POWER SERIES RINGS OVER A KRULL DOMAIN

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Let $D$ be a Krull domain and let $\{X_\lambda\}_{\lambda \in A}$ be a set of indeterminates over $D$. This paper shows that each of three “rings of formal power series in $\{X_\lambda\}$ over $D$” are also Krull domains; also, some relations between the structure of the set of minimal prime ideals of $D$ and the set of minimal prime ideals of these rings of formal power series are established.

In considering formal power series in the $X_j$'s over $D$, there are three rings which arise in the literature and which are of importance. We denote these here by $D[[{x_\lambda}]]_1$, $D[[{X_\lambda}]]_2$, and $D[[{X_\lambda}]]_3$. $D[[{x_\lambda}]]_1$ arises in a way analogous to that of $D[[X]]$—namely, $D[[{X_\lambda}]]_1$ is defined to be $\bigcup_{\varnothing \neq F \in \mathcal{F}} D[[F]]$, where $\mathcal{F}$ is the family of all finite nonempty subsets of $A$. $D[[{X_\lambda}]]_2$ is defined to be $\{ \sum_{i=0}^{\infty} f_i \in D[{X_\lambda}] : f_i = 0$ or a form of degree $i \}$, where equality, addition, and multiplication are defined on $D[[{x_\lambda}]]_2$ in the obvious ways. $D[[{X_\lambda}]]_2$ arises as the completion of $D[[X]]$ under the $(\{X_\lambda\})$-adic topology; the topology on $D[[{X_\lambda}]]_2$ is induced by the decreasing sequence $\{A_i\}_i$ of ideals, where $A_i$ consists of those formal power series of order $\geq i$—that is, those of the form $\sum_{j=i}^{\infty} f_j$. If $A$ is infinite, $A_i$ properly contains the ideal of $D[[{X_\lambda}]]_2$ generated by $\{X_\lambda\}$. Finally, $D[[{X_\lambda}]]_3$ is the full ring of formal power series over $D$, and is defined as follows (cf. [1, p. 66]): Let $N$ be the set of nonnegative integers, considered as an additive abelian semigroup, and let $S$ be the weak direct sum of $N$ with itself $|A|$ times. $S$ is an additive abelian semigroup with the property that for any $s \in S$, there are only finitely many pairs $(t, u)$ of elements of $S$ whose sum is $s$. $D[[{X_\lambda}]]_3$ is defined to be the set of all functions $f : S \rightarrow D$, where $(f + g)(s) = f(s) + g(s)$ and where $(fg)(s) = \sum_{t+u=s} f(t)g(u)$ for any $s \in S$, the notation $\sum_{t+u=s}$ indicating that the sum is taken over all ordered pairs $(t, u)$ of elements of $S$ with sum $s$. To within isomorphism we have $D[[{X_\lambda}]]_3 \subseteq D[[{X_\lambda}]]_2 \subseteq D[[{X_\lambda}]]_1$, and each of these containments is proper if and only if $A$ is infinite. Our method of attack in showing that $D[[{X_\lambda}]]_i$, $i = 1, 2, 3$, is a Krull domain if $D$ is consists in showing that $D[[{X_\lambda}]]_i$ is a Krull domain and that $D[[{X_\lambda}]]_3 \cap K_i = D[[{X_\lambda}]]_i$, for $i = 1, 2$, where $K_i$ denotes the quotient field of $D[[{X_\lambda}]]_i$.

1. The proof that $D[[{X_\lambda}]]_3$ is a Krull domain. Using the
notation of the previous section, we introduce some terminology which will be helpful in showing that $D[[X]]_3$ is a Krull domain. We think of the elements of $S$ as $\{n\}_\lambda$-tuples $\{n_\lambda\}_{\lambda \in \Lambda}$ which are finitely nonzero. For $s = \{n_\lambda\} \in S$, we define $\pi(s)$ to be $\sum_{\lambda \in \Lambda} n_\lambda$ and we denote by $S_t$ the set of elements $s$ of $S$ such that $\pi(s) = t$; clearly $\pi$ is a homomorphism from $S$ onto $N$. Given a well-ordering on the set $A$, we well-order the set $S$ as follows: if $s = \{m_\lambda\} = \{n_\lambda\}$ are distinct elements of $S$, then $s < t$ if $\pi(s) < \pi(t)$ or if $\pi(s) = \pi(t)$ and $m_\lambda < n_\lambda$ for the first $\lambda$ in $A$ such that $m_\lambda$ and $n_\lambda$ are unequal. It is clear that this ordering on $S$ is compatible with the semigroup operation—that is, $s_1 < s_2$ implies that $s_1 + t < s_2 + t$ for any $t$ in $S$. Also, $S$ is cancellative and $s_1 + t < s_2 + t$ implies that $s_1 < s_2$.

