THE ARENS PRODUCTS AND AN IMBEDDING THEOREM

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Let $X$ be a separable Banach space, $B(X)$ be the algebra of all bounded linear operators on $X$, and $\mathcal{C}$ be the algebra of all compact linear operators. This paper centers around the general question of giving a construction of $B(X)$ as a Banach algebra starting from $\mathcal{C}$.

It is a result of Schatten and von Neumann that if $H$ is a Hilbert space, then there is an isometric imbedding of $B(H)$ onto $\mathcal{C}^{**}$, where $\mathcal{C}^{**}$ denotes the second dual of $\mathcal{C}$. Moreover, each of the two Arens products on $\mathcal{C}^{**}$ coincides with the multiplication induced on $\mathcal{C}^{**}$ by operator multiplication on $B(H)$. The proofs of these results make strong use of the Hilbert space structure.

In this paper we generalize these results to a large class of uniformly convex spaces. Moreover, we show that even when $B(X)$ is not equal to $\mathcal{C}^{**}$ it is still possible to construct $B(X)$ as a Banach algebra starting from $\mathcal{C}$.

We now amplify the above statements. The theorem of Schatten and von Neumann is proved in [9, p. 48]. See Civin and Yood [2, p. 869] or Rickart [8, p. 289] for the result on the Arens products.

In § 2 we give basic definitions and elementary results concerning Banach space bases and linear operators. In § 3 we prove the existence of an isometric imbedding from $B(X)$ into $\mathcal{C}^{**}$, under the assumption that $X$ has a shrinking, unconditionally monotone basis. Also, we show that under the same assumptions, a sufficient condition for the imbedding to be surjective is that $X$ be uniformly convex. In § 4 we prove that the imbedding is surjective $\iff$ the two Arens products on $\mathcal{C}^{**}$ coincide, and in that case they coincide with the multiplication on $\mathcal{C}^{**}$ induced by operator multiplication on $B(X)$. Finally, we show that for a certain class of Banach spaces, $B(X)$ is characterized as the largest subset of $\mathcal{C}^{**}$ in which $\mathcal{C}$ is a 2-sided ideal.

2. Preliminary definition and results.

Definition 2.1. A basis $(e_j)$ in a Banach space $X$ is a sequence of elements of $X$, such that for each $x \in X$, there is a unique sequence of scalars $(a_j)$ depending on $x$ such that $\lim_{n \to \infty} \| \sum_{j=1}^{n} a_j e_j - x \| = 0$. The coefficient $a_j$ is called the $j^{th}$ coordinate of $x$. It is a theorem of Banach's that if you define $e^*_i$ by $e^*_i(e_j) = \delta_{ij}$, then $e^*_i$ is in $X^*$. A
basis is called shrinking if \((e_i)\) is a basis for \(X^*\). A basis is called unconditional if for each \(x \in X\), the series \(\sum_{j=1}^{\infty} e_j(x)e_j\) is unconditionally convergent.

**Definition 2.2.** If \((e_j)\) is a basis for \(X\), let \(U_m x = \sum_{i \leq m} e_i(x)e_i\). Then \((e_j)\) is called a monotone basis if \(\| U_m x \| \leq \| x \|\) for all \(x \in X\) and integers \(m\).

**Definition 2.3.** If \((e_j)\) is an unconditional basis and \(D\) is a subset of the positive integers, let \(x^D = \sum_{i = 1, i \in D} e_i(x)e_i\). It is clear that \(x^D\) is convergent, since in a Banach space an unconditionally convergent series is also subseries convergent. Then \((e_j)\) is called unconditionally monotone if \(\| x^D \| \leq \| x \|\) for all \(x \in X\) and subsets \(D \subseteq \omega\).

**Proposition 2.1.** If \(X\) is a Banach space with an unconditional basis \((e_j)\), then \(X\) can be renormed isomorphically so that \((e_j)\) is an unconditionally monotone basis.

**Proof.** The norm \(\| x \|' = \sup \{ \| x^D \| : D\) is a finite subset of \(\omega\}\) is isomorphic to the original norm, and has the property that every rearrangement of \((e_j)\) is a monotone basis for \(X[4, p. 73]\). Suppose that \((e_j)\) is not unconditionally monotone with respect to the new norm. Then there exists a subset \(S \subseteq \omega\) such that

\[
\left\| \sum_{j=1}^{\infty} a_j e_j \right\|' < \left\| \sum_{j \in S} a_j e_j \right\|'.
\]

Hence, for \(n\) large enough

\[
\left\| \sum_{j=n+1}^{\infty} a_j e_j \right\|' < \left\| \sum_{j \leq n, j \in S} a_j e_j \right\|'.
\]

But this contradicts the fact that if we rearrange the basis \((e_j)\) so that we take first all the \(j\) in \(S\) and \(\leq n\), then it is a monotone basis.

