DEFINITION 1. A sequence $\{V_n\}$ of finite nonempty subsets of Γ has the *translation property*, written $\{V_n\} \in T_r$, if and only if to every finite subset $\Gamma_0 \subseteq \Gamma$ there corresponds an $n_0 = n_0(\Gamma_0)$ such that for $n \geq n_0$ there exists a $\tau_n = \tau_n(\Gamma_0) \in \Gamma$ with the property that $\tau_n + \Gamma_0 \subseteq V_n$.

PROPOSITION 1. (i) $\{V_n\} \in T_r$ and $V_n \subseteq V_n^*$ for $n \in N^+$ implies $\{V_n^*\} \in T_r$. (ii) W_r is properly contained in T_r .

Proof. (i) is immediate from the definition. To prove (ii), first assume $\{V_n\} \in W_r$ and fix any finite nonempty subset $\Gamma_0 \subseteq \Gamma$. Set $C = \Gamma_0 \cup \{0\}$. We then readily conclude (see [2] where many properties of W_r are established)

$$(1) \quad \lim_{n \rightarrow \infty} \frac{|C + V_n|}{|V_n|} = 1.$$

But $C + V_n = \bigcup_{\tau \in V_n} (\tau + \Gamma_0) \cup V_n$. Hence if $\tau + \Gamma_0 \not\subseteq V_n$ for all $\tau \in V_n$, we have $|(\tau + \Gamma_0) \sim V_n| \geq 1$ for all $\tau \in V_n$ and consequently

$$(2) \quad |C + V_n| \geq |V_n| + \frac{|V_n|}{|\Gamma_0|}$$

since we may choose $|V_n|$ elements $\tau + \gamma_\tau \in (\tau + \Gamma_0) \sim V_n$, where $\tau \in V_n$ and $\gamma_\tau \in \Gamma_0$, and no element is duplicated more than $|\Gamma_0|$ times. But for sufficiently large n (2) violates (1) and therefore there is a $\tau_n \in V_n$ for which $\tau_n + \Gamma_0 \subseteq V_n$. Hence $W_r \subseteq T_r$.

We now show inclusion is proper. For let $\{V_n\} \in W_r$ ($\neq \emptyset$, by [2]); we shall construct a sequence $V_n^* \supseteq V_n$ such that $\{V_n^*\} \notin W_r$, which completes the proof of (ii) upon appealing to (i). Fix any $\gamma \in \Gamma \sim \{0\}$. We inductively construct a sequence $\nu_1^{(n)}, \dots, \nu_{|V_n|}^{(n)}$ as follows: $\nu_1^{(n)} \in V_n + \{0, \pm\gamma\}$, and

$$(I) \quad \nu_k^{(n)} \in (V_n \cup \{\nu_1^{(n)}, \dots, \nu_{k-1}^{(n)}\}) + \{0, \pm\gamma\} \quad (2 \leq k \leq |V_n|).$$

We set $V_n^* = V_n \cup \{\nu_1^{(n)}, \dots, \nu_{|V_n|}^{(n)}\}$ and verify that

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} \leq \frac{1}{2} \quad (n \in N^+)$$

implying $\{V_n^*\} \notin W_r$. For

$$\begin{aligned} & (\gamma + V_n^*) \cap V_n^* \\ &= ((\gamma + V_n) \cap V_n^*) \cup (\{\gamma + \nu_1^{(n)}, \dots, \gamma + \nu_{|V_n|}^{(n)}\} \cap (V_n \cup \{\nu_1^{(n)}, \dots, \nu_{|V_n|}^{(n)}\})) \\ &= ((\gamma + V_n) \cap V_n^*) \end{aligned}$$

since the second term in the union is empty by (I). Hence $|(\gamma + V_n^*) \cap V_n^*| \leq |\gamma + V_n| = |V_n|$, and therefore for $n \in N^+$

$$\frac{|(\gamma + V_n^*) \cap V_n^*|}{|V_n^*|} = \frac{|(\gamma + V_n^*) \cap V_n^*|}{2|V_n|} \leq \frac{|V_n|}{2|V_n|} = \frac{1}{2}.$$

We now prove a result, of independent interest, which is critical in the proof of Theorem 1.

PROPOSITION 2. *Let G be a compact Abelian group (with measure normalized to one), let $f \in L^1(G)$, and let ρ be any positive number. Then*

$$\|f\|_\infty = \sup_\omega \left| \int_G |\omega(x)|^\rho f(x) dx \right|$$

where ω ranges over all trigonometric polynomials on G satisfying $\|\omega\|_\rho \leq 1$.

Proof. Recall that a trigonometric polynomial is a finite linear combination of characters on G . Clearly

$$\left| \int_G |\omega(x)|^\rho f(x) dx \right| \leq \|f\|_\infty \int_G |\omega(x)|^\rho dx = \|f\|_\infty \|\omega\|_\rho^\rho \leq \|f\|_\infty .$$

We divide the proof of the converse inequality into two cases:

To prove the converse inequality, we first consider the case $\|f\|_\infty < +\infty$. Fix any $\delta > 0$ (until the conclusion of the argument). Let $S = S(\delta)$ be a measurable subset of the complex plane of diameter less than δ and such that

$$E = f^{-1}(S), \quad \|\chi_E f\|_\infty = \|f\|_\infty ,$$

where χ_E denotes the characteristic function of E . Hence for $s \in S$ and $x \in E$ we have

$$||s| - |(\chi_E f)(x)|| \leq |s - (\chi_E f)(x)| < \delta ,$$

and consequently also

$$||s| - \|f\|_\infty| = ||s| - \|\chi_E f\|_\infty| \leq \delta .$$

Therefore, if $g = \chi_E/|E|$ then

$$\begin{aligned} 0 &\leq \|f\|_\infty - \left| \int_G fg dx \right| \leq ||\|f\|_\infty - |s|| + \left| |s| - \left| \int_G fg dx \right| \right| \\ (3) \quad &\leq ||\|f\|_\infty - |s|| + \left| s - \int_G fg dx \right| \\ &= ||\|f\|_\infty - |s|| + \left| \frac{1}{|E|} \int_E (s - \chi_E f) dx \right| \leq 2\delta . \end{aligned}$$