If $f \in D[[X]]_3 - \{0\}$, we say that $f$ is a form of degree $i$, where $i \in N$, provided $f$ vanishes on $S - S_i$; the order of $f$, denoted by $\mathcal{O}(f)$, is defined to be the smallest nonnegative integer $t$ such that $f$ does not vanish on $S_t$. If $\mathcal{O}(f) = k$, then the initial form of $f$ is defined to be that element $f_k$ of $D[[X]]_3$ which agrees with $f$ on $S_k$ and which vanishes on $S - S_k$.

**Lemma 1.1.** If $f, g \in D[[X]]_3 - \{0\}$, then

1. If $f + g \neq 0$, $\mathcal{O}(f + g) \geq \min \{\mathcal{O}(f), \mathcal{O}(g)\}$.
2. $\mathcal{O}(fg) = \mathcal{O}(f) + \mathcal{O}(g)$.
3. If $f$ and $g$ are forms of degree $m$ and $n$, respectively, then $fg$ is a form of degree $m + n$.
4. The initial form of $fg$ is the product of the initial forms of $f$ and of $g$.

**Proof.** In a less general context, Lemma 1.1 is a well known result; we prove only (2) and (3) here.

(2): We let $s$ be the smallest element of $S$ on which $f$ does not vanish and we let $t$ be the corresponding element for $g$. By definition of $\pi$ and $\mathcal{O}$, $\pi(s) = \mathcal{O}(f) = i$ and $\pi(t) = \mathcal{O}(g) = j$. To show that $\mathcal{O}(fg) = i + j$, we prove that $(fg)(s + t) \neq 0$ and that $(fg)(u) = 0$ for $u < s + t$. The second statement is clear, for if $s' + t' = u$, then either $s' < s$ or $t' < t$ so that $f(s') = 0$ or $g(t') = 0$ and $f(s')g(t') = 0$ in either case. By similar reasoning, we see that $(fg)(s + t) = f(s)g(t) = 0$. Hence $\mathcal{O}(fg) = i + j$.

(3): By (2), $\mathcal{O}(fg) = m + n$. To see that $fg$ is a form, we need only observe that $fg$ vanishes on $S_k$ for any $k > m + n$. Thus if $w \in S_k$, then $(fg)(w) = \sum_{u + v = w} f(u)g(v)$ and for each such pair $(u, v)$ either $\pi(u) > m$ or $\pi(v) > n$ so that $f(u) = 0$ or $g(v) = 0$ so that $(fg)(w) = \sum_{u + v = w} f(u)g(v) = 0$.

**Lemma 1.2.** Let $K$ be a field and let $\{D_a\}$ be a family of sub-
domains of $K$ such that each $D_\alpha$ is a Krull domain. Let $D = \bigcap_\alpha D_\alpha$ and suppose that each nonzero element of $D$ is a nonunit in only finitely many $D_\alpha$'s. Then $D$ is a Krull domain.