Next we use a theorem of Maddaus to investigate \(\mathcal{E}\), the space of compact operators and its dual.

**Notation 2.1.** \(E_{ij}\) will denote the elementary matrix with a one in the \(ij^{th}\) coordinate and zeros elsewhere.

**Definition 2.4.** By a matrix concentrated in the \(j^{th}\) column (row), we will mean a matrix whose entries outside the \(j^{th}\) column (row), are all zero.

**Theorem 2.1.** Let \(X\) be a Banach space with a basis \((e_j)\). For
each compact operator $A$, let $A_n$ be the operator whose matrix consists of the first $n$ rows of $A$ and zeros elsewhere. Then $A$ is the uniform limit of the $A_n$.

Proof. This is proved in Maddaus [6].

**PROPOSITION 2.2.** Let $X$ be a Banach space with a basis $(e_k)$. Then for each fixed $j$, the set of matrices of $\mathcal{C}$ concentrated in the $j^{th}$ row is linearly isometric as a Banach space to $X^*$.

Proof. Let $R$ be the matrix of an operator in $\mathcal{C}$ concentrated in the $j^{th}$ row. Define $\alpha(e_k) = R_{jk}$ and extend $\alpha$ linearly to finite linear combinations of $(e_k)$. Let $x = \sum_{k=1}^n b_k e_k$. Then $\alpha(x) = \sum_{k=1}^n b_k R_{jk}$ and $R(x) = (\sum_{k=1}^n b_k R_{jk}) e_j$. Then since $|\alpha(x)| = ||R(x)||$ for each such $x$, $\alpha$ can be extended to a functional $\alpha \in X^*$ and the map $R \mapsto \alpha$ is isometric. This map is surjective because given $\alpha \in X^*$, define the matrix $R$ concentrated in the $j^{th}$ row with $R_{jk} = \alpha(e_k)$.

**PROPOSITION 2.3.** Let $X$ be a Banach space with an unconditionally monotone basis $(e_k)$. Then for each fixed $j$ the set of matrices of $\mathcal{C}$ concentrated in the $j^{th}$ column is linearly isometric as a Banach space to $X$.

Proof. Let $C_j$ be a matrix in $\mathcal{C}$ concentrated in the $j^{th}$ column. Consider the map $C_j \mapsto C_j e_j$. Clearly $||C_j e_j|| \leq ||C_j||$. For the other inequality, consider $x = b_j e_j + \sum_{i \neq j} b_i e_i$ with $||x|| = 1$. Then by unconditional monotonicity $|b_j| \leq 1$. Hence,

$$||C_j x|| = ||C_j(b_j e_j)|| \leq ||C_j e_j||.$$

**PROPOSITION 2.4.** Let $X$ be a Banach space with a shrinking basis $(e_j)$. Then, with each $f$ in $\mathcal{C}^*$ we can associate a matrix so that $f = g \langle = \rangle$ their matrices coincide.

Proof. First, we will show that the matrices with a finite number of nonzero entries span a dense linear manifold of $\mathcal{C}$.

Given a compact operator $A$ and $\varepsilon > 0$, choose $n$ so that $||A - A_n|| < (\varepsilon/2)$, where $A_n$ is the matrix consisting of the first $n$ rows of $A$. Let $R_j$ be the operator $A_n$ followed by the canonical projection onto the 1-dimensional subspace spanned by $[e_j]$, for $j = 1, 2, \ldots, n$. The matrix for $R_j$ is simply the $j^{th}$ row of $A_n$ and all other rows zero. Using the fact that the map in Proposition 2.2. is isometric and the hypothesis that $(e_k)$ is a shrinking basis, it follows that each of the matrices $R_j$ can be approximated to within $\varepsilon/2n$ by deleting (i.e., re-
placing by zeros) the tail of the $j^{th}$ row. Therefore, by the triangle inequality $A$ can be approximated to within $\varepsilon$ by a finite matrix.

For $f$ in $C^*$ we can define the matrix $(f_{ij})$ by $f_{ij} = f(E_{ij})$. Then if $f$ and $g$ have the same matrices they are equal.