We next wish to approximate g by a continuous function h , and at this point the estimate is rather delicate because this is also needed later in the case $\|f\|_\infty = +\infty$ and consequently we must avoid $\|f\|_\infty$ as a factor in the error of estimation. Now since $f \in L^1(G)$, to every $\varepsilon > 0$ there corresponds an $\eta = \eta(\varepsilon)$ such that for all measurable subsets T of G of measure at most η

$$\int_T |f| dx < \varepsilon .$$

We now choose $\gamma = \gamma(\delta)$ satisfying

$$(4) \quad (i) \quad \gamma < \delta |E|, \quad (ii) \quad \int_T |f| dx < \delta |E| \text{ if } |T| < \gamma .$$

Furthermore, since Haar measure is regular, we may find an open set E^+ and a closed set E^- such that

$$(4') \quad E^- \subseteq E \subseteq E^+, \quad |E^+ \sim E^-| < \gamma .$$

Finally (by Urysohn's Lemma, since G is a normal topological space) there exists a continuous $h_0: G \rightarrow [0, 1]$ such that $h_0|_{E^-} \equiv 1$ and $h_0|_{G \sim E^+} \equiv 0$. Our candidate for h is then defined to be the nonnegative function $h = h_0/|E|$. Let us now estimate $\int_G fg dx - \int_G fh dx$:

$$\begin{aligned} & \left| \int_G fg dx - \int_G fh dx \right| \leq \int_G |f| |g - h| dx \\ (5) \quad & = \left(\int_{E^+} + \int_{E^+ \sim E^-} + \int_{G \sim E^+} \right) |f| |g - h| dx = \int_{E^+ \sim E^-} |f| |g - h| dx \\ & \leq \max |g - h| \int_{E^+ \sim E^-} |f| dx \leq \frac{1}{|E|} \cdot \delta |E| = \delta \end{aligned}$$

by (4), (4') and the definitions of g and h . Also, we have

$$\int_{E^-} h dx \leq \int_G h dx = \int_{E^+} h dx \leq \|h\|_\infty |E^+| ,$$

implying the estimate

$$(6) \quad \frac{|E^-|}{|E|} \leq \|h\|_1 \leq \frac{|E^+|}{|E|} \leq 1 + \delta \text{ by virtue of (4) and (4') .}$$

Lastly, to any $\alpha > 0$ we may correspond a trigonometric polynomial ω_α satisfying $\|h^{1/\rho} - \omega_\alpha\|_\infty < \alpha$, and consequently $\|h^{1/\rho} - |\omega_\alpha|\|_\infty < \alpha$ since $h^{1/\rho} \geq 0$. Thus by choosing $\alpha_0 = \alpha_0(\delta)$ sufficiently small we may conclude

$$(7) \quad \|h - |\omega_{\alpha_0}|^\rho\|_1 \leq \|h - |\omega_{\alpha_0}|^\rho\|_\infty < \delta .$$

Also,

$$\|\omega_{\alpha_0}^\rho\|_1 \leq \|h - |\omega_{\alpha_0}|^\rho\|_1 + \|h\|_1 \leq \delta + (1 + \delta) = 1 + 2\delta .$$

We now let

$$\omega = \omega_{\alpha_0}/(1 + 2\delta)^{1/\rho}, \text{ implying } \|\omega\|_\rho \leq 1 .$$

Finally,

$$\begin{aligned}
 & \left| \int_G |\omega|^p f dx \right| = \frac{1}{1 + 2\delta} \left| \int_G |\omega_{\alpha_0}|^p f dx \right| \\
 (8) \quad & \geq \frac{1}{1 + 2\delta} \left(\left| \int_G f g dx \right| - \left| \int_G f g dx \right| - \left| \int_G f h dx \right| - \left| \int_G h f dx \right| - \int_G |\omega_{\alpha_0}|^p f dx \right) \\
 & \geq \frac{1}{1 + 2\delta} ((\|f\|_\infty - 2\delta) - \delta - \delta \|f\|_1) .
 \end{aligned}$$

By (3), (5), and (7). Our assertion follows upon letting $\delta \rightarrow 0$.

In case $\|f\|_\infty = +\infty$, we let S_n be a measurable subset of the complex plane of diameter less than δ and such that $E_n = f^{-1}(S_n)$, $\|\chi_{E_n} f\|_\infty > n$. Equations (3) – (8) still hold with $\|f\|_\infty$ replaced by $\|\chi_{E_n} f\|_\infty > n$ wherever it occurs, and we readily construct trigonometric polynomials ω_n with $\|\omega_n\|_\rho \leq 1$ and such that $\int_G |\omega_n|^p f dx$ is unbounded as $n \rightarrow +\infty$.

We now are ready to prove the basic theorem.