Proof. For each $\alpha$ we consider a defining family $\{V_{\beta_\alpha}\}$ of rank one discrete valuation rings for $D_\alpha$. If $L$ is the quotient field of $D$ and $\mathcal{S} = \{V_{\beta_\alpha} \cap L\}_{\alpha, \beta}$, $\mathcal{S}$ is a family of discrete valuation rings of rank $\leq 1$, and the intersection of the members of the collections $\mathcal{S}$ is $D$. If $d$ is a nonzero element of $D$, then $d$ is a nonunit in only finitely many $D_\alpha$'s, say $D_{\alpha_1}, \ldots, D_{\alpha_n}$. Because $D_{\alpha_i}$ is a Krull domain and $\{V_{\beta_i}^{(\alpha_i)}\}$ is a defining family for $D_{\alpha_i}$, $d$ is a nonunit in only finitely many of the $V_{\beta_i}^{(\alpha_i)}$'s. Therefore $D$ is a Krull domain and the family of essential valuations for $D$ is a subfamily of $\{V_{\beta_\alpha} \cap L\}_{\alpha, \beta}$ [6, p. 116].

We now give an outline of our proof that $D[[\{X_i\}]_a]$ is a Krull domain when $D$ is a Krull domain. Let $K$ be the quotient field of $D$ and let $\{V_a\}$ be the family of essential valuation rings for $D$ [7, p. 82]. By a result due to Cashwell and Everett [3] (see also [4]), $J[[\{X_i\}]_a]$ is a unique factorization domain (UFD), where $J$ is an integral domain with identity, if and only if $J[[(Y_1, \ldots, Y_n)]_a]$ is a UFD for any positive integer $n$. If $J$ is a principal ideal domain, then $J[[(Y_1, \ldots, Y_n)]_a]$ is a UFD for any $n$ [2, pp. 42, 100]; in particular, $K[[\{X_i\}]_a]$ and $V_a[[\{X_i\}]_a]$ are then UFD's for each $\alpha$. Consequently, $(V_a[[\{X_i\}]_a])_{N_\alpha}$ is a UFD for any multiplicative system $N_\alpha$ in $V_a[[\{X_i\}]_a]$. To show that $D[[\{X_i\}]_a]$ is a Krull domain, it will be sufficient, in view of Lemma 1.2, to show that by appropriate choices of the multiplicative systems $N_\alpha$, we can express $D[[\{X_i\}]_a]$ as

$$K[[\{X_i\}]_a] \cap \left( \bigcap_\alpha (V_a[[\{X_i\}]_a])_{N_\alpha} \right),$$

where each nonzero element of $D[[\{X_i\}]_a]$ is a nonunit in only finitely many $(V_a[[\{X_i\}]_a])_{N_\alpha}$'s. We define $N_\alpha$ as follows: $N_\alpha = \{f \in V_a[[\{X_i\}]_a] \mid 0 \in \mathcal{O}(f) = \mathcal{O}(f) = i \text{ and there exists } s \in S_i \text{ such that } f(s) \text{ is a unit of } V_a \}$, and we prove

**Proposition 1.3.** $N_\alpha$ is a multiplicative system in $V_a[[\{X_i\}]_a]$. 

$$(V_a[[\{X_i\}]_a])_{N_\alpha} \cap K[[\{X_i\}]_a] = V_a[[\{X_i\}]_a],$$

so that

$$D[[\{X_i\}]_a] = K[[\{X_i\}]_a] \cap \left( \bigcap_\alpha (V_a[[\{X_i\}]_a])_{N_\alpha} \right).$$

Each nonzero element of $D[[\{X_i\}]_a]$ is in all but a finite number of the $N_\alpha$'s.
Before giving the proof of Proposition 1.2, we recall a result concerning the content of the product of two polynomials. Let \( J \) be an integral domain with identity having quotient field \( F \) and for \( f \in F[[X_i]] \), let \( A_f \) denote the fractional ideal of \( J \) generated by the set of coefficients of \( f \). In order that \( A_{fg} = A_f A_g \) for each pair \( f, g \) of elements of \( F[[X_i]] \), it is necessary and sufficient that \( J \) be a Prüfer domain\(^1\) [5, Th. 1]. In particular \( A_{fg} = A_f A_g \) for each \( f, g \in F[[X_i]] \) if \( J \) is a valuation ring.