**Proposition 2.5.** Suppose $X$ is a Banach space with an unconditionally monotone basis $(e_j)$ and $T$ is in $B(X)$. Then the matrix obtained by deleting (i.e., replacing by zeros) any set of rows or columns from $T$ is in $B(X)$ and has norm $\leq || T ||$.

**Proof.** Fix a subset $D \subset \omega$. Define $P_x = \sum_{j \in D} e_j^*(x)e_j$. Then, $|| TP(x) || \leq || T || || P_x || \leq || T || || x ||$. Thus, $|| TP || \leq || T ||$. Also note that the matrix for $TP$ is formed by deleting the $j^{th}$ column from $T$ for every $j \in D$.

Similarly, $|| PT || \leq || T ||$ and the matrix for $PT$ is formed by deleting the $j^{th}$ row from $T$ for every $j \in D$.

**Proposition 2.6.** Suppose $X$ is a Banach space with an unconditionally monotone, shrinking basis $(e_j)$, and that $f$ is in $C^*$. Then the matrix obtained by deleting any set of rows or columns from the associated matrix for $f$, is the matrix associated with a functional in $C^*$ with norm $\leq || f ||$.

**Proof.** Fix a subset $D \subset \omega$. Let $d: C \rightarrow C$ be the linear transformation which deletes the $j^{th}$ column for each $j \in D$. Then its adjoint $d^*$ has norm 1. Note that $(d^*f)A = f(dA)$. Hence, the matrix for $d^*f$ is formed by deleting every $j^{th}$ column for $j \in D$.

The argument for deleting rows is similar.

**Proposition 2.7.** Let $X$ be a Banach space with an unconditionally monotone, shrinking basis.

1. For each fixed $j$, the set of matrices in $C^*$ which are concentrated in the $j^{th}$ row is linearly isometric as a Banach space to $X^{**}$.
2. For each fixed $j$, the set of matrices in $C^*$ which are concentrated in the $j^{th}$ column is linearly isometric to $X^*$.

**Proof.** (1) Let $f_j \in C^*$ be concentrated in the $j^{th}$ row. Define $\phi(e_k^*) = f_{jk}$. Extend $\phi$ linearly to finite linear combinations of $(e_k^*)$. It follows from Proposition 2.2 that $\phi$ can be extended to a functional in $X^{**}$. Moreover, $|| \phi || = || f_j ||$ since $f_j$ approaches its norm on compact operators of norm one, concentrated in the $j^{th}$ row. The map $f_j \mapsto \phi$ is surjective because given $\phi \in X^{**}$, the matrix whose $j^{th}$ row is given by $f_{jk} = (e_k^*)$ and whose other rows are zero is in $C^*$.
(2) The proof is similar.

3. An imbedding theorem. We are now ready to give an isometric imbedding of $B(X)$ into $C^{**}$.

**Theorem 3.1.** If $(e_j)$ is an unconditionally monotone, shrinking basis for the Banach space $X$, then there is a linear isometric map from $B(X) \to C^{**}$ such that each $A$ in $C$ is taken onto its usual image under the evaluation map of $C \to C^{**}$.

**Proof.** Given $T$ in $B(X)$ let $R_j$ be the matrix consisting of the $j^{th}$ row of $T$ with zeros elsewhere. Define $\Phi_T$ in $C^{**}$ by $\Phi_T(f) = \sum_{j=1}^{\infty} f(R_j)$, where $f$ is in $C^*$ and $\|f\| = 1$. We must show that the series $\sum_{j=1}^{\infty} f(R_j)$ is convergent. By Proposition 2.5,

$$|f(R_{j_1} + \cdots + R_{j_n})| \leq \|T\|$$

for an arbitrary set of integers $\{j_1, \cdots, j_n\}$, since the left side represents $f$ applied to a compact operator formed by deleting rows from $T$. It is clear then that the series $\sum_{j=1}^{\infty} f(R_j)$ is unconditionally convergent.