THEOREM 1. *Let G be a compact group (with measure normalized to one), let $f \in L^1(G)$, and let $F = \hat{f} \in L^\infty(\Gamma)$, the Fourier Transform of f . Furthermore, let $\{V_n\} \in T_\Gamma$ and let F_{V_n} be the Hilbert-Schmidt operator on $L^2(V_n)$:*

$$\begin{aligned}
 (F_{V_n} \psi)(\gamma) &= \int_{V_n} F(\gamma - \tau) \psi(\tau) d\tau = \sum_{\tau \in V_n} F(\gamma - \tau) \psi(\tau) \\
 & \quad (\psi \in L^2(V_n), \gamma \in V_n) .
 \end{aligned}$$

Let $(F_{V_n} \psi, \psi)_{V_n}$ denote the inner product of $F_{V_n} \psi$ and ψ over V_n , and let $|F_{V_n}|$ denote the maximal characteristic number of F_{V_n} as an operator on the Hilbert space $L^2(V_n)$. Then

- (i) $\lim_{n \rightarrow \infty} \max_{\|\psi\|_2=1} |(F_{V_n} \psi, \psi)_{V_n}| = \|f\|_\infty .$
- (ii) $\lim_{n \rightarrow \infty} |F_{V_n}| = \|f\|_\infty .$

Proof. (i) By definition,

$$\begin{aligned}
 (F_{V_n} \psi, \psi)_{V_n} &= \sum_{\tau, \gamma \in V_n} F(\gamma - \tau) \psi(\tau) \overline{\psi(\gamma)} \\
 &= \sum_{\tau, \gamma \in V_n} \left[\int_G \overline{(\gamma - \tau, x)} f(x) dx \right] \psi(\tau) \overline{\psi(\gamma)} \\
 &= \int_G \left[\sum_{\tau, \gamma \in V_n} (\tau, x) \psi(\tau) \overline{(\gamma, x) \psi(\gamma)} \right] f(x) dx \\
 &= \int_G \left| \sum_{\tau \in V_n} (\tau, x) \psi(\tau) \right|^2 f(x) dx .
 \end{aligned}$$

Note that

$$\omega_\psi(x) = \sum_{\tau \in V_n} (\tau, x)\psi(\tau)$$

is a trigonometric polynomial on G , and $\psi \rightarrow \omega_\psi$ is an isometry of $L^2(V_n)$ into $L^2(G)$ since $\|\omega_\psi\|_2^2 = \sum_{\tau \in V_n} |\psi(\tau)|^2 = \|\psi\|_2^2$. Therefore

$$(\dagger) \quad \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \max_{\|\omega\|_2=1} \left| \int_G |\omega(x)|^2 f(x) dx \right|$$

where ω ranges over linear combinations of characters on G generated by elements in V_n . Hence by Proposition 2 ($\rho = 2$),

$$\lim_{n \rightarrow \infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| \leq \|f\|_\infty .$$

On the other hand, let ω be *any* trigonometric polynomial on G , say

$$\omega(x) = \sum_{1 \leq i \leq k} (\gamma_i, x)c_i \quad (c_i \in \mathbb{C}, \gamma_i \in \Gamma) .$$

Let $\Gamma_0 = \{\gamma_1, \dots, \gamma_k\}$, a finite subset of Γ . Now since $\{V_n\} \in T_\Gamma$ there exists an n_0 such that for $n \geq n_0$ there exists $\tau_n \in \Gamma$ such that $\tau_n + \Gamma_0 \subseteq V_n$. Hence for $n \geq n_0$,

$$\omega_n(x) = (\tau_n, x)\omega(x) = \sum_{1 \leq i \leq k} (\tau_n + \gamma_i, x)c_i$$

is a linear combination of characters on G generated by elements of V_n . Since $|\omega(x)| = |\omega_n(x)|$ for all $x \in G$, the proof of (i) is completed by again applying Proposition 2 with $\rho = 2$.

(ii) Recall that $|F_{V_n}|$ is the norm of F_{V_n} considered as an operator on $L^2(V_n)$, i.e.,

$$|F_{V_n}| = \max_{\|\psi\|_2=1} \|F_{V_n}\psi\|_2 .$$

but by the Cauchy-Schwarz Inequality, for $\|\psi\|_2 = 1$

$$|(F_{V_n}\psi, \psi)_{V_n}| \leq \|F_{V_n}\psi\|_2 \|\psi\|_2 = \|F_{V_n}\psi\|_2 \leq |F_{V_n}|$$

and therefore by (i),

$$\lim_{n \rightarrow \infty} |F_{V_n}| \geq \lim_{n \rightarrow \infty} \max_{\|\psi\|_2=1} |(F_{V_n}\psi, \psi)_{V_n}| = \|f\|_\infty .$$

Thus, if $\|f\|_\infty = +\infty$ nothing remains to be proved. If $\|f\|_\infty < +\infty$ we have $f \in L^1(G) \cap L^\infty(G)$, and therefore by [3], p. 445, $|F_{V_n}| \leq \|f\|_\infty$ for all $n \in N^+$. Hence $\overline{\lim}_{n \rightarrow \infty} |F_{V_n}| \leq \|f\|_\infty$, and consequently $\lim_{n \rightarrow \infty} |F_{V_n}| = \|f\|_\infty$ in this case as well.

We now conversely prove that the hypothesis $\{V_n\} \in T_r$ is in fact necessary for the conclusion of Theorem 1. More precisely,

THEOREM 1'. *Using the notation of Theorem 1, if $\{V_n\}$ is any sequence of finite subsets of Γ for which conclusion (i) holds for all trigonometric polynomials f on G , then $\{V_n\} \in T_r$.*

Proof. Assume $\{V_n\} \notin T_r$, i.e., there exists a finite subset Γ_0 of Γ such that no translate of Γ_0 lies in V_m for an appropriate subsequence $m \rightarrow \infty$. We then assert that

$$f(x) = \frac{1}{|\Gamma_0|} \sum_{\tau \in \Gamma_0} (\tau, x) \quad (\|f\|_\infty = f(0) = 1)$$

is a trigonometric polynomial for which (i) fails. More precisely we show for all these m :

$$\text{Max}_{\|\psi\|_2=1} |(F_{V_m}\psi, \psi)_{V_m}| \leq \left(1 - \frac{1}{2|\Gamma_0|}\right) \|f\|_\infty.$$

Recalling relation (†) of the proof of Theorem 1. We have:

$$(†) \quad \text{Max}_{\|\psi\|_2=1} |(F_{V_m}\psi, \psi)_{V_m}| = \text{Max}_{\|\omega\|_2=1} \left| \int_G |\omega(x)|^2 f(x) dx \right|$$

where ω ranges over all linear combinations of characters on G generated by elements in V_m .