**Proof of Proposition 1.3.** To show that \( N_a \) is a multiplicative system, let \( f, g \in N_a \). Then the initial forms \( f_i, g_j \) of \( f \) and \( g \) are in \( N_a \). \( f_i g_j \) is the initial form of \( f g \) and \( \mathcal{O}(fg) = i + j = \mathcal{O}(f) + \mathcal{O}(g) \). Therefore we need only show that \( (fg)(s) \) is a unit of \( V_a \) for some \( s \in S_{i+j} \). The smallest element \( u \) of \( S \) for which \( f(u) \) is a unit in \( V_a \) is an element of \( S_i \) and the smallest element \( v \) of \( S \) for which \( g(v) \) is a unit of \( V_a \) is an element of \( S_j \). \( u + v \in S_{i+j} \) and \( (fg)(u + v) = \sum u^i v^j f(u') g(v') \) is a unit of \( V_a \). For if \( u' + v' = u + v \) and if \( \{u', v'\} \neq \{u, v\} \), then either \( u' < u \) or \( v' < v \) so that \( f(u') \) or \( g(v') \), and hence \( f(u') g(v') \), is a nonunit of \( V_a \). It follows that \( (fg)(u + v) \) is the unit \( f(u) g(v) \) plus a nonunit of \( V_a \). Therefore \( (fg)(u + v) \) is a unit of \( V_a \), \( fg \in N_a \), and \( N_a \) is a multiplicative system.

To prove that \( K[[\{X_i\}]]_3 \cap (V_a[[\{X_i\}]]_3)_{N_a} \subseteq V_a[[\{X_i\}]]_3 \), (the opposite containment is clear), we must show that if \( f \in K[[\{X_i\}]]_3 \setminus \{0\} \) and if there is an element \( g \) of \( N_a \) such that \( fg \in V_a[[\{X_i\}]]_3 \), then \( f \in V_a[[\{X_i\}]]_3 \). By induction, it suffices to show that the initial form \( f_i \) of \( f \) is in \( V_a[[\{X_i\}]]_3 \). If \( g_j \) is the initial form of \( g \), then \( g_j \in N_i \) and \( f_i g_j \), the initial form of \( fg \), is in \( V_a[[\{X_i\}]]_3 \). We can therefore assume without loss of generality that \( f \) and \( g \) are forms of degree \( i \) and \( j \), respectively. Let \( s \in S_i \). We must show that \( f(s) \in V_a \). Let \( t \) be an element of \( S_j \) such that \( g(t) \) is a unit of \( V_a \). If \( s = \{m_i\} \) and if \( t = \{n_i\} \) there are only finitely many elements \( \tau \) of \( \Lambda \) such that \( m_\tau \neq 0 \) or \( n_\tau \neq 0 \); let \( \lambda_1, \lambda_2, \ldots, \lambda_u \) be this finite set of elements of \( \Lambda \). There are only finitely many elements \( \{k_\lambda\} \) of \( S_i \) such that \( k_\lambda = 0 \) for each \( \lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_u\} \); let these elements be \( s_1, s_2, \ldots, s_v \). Also, there are only finitely many elements \( \{k_\lambda\} \) of \( S_j \) such that \( k_\lambda = 0 \) for each \( \lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_u\} \), and we let these elements be \( t_1, t_2, \ldots, t_v \). If \( f^* \) is the polynomial \( \sum_{q=1}^r f(s_q)X_{i_1}^{n_{q1}} \cdots X_{i_u}^{n_{qu}} \), where \( s_q = \{n_\lambda^{(q)}\} \) and if \( g^* = \sum_{q=1}^r g(t_q)X_{i_1}^{m_{q1}} \cdots X_{i_u}^{m_{qu}} \), where \( t_q = \{m_\lambda^{(q)}\} \), then by definition of addition in \( S \), it is true that \( (fg)(\{k_\lambda\}) \) is equal to the coefficient of \( X_{i_1}^{k_{i_1}} \cdots X_{i_u}^{k_{i_u}} \) in \( f^* g^* \) for any \( \{k_\lambda\} \) in \( S_{i+j} \) such that \( k_\lambda = 0 \) for \( \lambda \in \{\lambda_1, \lambda_2, \ldots, \lambda_u\} \).