The map $T \mapsto \Phi_T$ is obviously linear, since matrix addition and taking limits are linear operations.

$$|\Phi_T(f)| = \left|\sum_{j=1}^{\infty} f(R_j)\right| = \lim_{n \to \infty} \left|f\left(\sum_{j=1}^{n} R_j\right)\right| \leq \|f\| \|T\|,$$

since $\sum_{j=1}^{n} R_j$ is a compact operator of norm $\leq \|T\|$. Hence, $\Phi_T$ is bounded and $\|\Phi_T\| \leq \|T\|$. To prove the reverse, first, we note that $\|\sum_{j=1}^{n} R_j\|$ approaches $\|T\|$ as $n$ approaches $\infty$. Then, given $\varepsilon > 0$, take $\|\sum_{j=1}^{n} R_j\| > \|T\| - \varepsilon$. Since $\sum_{j=1}^{n} R_j$ is compact, we can find by the Hahn Banach theorem a $g$ in $C^*$ of norm 1, such that

$$g\left(\sum_{j=1}^{n} R_j\right) > \|T\| - \varepsilon.$$

Then let $g^0$ be the matrix formed by deleting the columns of $g$ past the $n^{th}$. By Proposition 2.6., $\|g^0\| \leq 1$, and we have that $\Phi_T(g^0) > \|T\| - \varepsilon$. Hence, $\|\Phi_T\| \geq \|T\|$ and the imbedding is isometric.

Then as we noted in Proposition 2.4., the finite matrices form a dense manifold of $C$. It is clear that $\Phi$ and the evaluation map agree on all finite matrices in $C$ and hence on all of $C$.

**Proposition 3.1.** Let $X$ be a Banach space with an unconditionally monotone, shrinking basis. Then $B(X) = C^{**}$ under the previous imbedding $\langle\langle\rangle\rangle$ the set of finite matrices in $C^*$ is a dense
linear manifold. Moreover, in that case \( X \) is reflexive.

**Proof.** If the set of finite matrices is not dense in \( C^* \), then there exists a nonzero \( F \) in \( C^{**} \), which is 0 on all finite matrices. However no \( \Phi_T \) for nonzero \( T \) in \( B(X) \) can have this property, since if \( T \) has the entry \( T_{ij} \neq 0 \), then \( \Phi_T(f_{ij}) = T_{ij} \) where \( f_{ij} \) is an elementary matrix in \( C^* \).

Assume the finite matrices are dense in \( C^* \). Let \( \pi \) be an arbitrary functional in \( X^{**} \). Then by Proposition 2.7, \( \pi \) can be identified with an \( f \in C^* \) which is concentrated in the \( j^{th} \) row. Since the finite matrices are dense in \( C^* \), \( \sum_{k=1}^{\infty} f_{jk} e_k \) converges in norm to \( \pi \) and hence \( X \) is reflexive.

Given \( F \in C^{**} \), define the matrix \( (F_{ij}) \) by \( F_{ij} = F(f_{ij}) \). \( F \) is determined by this associated matrix. By reflexivity and Proposition 2.7, it follows that each column of \( F \) represents an element of \( X \) with respect to \( (e_j) \). Then let \( T_n \) be the matrix consisting of the first \( n \) columns of \( F \). It is the matrix of a compact operator. Furthermore \( \Phi_{T_n}(f) = F(f^n) \) for each \( f \in C^* \), where \( f^n \) is the matrix formed from \( f \) by deleting all the columns past \( n^{th} \). Hence, \( ||T_n|| = ||\Phi_{T_n}|| \leq ||F||. \) Define the operator \( T \) by \( T(\sum_{j=1}^{n} a_j e_j) = T_n(\sum_{j=1}^{n} a_j e_j) \). \( T \) is well defined on the set of all finite linear combinations of the \( (e_j) \), and has norm \( \leq ||F||. \) Hence, it can be extended uniquely to a bounded operator on all of \( X \). It is clear that \( F = \Phi_T \), since \( F \) and \( \Phi_T \) agree on all finite matrices in \( C^* \).

The next proposition puts Proposition 3.2 into a more workable form for applications.

**Proposition 3.2.** Let \( X \) be a Banach space with an unconditionally monotone shrinking basis \( (e_j) \). Then, \( B(X) = C^{**} \langle = \rangle \) for each \( f \in C^* \), \( ||f^N|| \to 0 \), where \( f^N \) is the matrix formed from \( f \) by deleting the first \( N \) rows and \( N \) columns.

**Proof.** We will show that the condition on the right is satisfied \( \langle = \rangle \) the set of finite matrices in \( C^* \) span a dense manifold.

Suppose that the finite matrices are norm dense in \( C^* \). Given \( \varepsilon > 0 \) and \( f \in C^* \) there exists a finite \( g \) such that \( ||f - g|| < \varepsilon \). Then since \( g \) is finite we can pick \( N \) large enough so that \( f^N = (f - g)^N \). By Proposition 2.6. \( ||(f - g)^N|| \leq ||f - g|| < \varepsilon. \)

Conversely, suppose \( ||f^N|| \to 0 \). Given \( \varepsilon > 0 \) choose \( N \) large enough: \( ||f^N|| = ||f - (f - f^N)|| < \varepsilon/2. \) The matrix for \( f - f^N \) is not finite, but can be approximated to within \( \varepsilon/2 \) by a finite matrix.