However, any such ω is of the form

$$\omega(x) = \sum_{\tau \in V_m} (\tau, x) a_\tau$$

where

$$\sum_{\tau \in V_m} |a_\tau|^2 = \|\omega\|_2^2 \leq 1,$$

implying

$$|\omega(x)|^2 = \sum_{\tau_1, \tau_2 \in V_m} (\tau_1 - \tau_2, x) a_{\tau_1} \bar{a}_{\tau_2},$$

and finally

$$\int_G |\omega(x)|^2 f(x) dx = \frac{1}{|\Gamma_0|} \sum_{\substack{\tau_1, \tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} a_{\tau_1} \bar{a}_{\tau_2}.$$

Consequently,

$$\begin{aligned}
 \left| \int_G |\omega(x)|^2 f(x) dx \right| &\leq \frac{1}{|\Gamma_0|} \sum_{\substack{\tau_1, \tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} |a_{\tau_1}| |a_{\tau_2}| \leq \frac{1}{2|\Gamma_0|} \sum_{\substack{\tau_1, \tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} (|a_{\tau_1}|^2 + |a_{\tau_2}|^2) \\
 &= \frac{1}{2|\Gamma_0|} \left(\sum_{\tau_1 \in V_m} |a_{\tau_1}|^2 \left(\sum_{\substack{\tau_2 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} 1 \right) + \sum_{\tau_2 \in V_m} |a_{\tau_2}|^2 \left(\sum_{\substack{\tau_1 \in V_m \\ \tau_2 - \tau_1 \in \Gamma_0}} 1 \right) \right) \\
 &= \frac{1}{2|\Gamma_0|} \left(\sum_{\tau_1 \in V_m} |a_{\tau_1}|^2 |(V_m - \tau_1) \cap \Gamma_0| \right. \\
 &\quad \left. + \sum_{\tau_2 \in V_m} |a_{\tau_2}|^2 |(\tau_2 - V_m) \cap \Gamma_0| \right) \\
 &\leq \frac{1}{2|\Gamma_0|} \left\{ (|\Gamma_0| - 1) \sum_{\tau_1 \in V_m} |a_{\tau_1}|^2 + |\Gamma_0| \sum_{\tau_2 \in V_m} |a_{\tau_2}|^2 \right\} \\
 &= \left(1 - \frac{1}{2|\Gamma_0|} \right) \sum_{\tau \in V_m} |a_{\tau}|^2 \leq \left(1 - \frac{1}{2|\Gamma_0|} \right) \|f\|_\infty \\
 &= \left(1 - \frac{1}{2|\Gamma_0|} \right) \|f\|_\infty
 \end{aligned}$$

since no translate $V_m - \tau_1$ contains Γ_0 by hypothesis. Our assertion now readily follows.

2. A class of doubly-infinite matrices. We now translate the theorem of the preceding section into a statement concerning a class of doubly-infinite complex matrices $M = (\alpha_{i,j})_{i,j=1}^\infty$ whose entries $\alpha_{i,j}$ are determined by a "group law".

DEFINITION 2. Let $M = (\alpha_{i,j})_{i,j=1}^\infty$ be a matrix with complex entries. We then write

$$M \sim (\Gamma, A, F)$$

if and only if

- (i) Γ is a countable Abelian group.
- (ii) A is a subset of Γ .
- (iii) $F: \Gamma \rightarrow \mathcal{C}$.
- (iv) There exists an ordering of $A = \{\lambda_1, \dots, \lambda_n, \dots\}$ such that for all $i, j \in N^+$,

$$\alpha_{i,j} = F(\lambda_i - \lambda_j).$$

REMARK. For any $M = (\alpha_{i,j})_{i,j=1}^\infty$ with complex entries we may take Γ to be Q^\times , the multiplicative group of rational numbers, and A to be $P = \{p_n: n \in N^+\}$, the set of all positive integral primes, upon defining F by

$$F(r) = \begin{cases} \alpha_{i,j} & \text{if } r = p_i/p_j \\ 0 & \text{otherwise.} \end{cases}$$

We then have $M \sim (Q^\times, P, F)$.

Under suitable restrictions on (Γ, A, F) we shall be able to compute the norm and quadratic norm of the matrix M , which are defined as follows:

DEFINITION 3. The norm of M , $|M|$, and the quadratic norm of M , $|M|_I$, are defined by

$$|M| = \sup_{\|X\|_2 \leq 1} \|MX\|_2, \quad |M|_I = \sup_{\|X\|_2 \leq 1} |(MX, X)|,$$

where $X = (\{x_i\})$ ranges over elements of the complex Hilbert space l^2 with only finitely many $x_i \neq 0$, and $MX = (\{\sum_j \alpha_{i,j} x_j\})$.

LEMMA 1. If M induces a bounded operator on l^2 , then

$$(i) \quad |M| = \sup_{\|X\|_2 \leq 1} \|MX\|, \quad (ii) \quad |M|_I = \sup_{\|X\|_2 \leq 1} |(MX, X)|,$$

where $X = (\{x_i\})$ ranges over all elements of l^2 (with $\|X\|_2 \leq 1$). Hence in this case $|M|$ is the standard norm of M considered as a bounded linear operator on l^2 .

Proof. For $x \in l^2$, let X_n be the projection of X on its first n components (0 elsewhere). Since M is bounded and consequently closed, $\lim_{n \rightarrow \infty} MX_n = MX$ and (i) follows since X_n has at most n nonzero components. Also

$$(MX, X) = (MX_n, X_n) + (M(X - X_n), X_n) + (MX, X - X_n),$$

and therefore

$$\begin{aligned} & |(MX, X) - (MX_n, X_n)| \\ & \leq \|M(X - X_n)\|_2 \|X_n\|_2 + \|MX\|_2 \|X - X_n\|_2 \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$, and (ii) clearly follows.