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\(^1\) A Prüfer domain is an integral domain with identity in which each nonzero finitely generated ideal is invertible.
Therefore, \( f^*g^* \in V_a[[X_i]] \) since \( f^*g^* \in V_a[[X_i]] \). Further, \( A_{\alpha'} = V_\alpha \) since \( t \in \{t_1, \ldots, t_r\} \) and since \( g(t) \) is a unit of \( V_\alpha \). Therefore \( A_{\alpha'}^* = A_{\alpha'} = A_{\alpha'} \subseteq V_\alpha \). But \( f(s) \in A_{\alpha'} \) since \( s \in \{s_1, s_2, \ldots, s_r\} \). Hence \( f(s) \in V_\alpha \) and our proof is complete.

Finally, if \( h \) is a nonzero element of \( D[[X_i]] \), of order \( i \), then we choose \( s \in S_i \) such that \( h(s) \neq 0 \). Since \( \{V_\alpha\} \) is the family of essential valuation rings for the Krull domain \( D \), \( h(s) \) is a unit in all but a finite set \( \{V_{\alpha_1}, \ldots, V_{\alpha_w}\} \) of the \( V_\alpha \)'s. Hence \( h \) is in each \( \alpha \) save \( N_{\alpha_1}, \ldots, N_{\alpha_w} \).

**Theorem 1.4.** If \( D \) is a Krull domain, then \( D[[X_i]] \) is also a Krull domain.

2. The proofs that \( D[[X_i]] \) and \( D[[X_i]] \) are Krull domains. In view of Theorem 1.4, in order to show that \( D \) Krull implies that \( D[[X_i]] \), \( i = 1, 2 \), is Krull, it is sufficient to show that for any integral domain \( J \) with identity, \( J[[X_i]] \cap K_i = J[[X_i]] \), where \( K_i \) denotes the quotient field of \( J[[X_i]] \). Thus we need to show that if \( f \in J[[X_i]] \) \( \neq 0 \) and if \( g \) is a nonzero element of \( J[[X_i]] \) \( \neq 0 \) such that \( f^*g \in J[[X_i]] \), then \( f \in J[[X_i]] \). We consider first the case when \( i = 2 \). By induction, it suffices to show that the initial form of \( f \) is in \( J[[X_i]] \), and since the product of the initial form of \( f \) and the initial form of \( g \) is the initial form of \( fg \) and is in \( J[[X_i]] \), we need consider only the case when \( f \) and \( g \) are forms of degrees \( i \) and \( j \), respectively. Since \( fg \) and \( g \) are in \( J[[X_i]] \), there is a finite subset \( \{\lambda_1, \ldots, \lambda_n\} \) of \( A \) such that \( g \) vanishes on each element \( \{n_i\} \) of \( S_j \) for which \( n_i \neq 0 \) for some \( \lambda \in A - \{\lambda_1\} \) and such that \( fg \) vanishes on each element \( \{m_i\} \) of \( S_{i+j} \) for which \( m_i \neq 0 \) for some \( \lambda \in A - \{\lambda_1\} \). We observe that this implies that \( f \) vanishes on each element \( \{p_i\} \) of \( S_i \) such that \( p_i \neq 0 \) for some \( \lambda \in \{\lambda_1, \ldots, \lambda_n\} \), for if this were not the case, then there would be a smallest element \( p = \{p_i\} \) of \( S_i \) with \( p_i \neq 0 \) for some \( \mu \in \{\lambda_1, \ldots, \lambda_n\} \) for which \( f(p) \neq 0 \). Then if \( s = \{s_i\} \) is the smallest element of \( S_j \) for which \( g(s) \neq 0 \), we observe that \( (fg)(p + s) = f(p)g(s) \neq 0 \) and that \( p + s = \{p_i + s_i\} \), where \( p_i + s_i \) \( \geq p_i > 0 \), contrary to the hypothesis on \( fg \). We see that \( (fg)(p + s) = f(p)g(s) \) as follows: If \( p' + s' = p + s \) where \( p' \in S_i \) and \( s' \in S_j \), then \( s' < s \) implies that \( g(s') = 0 \) so that \( f(p')g(s') = 0 \). On the other hand, if \( s' > s \), then \( p' < p \) so that \( f(p') = 0 \) if \( p' = \{p_i\} \) and \( p'_i \neq 0 \), while \( g(s') = 0 \) if \( p'_i = 0 \) since the \( \mu \)-th coordinate of \( s' \) is then nonzero. Consequently, \( (fg)(p + s) = f(p)f(s) \), and the contradiction which this equality implies shows that it is indeed the case that \( f(\{p_i\}) = 0 \) for each \( \{p_i\} \) in \( S_i \) such that \( p_i \neq 0 \) for some \( \lambda \in \{\lambda_1, \ldots, \lambda_n\} \). Hence \( f \in J[[X_i]] \) as we wished to show.