The next proposition shows that if \( B(X) \neq C^{**} \), then the Banach space \( X \) behaves very much like \( (c_0) \), the space of sequences which
converge to 0.

**PROPOSITION 3.3.** Let $X$ be a Banach space with an unconditionally monotone shrinking basis $(e_i)$. If $B(X) \neq \mathscr{C}^{**}$, then for every $\varepsilon > 0$, and integer $n$, we can find an $x$ of norm 1, such that $x = x_1 + \cdots + x_n$, where each $x_i$ is a finite linear combination of distinct sets of basis vectors and $\|x_i\| \leq 1 - \varepsilon$.

**Proof.** By the previous proposition there exists an $f$ in $\mathscr{C}^*$ such that $\|f^N\|$ does not approach 0. The $f^N$ decrease in norm, since $f^{N+1}$ is formed by deleting a row and a column from $f^N$. We can assume without loss of generality that $\|f^N\| \rightarrow 1$ and never achieve it as $N \rightarrow \infty$. Then, given $\lambda > 0$, there exists an integer $N_1$: $\|f^{N_1}\| < 1 + \lambda$. Since the finite operators are dense in the compact operators there exists an integer $N_1 > N_1$, and a finite operator $T_1$ of norm 1: $T_1$ is concentrated on the manifold $X_1$ spanned by $[e_{N_1}, \cdots, e_{N_1'}]$. Let $N_2 = N_1 + 1$. For $f^{N_2}$ there exists a finite operator $T_2$ of norm 1, concentrated on the manifold $X_2 = [e_{N_2}, \cdots, e_{N_2'}]$. Repeating this process $n$ times, we can construct $T_1, \cdots, T_n$ such that $f^{N_k}(T_k) > 1$, and the $T_k$ are concentrated on disjoint basic blocks of $X$. Hence

$$n < f^{N_1}(T_1) + \cdots + f^{N_n}(T_n) = f^{N_1}(T_1 + \cdots + T_n) \leq \|f^{N_1}\| \|T_1 + \cdots + T_n\|,$$

and $n/1 + \lambda < \|T_1 + \cdots + T_n\|$. This means that there exists an $x$ of norm 1, where $x = x_1 + \cdots + x_n$, each $x_i$ is in $X_i$, and such that

$$\frac{n}{1 + \lambda} < \|(T_1 + \cdots + T_n)x\| \leq \|T_1x_1\| + \cdots + \|T_nx_n\|.$$

However, $\lambda > 0$ was arbitrary. By picking $\lambda > 0$ small enough, we can find $T_1, \cdots, T_n$: the sum $\|T_1x_1\| + \cdots + \|T_nx_n\|$ is as close to $n$ as we wish. By unconditional monotonicity, each $\|x_i\| \leq 1$. Thus, $\|T_i x_i\| \leq 1$. Hence, each $\|T_i x_i\|$ and $\|x_i\|$ will be close to 1.

**LEMMA 3.1.** A uniformly convex Banach space is reflexive.

**Proof.** See Wilansky [10, p. 109].

**LEMMA 3.2.** If $X$ is a reflexive Banach space with a basis, then the basis is shrinking.

**Proof.** See [10, p. 213].

**THEOREM 3.2.** If $X$ has an unconditionally monotone basis $(e_i)$
and $X$ is isomorphic to a uniformly convex Banach space $Z$, then $B(X) = \mathcal{C}^{**}$.

Proof. For each $x$ in $X$ call its norm $\|x\|$, and for its image in $Z$ call its norm $|x|$. Uniform convexity means that for every $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that if $x, x'$ are in the unit ball of $Z$, and $|x - x'| > \varepsilon$, then $|x + x'|/2 \leq 1 - \delta(\varepsilon)$. Clearly, if we renorm $Z$ by multiplying the old norm by some constant, the renormed $Z$ will still be uniformly convex. Hence, we may assume without loss of generality that there exists a constant $M$: $\|x\| \leq |x| \leq M\|x\|$. Let $t = \delta(1/2M)$. Choose $r$ large enough so that, $(1/1 - t)^r(1/2M) > 1$. Suppose $B(X) \neq \mathcal{C}^{**}$. By Proposition 3.3, there exists an $x$ of norm 1, such that $x = x_1 + \cdots + x_r$, where each $\|x_i\| \geq 1/2$ and where each $x_i$ is a linear combination of distinct $(e_i)$. We want to construct an element $v$: $\|v\| > 1$ and $|v| \leq 1$. This will contradict the fact that $\|v\| \leq |v|$.