THEOREM 2. Let $M \sim (\Gamma, A, F)$ where

(i) $F \in A(\Gamma)$, i.e., $F = \hat{f}$ for some $f \in L^1(G)$.

(ii) To each finite subset $\Gamma_0 \subseteq \Gamma$ there corresponds a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteq A$.

Then $|M| = |M|_I = \|f\|_\infty$.

Proof. Assume $A = \{\lambda_1, \dots, \lambda_n, \dots\}$ as in Definition 2, and set $V_n = \{\lambda_1, \dots, \lambda_n\}$. Then hypothesis (ii) clearly implies $\{V_n\} \in T_\Gamma$. The theorem will follow from the two inequalities

- (i) $|M| \leq \|f\|_\infty$
- (ii) $\|f\|_\infty \leq |M|_I$,

since $|M|_I \leq |M|$ by the Cauchy-Schwarz inequality.

(i): If $\|f\|_\infty = +\infty$ there is nothing to prove. Otherwise $f \in L^1(G) \cap L^\infty(G)$, and therefore the operator $M': L^2(\Gamma) \rightarrow L^2(\Gamma)$ defined by

$$(M'\varphi)(\gamma) = \sum_{\tau \in \Gamma} F(\gamma - \tau)\varphi(\tau) \quad (\varphi \in L^2(\Gamma), \gamma \in \Gamma)$$

has norm $|M'| = \|f\|_\infty$ by [3], § 3.2., p. 441. Hence if we only consider φ with support in A and restrict γ to A , M' restricts to an operator $M'': L^2(A) \rightarrow L^2(A)$ with $|M''| \leq |M'|$. Now consider the isometry of l^2 onto $L^2(A)$ given by $X = (\{x_n\}) \rightsquigarrow \varphi_X$ where $\varphi(\lambda_n) = x_n$ for $n \in N^+$. Then for this $\varphi = \varphi_X$ and $\lambda_i \in A$,

$$(M''\varphi)(\lambda_i) = \sum_{\lambda \in A} F(\lambda_i - \lambda)\varphi(\lambda) = \sum_j F(\lambda_i - \lambda_j)\varphi(\lambda_j) = \sum_j \alpha_{i,j}x_j$$

which is the i^{th} component of MX , and therefore $|M| = |M''| \leq |M'| = \|f\|_\infty$ (and thus M induces a bounded linear operator if $\|f\|_\infty < +\infty$).

(ii): For $n \in N^+$, consider the isometry of $L^2(V_n)$ (which is none other than n -dimensional Euclidean space) into l^2 given by $\varphi \rightarrow X_\varphi$ where $X_\varphi = (\{x_j^\varphi\})$ and $x_j^\varphi = \varphi(\lambda_j)$ for $1 \leq j \leq n$ and 0 otherwise. Hence X_φ has only finitely many nonzero components, and each $X \in l^2$ with only finitely many nonzero components is in the image of $L^2(V_n)$ under the above isometry for $n = n(X)$ sufficiently large. Now consider F_{V_n} on $L^2(V_n)$:

$$\begin{aligned} (F_{V_n}\varphi, \varphi)_{V_n} &= \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} F(\lambda_i - \lambda_j)\varphi(\lambda_j)\overline{\varphi(\lambda_i)} \\ &= \sum_{1 \leq i, j \leq n} \alpha_{i,j}x_j\bar{x}_i = (MX_\varphi, X_\varphi) \quad (\text{in } l^2). \end{aligned}$$

But by Theorem 1 (i) $\lim_{n \rightarrow \infty} \max_{\|\varphi\|_2=1} |(F_{V_n}\varphi, \varphi)_{V_n}| = \|f\|_\infty$ and therefore $|M|_I = \|f\|_\infty$ since

$$|M|_I = \sup_{\substack{\|x\|_2 \leq 1 \\ x_i \neq 0 \text{ finitely}}} |(MX, X)| = \lim_{n \rightarrow \infty} \max_{\|\varphi\|_2 \leq 1} |(F_{V_n}\varphi, \varphi)_{V_n}| = \|f\|_\infty.$$

COROLLARY 1. (1) Hypothesis (i) of Theorem 2 is satisfied if

$$(i)' \quad \sum_{\gamma \in \Gamma} |F(\gamma)|^2 < +\infty.$$

(2) Hypothesis (ii) of Theorem 2 is satisfied if

$$(ii)' \quad A + A \subseteq A \quad \text{and} \quad (ii)'' \quad A \text{ generates } \Gamma.$$

Proof. (1), (i)' implies $f(x) = \sum_{\gamma \in \Gamma} (\gamma, x)F(\gamma) \in L^2(G)$ and therefore also $f \in L^1(G)$ since G is compact and hence of finite measure. Clearly $F = \hat{f} \in A(\Gamma)$.

(2) First note that any element of Γ is the difference of two elements in A , i.e., $\Gamma = A - A$. For by (ii)'', if $\gamma \in \Gamma$ we have $\gamma = \lambda_{i_1} + \dots + \lambda_{i_k} - \dots - \lambda_{i_n}$ for some suitable finite sequence of integers i_1, \dots, i_n (if no terms with a plus sign occur we may take $k = 0$, if none with a minus sign occur take $k = n$). But by (ii)', $\gamma^+ = \lambda_{i_1} + \dots + \lambda_{i_k} \in A$ if $k > 0$, $\gamma^- = \lambda_{i_{k+1}} + \dots + \lambda_{i_n} \in A$ if $k < n$. If $k = 0$ we may write $\gamma = \lambda_1 - (\lambda_1 + \dots + \lambda_n + \lambda_1)$ and similarly $\gamma = (\lambda_1 + \dots + \lambda_n + \lambda_1) - \lambda_1$ if $k = n$.