Our proof for \( J[[X_i]] \) shows that if the set \( \{\lambda_1, \ldots, \lambda_n\} \) does
not depend on $i$, as is the case if $g$ and $fg$ are in $J[\{X_i\}]$, then each form $f_i$ associated with $f$ (that is, $f \cdot \chi_i$, where $\chi_i$ is the characteristic function of $S_i$) will also have the property that it vanishes on each element $\{s_i\}$ of $S_i$ such that $s_i \neq 0$ for some $\lambda \in \{\lambda_1, \ldots, \lambda_\alpha\}$. Consequently, $f \in J[\{X_i\}]$. We have proved

**Theorem 2.1.** If $D$ is a Krull domain, then $D[\{X_i\}]$ and $D[\{X_i\}]$ are also Krull domains.

3. Minimal primes of $D[\{X_i\}]$. Our proofs of Lemma 1.2 and Proposition 1.3 show the following, in case $D$ is a Krull domain with quotient field $K$. If $L$ is the quotient field of $D[\{X_i\}]$, then the set of essential valuation rings for $D[\{X_i\}]$ is a subset of $\{W_\alpha \cap L \cup \{W^{(a)}_i \cap L\}$, where $\{W_\alpha\}$ is the family of essential valuation rings for $K[\{X_i\}]$, and where $\{W^{(a)}_i\}$ is the family of essential valuation rings for $(V_\alpha[\{X_i\}])_\alpha$; $\{V_\alpha\}$ the family of essential valuation rings for $D$. Let $M_\alpha$ be the center of $W_\alpha \cap L$ on $D[\{X_i\}]$ and let $M^{(a)}_i$ be the center of $W^{(a)}_i \cap L$ on $D[\{X_i\}]$. Since $K \subset W_\alpha, M_\alpha \cap K = (0)$; in particular, $M_\alpha \cap D = (0)$. Further, $V_\alpha$ is clearly contained in $W^{(a)}_i \cap L$ so that $W^{(a)}_i \cap L = V_\alpha$ or $W^{(a)}_i \cap L = K$. In the first case $M^{(a)}_i \cap D = P_\alpha$, where $V_\alpha = D_{P_\alpha}$, and in the second $M^{(a)}_i \cap D = (0)$. Since $D[\{X_i\}]$ is a Krull domain, the set of minimal primes of $D[\{X_i\}]$ is a subset of $\{M_\alpha\} \cup \{M^{(a)}_i\}$. Hence we have proved

**Lemma 3.1.** Each minimal prime of $D[\{X_i\}]$ meets $D$ either in zero or in minimal prime of $D$.

Our main purpose in this section is to prove:

**Theorem 3.2.** If $P_\alpha$ is a minimal prime of $D$, there is a unique minimal prime of $D[\{X_i\}]$ which meets $D$ in $P_\alpha$.

Our proof of Theorem 3.2 proceeds as follows. Let $v_\alpha$ be a valuation associated with the valuation ring $D_{P_\alpha}$. We observe that the function $v_\alpha^*$ defined on $D[\{X_i\}]$, by $v_\alpha^*(f) = \min \{v_\alpha(f(s)) \mid s \in S\}$ induces a valuation on $L$, the quotient field of $D[\{X_i\}]$. To prove this, let $f, g \in D[\{X_i\}]$ and suppose that $v_\alpha((f + g)(t)) = v_\alpha(f) + v_\alpha(g)$. Since $v_\alpha(f(t) + g(t)) \geq \min \{v_\alpha(f(t)), v_\alpha(g(t))\} \geq \min \{v_\alpha^*(f), v_\alpha^*(g)\}$, it follows that $v_\alpha^*(f + g) \geq \min \{v_\alpha^*(f), v_\alpha^*(g)\}$. Also, if $s$ is the smallest element of $S$ such that $v_\alpha(f(s)) = v_\alpha^*(f)$ and if $u$ is the smallest element of $S$ such that $v_\alpha(g(u)) = v_\alpha^*(g)$, then it is straightforward to show that