Consider the following system of elements like the seeding chart of a tennis tournament. In the first round put the elements $w_1, \cdots, w_r$, where $w_k = (x_1 + \cdots + x_k)/M$ and $x_k$ as above. Then we construct the second round consisting of $2^{r-1}$ elements by letting the $n^{th}$ element of the second round be $u_n = (w_{zn-1} + w_{zn})/2(1 - t)$. To form the $n^{th}$ element $y_n$ of the third round, let

$$y_n = \frac{1}{2(1 - t)}(u_{zn-1} + u_{zn}).$$

The elements for the other rounds are formed in the same manner.

We claim that every element in this system lies in the unit ball of $Z$. For the first round, each $w_k$ is in the unit ball of $Z$, because $\|w_k\| \leq 1/M$ by unconditional monotonicity. We can assume that two paired elements $u$ and $u'$ from the $n^{th}$ round are in the unit ball of $Z$. Note that there exists an $x_k$: $u' = (1/M(1 - t)^{n-1})x_k + \text{other terms not involving } x_k$, whereas $u$ does not involve any of the $(e_i)$ used in expressing $(x_k)$. By unconditional monotonicity

$$\|u - u'\| \geq \frac{1}{M}\|x_k\| \geq \frac{1}{2M}.$$

Hence,

$$|u - u'| \geq \frac{1}{2M} \quad \text{and} \quad \left|\frac{1}{2(1 - t)}(u + u')\right| \leq 1.$$

Thus an arbitrary element of the $(n+1)^{st}$ round is in the unit ball of $Z$. Let $v$ be the element in the $r^{th}$ round. Then, $v = (1/(1 - t)^rM)x_1 + \text{other terms not involving } x_1$. Hence $\|v\| > 1$. This is impossible
since $|v| \leq 1$.

**COROLLARY 3.1.** If $X$ is isomorphic to a uniformly convex space and has an unconditional basis, then $B(X)$ is isomorphic to $\ell^*_{**}$.

*Proof.* Renorm $X$ so that the basis is unconditionally monotone.

**EXAMPLE 3.1.** The canonical basis for $l^p$ for $1 < p < \infty$ is unconditionally monotone and $l^p$ is uniformly convex, see Clarkson [3]. $L^p[0, 1]$ for $1 < p < \infty$, has an unconditional basis and is uniformly convex. See Pelczynski [7].

4. The Arens products. The two Arens products are defined in stages according to the following rules. Let $\mathcal{A}$ be a Banach algebra. Let $A, B \in \mathcal{A}; f \in \mathcal{A}^*; F, G \in \mathcal{A}^{**}$.

**DEFINITION 4.1.**

$(f^*A)B = f(AB)$. This defines $f^*A$ as an element of $\mathcal{A}^*$.

$(G^*f)A = G(f^*A)$. This defines $G^*f$ as an element of $\mathcal{A}^*$.

$(F^*G)f = F(G^*f)$. This defines $F^*G$ as an element of $\mathcal{A}^{**}$.

We will call $F^*G$ the first Arens product, or the $m_1$ product.

**DEFINITION 4.2.**

$(A^*_f)B = f(BA)$. This defines $A^*_f$ as an element of $\mathcal{A}^*$.

$(f^*_2F)A = F(A^*_f)$. This defines $f^*_2F$ as an element of $\mathcal{A}^*$.

$(F^*_2G)f = G(f^*_2F)$. This defines $F^*_2G$ as an element of $\mathcal{A}^{**}$.

$F^*_2G$ is the second Arens product or the $m_2$ product.

It is proved in Arens [1] that $m_1$ and $m_2$ are both Banach algebra products on $\mathcal{A}^{**}$, which extend the original multiplication on $\mathcal{A}$ when it is imbedded in $\mathcal{A}^{**}$.

**DEFINITION 4.3.** A Banach algebra $\mathcal{A}$ is called Arens regular if the two Arens products coincide on $\mathcal{A}^{**}$.

**DEFINITION 4.4.** Let $E_\alpha$ be a net of elements in the unit ball of $\mathcal{A}$. Then $E_\alpha$ is a weak identity if for every $A \in \mathcal{A}, f \in \mathcal{A}^*$, both $f(E_\alpha A) \to f(A)$ and $f(AE_\alpha) \to f(A)$.