Now let $\Gamma_0 = \{\gamma_1, \dots, \gamma_k\}$ be a nonempty finite subset of Γ . Then for appropriate $a_i, b_i \in N^+$ we have

$$\gamma_i = \lambda_{a_i} - \lambda_{b_i} \quad (1 \leq i \leq k).$$

Consequently, for $1 \leq i \leq k$,

$$\gamma_i = \lambda_{a_i} + \lambda_{b_1} + \dots + \widehat{\lambda_{b_i}} + \dots + \lambda_{b_k} - (\lambda_{b_1} + \dots + \lambda_{b_k})$$

where $(\widehat{})$ denotes deletion of a term. Hence if we set $\gamma = \lambda_{b_1} + \dots + \lambda_{b_k}$ we have $\gamma + \Gamma_0 \subseteq A$ since

$$\lambda_{a_i} + \lambda_{b_1} + \dots + \widehat{\lambda_{b_i}} + \dots + \lambda_{b_k} \in A$$

by (ii)'.

We now apply Theorem 2 to completely solve the norm evaluation in the case $M \sim (\Gamma, A, F)$ where $F \geq 0$ and A satisfies (ii). We make use of the following simple lemma:

LEMMA 2. If $M = (\alpha_{i,j})_{i,j=1}^\infty$ and $M' = (\alpha'_{i,j})_{i,j=1}^\infty$ where

$$\alpha_{i,j} \geq \alpha'_{i,j} \geq 0 \quad \text{for all } i, j \in N^+$$

then

$$(i) \quad |M| = \sup_{\|x\|_2 \leq 1} \|MX\|, \quad |M|_I = \sup_{\|x\|_2 \leq 1} (MX, X)$$

where $X = (\{x_i\})$ has only finitely many nonzero coordinates, all positive.

$$(ii) \quad |M'| \leq |M|, \quad |M'|_I \leq |M|_I.$$

Proof. For $X = (\{x_i\}) \in l^2$ we define $X^+ = (\{|x_i|\})$. Note $\|X\|_2 = \|X^+\|_2$ and X^+ and X have the same cardinality of nonzero coordinates. Also, $\alpha_{i,j} \geq 0$ clearly implies $\|MX\|_2 \leq \|MX^+\|_2$ and $|(MX, X)| \leq (MX^+, X^+)$ and (i) readily follows. But $\alpha_{i,j} \geq \alpha'_{i,j} \geq 0$ also implies each component of $M'X^+$ is dominated by the corresponding component of MX^+ and hence (ii) follows from (i).

THEOREM 3. Let $M \sim (\Gamma, A, F)$ where

(i) $F(\gamma) \geq 0$ for all $\gamma \in \Gamma$.

(ii) If Γ_0 is any finite subset of Γ there exists a $\gamma = \gamma(\Gamma_0)$ such that $\gamma + \Gamma_0 \subseteq A$. Then

$$|M| = |M|_I = \sum_{\gamma \in \Gamma} F(\gamma) \quad (\text{possibly } +\infty).$$

Proof. Since $|M|_I \leq |M|$ it suffices to show that

$$(1) \quad |M| \leq \sum_{\gamma \in \Gamma} F(\gamma), \quad (2) \quad |M|_I = \sum_{\gamma \in \Gamma} F(\gamma).$$

If $\sum_{\gamma \in \Gamma} F(\gamma) < +\infty$ there is nothing to prove since in this case the result is included in Theorem 2 because $f(x) = \sum_{\gamma \in \Gamma} (\gamma, x)F(\gamma)$ is a continuous function on G , $F = \hat{f}$, and $\|f\|_\infty = f(0) = \sum_{\gamma \in \Gamma} F(\gamma)$.

On the other hand, if $\sum_{\gamma \in \Gamma} F(\gamma) = +\infty$ then F may not be in $A(\Gamma)$ and hence we cannot apply Theorem 2 directly. Clearly (1) is true in this case and we need only verify (2). Let $\Gamma' = \{\gamma_1, \dots, \gamma_n\}$ be any finite subset of Γ and define

$$M_{\Gamma'} = (\alpha_{i,j}^{\Gamma'})_{i,j=1}^n$$

where

$$\alpha_{i,j}^{\Gamma'} = \begin{cases} F(\gamma_\nu) & \text{if } \lambda_i - \lambda_j = \gamma_\nu \in \Gamma' \\ 0 & \text{otherwise,} \end{cases}$$

i.e., $M_{\Gamma'} \sim (\Gamma, A, F_{\Gamma'})$ where $F_{\Gamma'}(\gamma) = F(\gamma)I_{\Gamma'}(\gamma)$. Since $F \geq 0$, $\alpha_{i,j} \geq \alpha_{i,j}^{\Gamma'} \geq 0$ for $i, j \in N^+$, and Lemma 2 implies $|M|_I \geq |M_{\Gamma'}|_I$. But by Theorem 2

$$|M_{\Gamma'}|_I = \text{ess sup}_{x \in G} \left| \sum_{\gamma \in \Gamma'} (\gamma, x)F_{\Gamma'}(\gamma) \right| = \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $\sum_{\gamma \in \Gamma'} (\gamma, x)F_{\Gamma'}(\gamma)$ is continuous and $F_{\Gamma'} \geq 0$. This in turn implies

$$|M|_I \geq \sup_{|\Gamma'| < +\infty} \sum_{\gamma \in \Gamma'} F(\gamma) = +\infty.$$

COROLLARY 2. *Under the hypothesis of Theorem 3.*

$$|M| = |M|_I = \sup_{i \in N^+} \left(\sum_{j \in N^+} \alpha_{i,j} \right) = \sup_{j \in N^+} \left(\sum_{i \in N^+} \alpha_{i,j} \right).$$

Proof. We prove only $|M| = |M|_I = \sup_{i \in N^+} (\sum_{j \in N^+} \alpha_{i,j})$ the proof of the other equality being similar. By Theorem 3. we need only verify $\sup_{i \in N^+} (\sum_{j \in N^+} \alpha_{i,j}) = \sum_{\gamma \in \Gamma} F(\gamma)$. First