$$v_\alpha((fg)(s + u)) = v_\alpha(f(s)) + v_\alpha(g(u)) = v_\alpha^*(f) + v_\alpha^*(g) = \min \{v_\alpha((fg)(t)) \mid t \in S\} = v_\alpha^*(fg).$$
We denote the extension of \( v^*_a \) to \( L \) by \( v^*_v \) also; it is clear that \( v^*_a \) and \( v^*_v \) have the same value group so that \( v^*_a \) is rank one discrete and is an extension of \( v_a \) to \( L \). The center of \( v^*_a \) on \( \mathcal{O}_a \) is \( \mathcal{O}_a \). The center of \( \mathcal{O}_a \) on \( \mathcal{O}_a \) is the prime ideal \( Q_a = \{ f \mid f(s) \in P_a \text{ for each } s \in S \} \); we next prove that \( (D[[X]])_{\mathcal{O}_a} \) is the valuation ring of \( v^*_a \). One containment is clear. To prove the reverse containment, we show that if \( f, g \in D[[X]] \) and if \( v^*_a(f) \geq v^*_a(g) \), then for some \( \xi \) in \( K \), \( f/\xi g \) where \( \xi g \in D[[X]] \) and \( \xi g \in D[[X]] - Q_a \). This is immediate from the approximation theorem for Krull domains \([2, \text{ P. 12}]\), which shows that there is an element \( \xi \) of \( K \) such that \( v_a(\xi) = -v^*_a(g) \) and such that \( v^*_a(\xi) \geq 0 \) for each essential valuation \( v^*_a \) of \( D \) distinct from \( v_a \). Hence \( (D[[X]])_{\mathcal{O}_a} \) is the valuation ring of \( v^*_a \). Before proving Theorem 3.2, we need to make one final observation: If \( P_a \) is finitely generated—say \( P_a = (p_1, \ldots, p_s) \)—then \( Q_a \) is the extension of \( P_a \) to \( D[[X]] \). For is \( f \in Q_a \), then \( f(s) \) can be written in the form \( \sum a_i p_i \) for some \( a_i \in D \). Hence if \( f \) is the element of \( D[[X]] \) such that \( f_i(s) = a_i(s) \) for each \( s \) in \( S \), then \( f = \sum_i f_i p_i \) and \( f \) is in the extension of \( P \) to \( D[[X]] \).

**Proof of Theorem 3.2.** That \( Q_a \) is a minimal prime of \( D[[X]] \), lying over \( P_a \) in \( D \) is clear. If \( M \) is any minimal prime of \( D[[X]] \), lying over \( P_a \), then our previous observations show that \( M \) must be of the form \( M_{(a)} \), since only the \( V_{(a)} \)'s meet \( K \) in \( V_a \). Hence \( V_{(a)} \supseteq (D_{P_a}[[X]]_{\mathcal{O}_a}, \mathcal{O}_a) \) and \( MV_{(a)} \), the maximal ideal of \( V_{(a)} \), contains \( P_a(D_{P_a}[[X]]_{\mathcal{O}_a}, \mathcal{O}_a) \). Now \( P_aD_{P_a} \) is principal so that \( Q_a(D_{P_a}[[X]]_{\mathcal{O}_a} = P_a(D_{P_a}[[X]]_{\mathcal{O}_a}) = \mathcal{O}_a \). Consequently

\[
Q_a \subseteq Q_a(D_{P_a}[[X]]_{\mathcal{O}_a} \cap D[[X]]) = MV_{(a)} \cap D[[X]] = M.
\]

But since \( M \) is a minimal prime of \( D[[X]] \), this implies that \( M = Q_a \) and our proof is complete.

**References**


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