**LEMMA 4.1.** If $\mathcal{A}$ has a weak identity $E_\alpha$, then there exists an element $I \in \mathcal{A}^{**}$, which is simultaneously (1) a right identity for $m_1$, (2) a left identity for $m_2$. Call such an element $I$ a simultaneous identity.
Proof. (1) is proved in [2, p. 855]. The proof of (2) is similar. A subnet of the \( \{E_a\} \) converges to \( I \) in the weak star topology.

Definition 4.5. Let \( X \) be a normed space. Then, \( f_a \to f \) in the bounded weak star topology \( \langle \Rightarrow \rangle \) the \( \{f_a\} \) constitute a bounded set and \( f_a \to f \) in the weak star topology.

Lemma 4.2. \( \mathcal{A} \) is Arens regular \( \langle \Rightarrow \rangle \) there is a multiplication \( m_\alpha \) on \( \mathcal{A}^{**} \) which extends the multiplication on \( \mathcal{A} \) to \( \mathcal{A}^{**} \) in a way such that (1) \( F \ast G \) is weak star bounded continuous in \( F \) for each fixed \( G \) and (2) \( F \ast G \) is weak star bounded continuous in \( G \) for each fixed \( F \).

Proof. Arens [1, p. 843].

Theorem 4.1. If \( X \) is a Banach space with an unconditionally monotone, shrinking basis \( (e_i) \), then \( B(X) = \mathcal{C}^{**} \langle \Rightarrow \rangle \mathcal{C} \) is Arens regular.

Proof. Assume \( B(X) = \mathcal{C}^{**} \). We claim that ordinary matrix multiplication satisfies (1) and (2) of the above lemma. Let \( S_\alpha, S, \) and \( T \) all be in the unit ball of \( B(X) \) and \( S_\alpha \to S \) weak star. Let \( f_{ij} \) be the matrix in \( \mathcal{C}^* \) with a 1 in the \( ij \)th coordinate and zeros elsewhere. First, we claim that \( (S_\alpha T)f_{ij} \to (ST)f_{ij} \). Clearly, only the \( i \)th rows of \( S_\alpha \) and \( S \) and the \( j \)th column of \( T \) are relevant. By Proposition 2.3, given \( \varepsilon > 0 \), there exists an integer \( n \) such that the tail of the \( j \)th column of \( T \) after the first \( n \) terms has norm \( < \varepsilon/2 \).

Since \( S_\alpha \to S \) weak star, it is clear that \( S_\alpha \) approaches \( S \) coordinate-wise. Let \( \alpha \) be large enough so that each of the first \( n \) entries of the \( i \)th row of \( S \) are within \( \varepsilon/2n \) of the corresponding entry of \( S \). Then \(|(S_\alpha T)f_{ij} - (ST)f_{ij}| \leq \varepsilon \). Hence, \( (S_\alpha T)f_{ij} \to (ST)f_{ij} \). Since \( B(X) = \mathcal{C}^{**} \) implies that the finite matrices are norm dense in \( \mathcal{C}^* \), it follows that for arbitrary \( g \in \mathcal{C}^* \), \( (S_\alpha T)g \to (ST)g \). The argument that (2) is satisfied is similar.

Now assume \( B(X) \neq \mathcal{C}^{**} \). Then the finite matrices do not span a dense manifold of \( \mathcal{C}^* \). Hence, there exists a nonzero \( F \) in \( \mathcal{C}^{**} \) which is 0 on all finite matrices. Let \( E_n \) be the matrix in \( \mathcal{C} \) with ones down the first \( n \) entires of the diagonal and zeros elsewhere. Then, \( (E_n) \) is a weak identity since it is actually an approximate identity by the fact that finite matrices are norm dense in \( \mathcal{C} \).

Let \( I \) be the simultaneous identity in Lemma 4.1., and \( f \in \mathcal{C}^* \). By Theorem 3.2. [1]

\[
(F^*_\varepsilon I)f = \lim [(F^*_\varepsilon E_n)f] = \lim [E_n(f^*_\varepsilon F)] = \lim [(f^*_\varepsilon F)E_n] = \lim [F(E_n^* f)].
\]
However, $E_{*j} f$ is the matrix in $C^*$ which consists of the first $n$ columns of $f$, and thus can be approximated in norm by a finite matrix, since the basis is shrinking. Hence $(F_i^* I) = 0$ whereas $F_i^* I = F$.