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{j \in N^+} F(\lambda_i - \lambda_j) = \sum_{\gamma \in \lambda_i - A} F(\gamma) \leq \sum_{\gamma \in \Gamma} F(\gamma).$$

Let $\Gamma' = \{\gamma_1, \dots, \gamma_n\}$ be any finite subset of Γ and let $\Gamma_0 = \{0, -\gamma_1, \dots, -\gamma_n\}$. Condition (ii) insures the existence of an $\alpha \in \Gamma$ such that $\alpha + \Gamma_0 \subseteq \Lambda$. In particular $\alpha \in \Lambda$, say $\alpha = \lambda_{k(\alpha)}$. But

$$\Gamma' \subseteq -\Gamma_0 = \alpha - (\alpha + \Gamma_0) \subseteq \lambda_{k(\alpha)} - \Lambda,$$

and thus for $i = k(\alpha)$ we have

$$\sum_{j \in N^+} \alpha_{i,j} = \sum_{\gamma \in \lambda_{k(\alpha)} - \Lambda} F(\gamma) \geq \sum_{\gamma \in \Gamma'} F(\gamma)$$

since $F \geq 0$, and our assertion follows.

3. **An application.** In this section we apply the results of § 2 to evaluate the norm of a special type of linear operator.

DEFINITION 4. Let T be the circle group, considered as the real numbers $R^+ \text{ mod } 2\pi$, and let $L^2 = L^2(T, dt)$ be the associated Hilbert function space with respect to normalized Lebesgue measure. Let $\mathcal{M} \subseteq L^2$ be the submanifold

$$\mathcal{M} = \left\{ f \in L^2: \int_T f(t) dt = 0 \right\}.$$

Furthermore, let $Z' = Z \sim \{0\}$ and for $a = \{a_n\}_{n \in Z'} \in L^1(Z')$ define $H_a: \mathcal{M} \rightarrow \mathcal{M}$ by

$$(H_a f)(t) = \sum_{n \in Z'} a_n f(nt)$$

(where equality of functions is to be taken in the L^2 sense).

We now show that the mapping $a \rightsquigarrow H_a$ is a one-to-one bounded linear transformation from $L^1(Z')$ into \mathcal{M}^* , the dual space of \mathcal{M} . For

$$\begin{aligned} \|H_a f\|_2^2 &= \left\| \sum_{n \in Z'} a_n f(nt) \sum_{m \in Z'} \overline{a_m f(mt)} \right\|_1 \\ &= \left\| \sum_{m, n \in Z'} a_n \overline{a_m} f(nt) \overline{f(mt)} \right\|_1 \leq \sum_{m, n \in Z'} |a_n| |a_m| \|f(nt) \overline{f(mt)}\|_1 \\ &\leq \sum_{m, n \in Z'} |a_n| |a_m| \|f(nt)\|_2 \|f(mt)\|_2 = \left(\sum_{n \in Z'} |a_n| \right)^2 \|f\|_2^2 = \|a\|_1^2 \|f\|_2^2 \end{aligned}$$

since $\|f(nt)\|_2 = \|f(t)\|_2$ for all $n \in Z'$. Therefore $\|H_a\|_{op} \leq \|a\|_1$. Also, $f \in \mathcal{M}$ implies $H_a f \in \mathcal{M}$ since

$$\int_T (H_a f)(t) dt = \sum_{n \in Z'} a_n \int_T f(nt) dt = 0.$$

Therefore, since H_a is clearly linear, $H_a \in \mathcal{M}^*$ and the mapping

$a \rightsquigarrow H_a$ is bounded and linear from $L^1(\mathbf{Z}')$ to \mathcal{M}^* . Finally, the mapping is one-to-one since

$$H_a(e^{it}) = \sum_{n \in \mathbf{Z}'} \alpha_n e^{int} = 0 \Leftrightarrow a = 0 .$$

We now apply Corollary 1 to evaluate the norm of H_a .

THEOREM 4. *Let $a = \{\alpha_n\}_{n \in \mathbf{Z}'}$ $\in L^1(\mathbf{Z}')$, and for $r \in \mathbf{Q}^\times$ let*

$$F(r) = \sum_{\substack{m/n=r \\ m, n \in \mathbf{Z}'}} \alpha_m \bar{\alpha}_n .$$

Then

$$\|H_a\|_{op} = \max_{x \in \hat{\mathbf{Q}}^\times} \left| \sum_{r \in \mathbf{Q}^\times} (r, x) F(r) \right|^{\frac{1}{2}}$$

where $\hat{\mathbf{Q}}^\times$ is the compact dual of the discrete group \mathbf{Q}^\times .

Proof. Let $f \in \mathcal{M}$, and let the Fourier expansion of f be

$$f(t) = \sum_{m \in \mathbf{Z}'} b_m e^{imt} .$$

Then

$$\begin{aligned} (H_a f)(t) &= \sum_{n \in \mathbf{Z}'} \alpha_n f(nt) = \sum_{n \in \mathbf{Z}'} \alpha_n \left[\sum_{m \in \mathbf{Z}'} b_m e^{imnt} \right] \\ &= \sum_{m, n \in \mathbf{Z}'} \alpha_n b_m e^{imnt} = \sum_{p \in \mathbf{Z}'} c_p e^{ipt} , \end{aligned}$$

where $c_p = \sum_{mn=p} \alpha_n b_m$, and L^2 convergence is the justification for the rearrangement of summation. Therefore

$$\begin{aligned} \|H_a f\|_2^2 &= \sum_{p \in \mathbf{Z}'} |c_p|^2 = \sum_{p \in \mathbf{Z}'} \left(\sum_{\substack{mn=p \\ m, n \in \mathbf{Z}'}} \bar{\alpha}_n \overline{b_m} \sum_{\substack{m'n'=p \\ m', n' \in \mathbf{Z}'}} \alpha_n b_{m'} \right) \\ &= \sum_{\substack{m, n, m', n' \in \mathbf{Z}' \\ mn=m'n'}} \bar{\alpha}_n \bar{b}_m \alpha_n b_{m'} = \sum_{m, m' \in \mathbf{Z}'} \left\{ \left(\sum_{\substack{n, n' \in \mathbf{Z}' \\ n'/n=m/m'}} \alpha_n \bar{\alpha}_n \right) \bar{b}_m \overline{b_{m'}} \right\} \end{aligned}$$

where the manipulation of the quadruple sum is justified by absolute convergence:

$$\begin{aligned} \sum_{\substack{m, n, m', n' \in \mathbf{Z}' \\ mn=m'n'}} |\bar{\alpha}_n \bar{b}_m \alpha_n b_{m'}| &= \sum_{n, n' \in \mathbf{Z}'} |\alpha_n| |\alpha_{n'}| \left(\sum_{\substack{m, m' \in \mathbf{Z}' \\ n'/n=m/m'}} |b_m| |b_{m'}| \right) \\ &\leq \sum_{n, n' \in \mathbf{Z}'} |\alpha_n| |\alpha_{n'}| \left(\sum_{m \in \mathbf{Z}'} |b_m|^2 \right) = \|\alpha\|_1^2 \|f\|_2^2 < +\infty \end{aligned}$$

by Cauchy-Schwarz. Upon setting

$$\alpha_{i,j} = \sum_{m/n=i/j} \alpha_m \bar{\alpha}_n = F\left(\frac{i}{j}\right) \quad \text{for } i, j \in \mathbf{Z}'$$

and

$$M = (\alpha_{ij}) \quad (\text{order } \mathbf{Z}' = (1, -1, 2, -2, \dots)),$$

we obtain

$$M \sim (Q^\times, \mathbf{Z}', F).$$

Also, upon identifying \mathcal{M} with \mathcal{L}^2 by $f \leftrightarrow X_f = (b_1, b_{-1}, b_2, b_{-2}, \dots)$ we have

$$\|H_a f\|_2^2 = (MX_f, X_f).$$

But

$$\begin{aligned} \sum_{r \in Q^\times} |F(r)| &= \sum_{\substack{i,j \in \mathbf{Z}' \\ \binom{i}{j}=1 \\ i>0}} \left| F\left(\frac{i}{j}\right) \right| \leq \sum_{\substack{i,j \in \mathbf{Z}' \\ \binom{i}{j}=1 \\ i>0}} \sum_{m/n=i/j} |\alpha_n| |\alpha_m| \\ &= \left(\sum_{n \in \mathbf{Z}'} |\alpha_n| \right)^2 = \|a\|_1^2 < +\infty, \end{aligned}$$

and hence

$$f(x) = \sum_{r \in Q^\times} (r, x) F(r)$$

is a continuous function on \hat{Q}^\times with Fourier transform F . The theorem follows upon applying Theorem 2 (Corollary 1 (i)) to $M \sim (Q, \mathbf{Z}', F)$.

COROLLARY 3. *If $a = \{\alpha_n\}_{n \in \mathbf{Z}'} \in L^1(\mathbf{Z}')$ and $\alpha_n \geq 0$ for all $n \in \mathbf{Z}'$, then*

$$\|H_a\|_{op} = \|a\|_1.$$

Proof. By Theorem 4,

$$\|H_{a_p}\|_{op} = \max_{x \in \hat{Q}^\times} \left| \sum_{r \in Q^\times} (r, x) F(r) \right|^{\frac{1}{2}} \leq \left(\sum_{r \in Q^\times} |F(r)| \right)^{\frac{1}{2}} = \|a\|_1$$

since

$$\alpha_n \geq 0, |F(r)| = \sum_{\substack{m/n=r \\ m, n \in \mathbf{Z}'}} \alpha_m \alpha_n, \text{ and } \sum_{r \in Q^\times} |F(r)| = \|a\|_1^2.$$

But upon setting $x = 0$ we obtain

$$\left| \sum_{r \in Q^\times} (r, 0)F(r) \right|^{\frac{1}{2}} = \left| \sum_{r \in Q^\times} F(r) \right|^{\frac{1}{2}} = \left(\sum_{r \in Q^\times} |F(r)| \right)^{\frac{1}{2}} = \|a\|_1,$$

and thus the proof is complete.

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Herbert James Alexander, <i>Extending bounded holomorphic functions from certain subvarieties of a polydisc</i>	485
Edward T. Cline, <i>On an embedding property of generalized Carter subgroups</i>	491
Roger Cuppens, <i>On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum. II</i>	521
William Richard Emerson, <i>Translation kernels on discrete Abelian groups</i>	527
Robert William Gilmer, Jr., <i>Power series rings over a Krull domain</i>	543
Julien O. Hennefeld, <i>The Arens products and an imbedding theorem</i>	551
James Secord Howland, <i>Embedded eigenvalues and virtual poles</i>	565
Bruce Ansgar Jensen, <i>Infinite semigroups whose non-trivial homomorphs are all isomorphic</i>	583
Michael Joseph Kascic, Jr., <i>Polynomials in linear relations. II</i>	593
J. Gopala Krishna, <i>Maximum term of a power series in one and several complex variables</i>	609
Renu Chakravarti Laskar, <i>Eigenvalues of the adjacency matrix of cubic lattice graphs</i>	623
Thomas Anthony Mc Cullough, <i>Rational approximation on certain plane sets</i>	631
T. S. Motzkin and Ernst Gabor Straus, <i>Divisors of polynomials and power series with positive coefficients</i>	641
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix.</i>	653
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix. II</i>	663
Donald Steven Passman, <i>Exceptional 3/2-transitive permutation groups</i>	669
Grigorios Tsagas, <i>A special deformation of the metric with no negative sectional curvature of a Riemannian space</i>	715
Joseph Zaks, <i>Trivially extending decompositions of E^n</i>	727