**Lemma 4.3.** If there is a continuous homomorphism of the Banach algebra $A$, onto the Banach algebra $A$, and if the multiplication in $A$ is regular, then so is the multiplication in $A$.

**Proof.** Civin and Yood [2], Corollary 6.4.

**Corollary 4.1.** If $X$ is a Banach space with an unconditional basis $(e_j)$, and which is isomorphic to a uniformly convex space, then its space of compact operators is Arens regular.

**Proof.** By Proposition 2.1., $X$ can be renormed isomorphically to $X'$ so that $(e_j)$ is an unconditionally monotone basis for $X'$. Let $i$ be an isomorphic map from $X$ to $X'$. Then the map $A \mapsto i^{-1} A i$, where $A \in C'$, is a continuous homomorphism from $C'$ onto $C$.

**Theorem 4.2.** Let $X$ be a Banach space with an unconditionally monotone, shrinking basis, and for which the matrices in $C^*$ with a finite number of rows are norm dense. Then $B(X) = \{ F \in C^{***} : F_i^* A$ and $A_i^* F$ are both in $C$ for all $A \in C \}$. Furthermore, each of the Arens products coincides with operator multiplication on $B(X)$.

**Proof.** Let $F$ be in $C^{**}$. Let $D_j$ denote the elementary matrix $E_{jj}$. Call $D_j^* F$ the $j^{th}$ row of $F$. Note that $D_j^* F$ is concentrated on the $j^{th}$ row of matrices in $C^*$. In fact,

$$(D_j^* F) f = D_j (F_i^* f) = (F_i^* f) D_j = F (f_i^* D_j) .$$

But the matrix for $f_i^* D_j$ is easily seen to be the matrix formed from $f$ by deleting all but the $j^{th}$ row. By Proposition 2.7., the $j^{th}$ row of $F$ can be identified with a functional in $X^{***}$.

Call $F_i^* D_j$ the $j^{th}$ column of $F$. It is concentrated on the $j^{th}$ column of matrices in $C^*$, because $D_j^* f$ is the matrix formed by deleting all but the $j^{th}$ column of $f$. Then by Proposition 2.7. it can be identified with an element of $X^{**}$.

We claim $F \in B(X) \iff$ each of its rows is in $X^*$ and each of its columns is in $X$. Suppose $F \in C^{**}$ with each of its rows in $X^*$ and columns in $X$. Let $T$ be the actual matrix formed by writing down the columns of $F$ as elements in $X$ with respect to the basis $(e_j)$. Let $T_n$ be the first $n$ columns of $T$. It is a compact operator since each column is in $X$. Also by Proposition 2.6.
\[ \| T_n \| = \| \Phi_{r_n} \| \leq \| F \| \]

where \( \Phi \) is the isometry defined in Theorem 3.1. Hence, the \( \{T_n\} \) define a single bounded operator on the dense linear manifold of finite linear combinations of \( (e_j) \). This bounded operator has the same matrix as \( T \).

Clearly \( \Phi_T \) and \( F \) agree on any elementary matrix in \( C^* \). Hence they agree on any matrix in \( C^* \) concentrated in a single row, since each row of \( F \) is in \( X^* \) and the \( (e_j^*) \) form a basis for \( X^* \). Then by the hypothesis that the matrices in \( C^* \) with a finite number of rows are dense, \( \Phi_T = F \).

Conversely, if \( F \in B(X) \) it is clear that its generalized rows and columns will be in \( X^* \) and \( X \) respectively.

Using this characterization of \( B(X) \) as a subspace of \( C^{**} \), it is clear that if \( F \in B(X) \), then for some \( j \) either \( D_{j1}^*F \) or \( F_{1j}^*D_j \) lies outside \( B(X) \) and hence outside \( C \). But \( D_j \) is a compact operator.

To finish the proof we will show that on \( B(X) \), \( m_1 \) is equal to operator multiplication. The proof for \( m_2 \) is similar.

Clearly it is enough to show that \( (ST)f_j = (S^*_jT)f_j \) for \( f_j \) a matrix in \( C^* \) concentrated in the \( j^{th} \) row and where \( \| S \| = \| T \| = \| f_j \| = 1 \).

Given \( \varepsilon > 0 \), we can approximate the \( j^{th} \) row of \( S \) in norm to within \( \varepsilon \) by deleting after the first \( n \) terms for \( n \) large enough.

Then

\[
(S^*T)f_j = (S^*_jT_{11} + S^*_jT_{j2} + \cdots + S^*_jT_{nj})f_j
\]
\[
\vdots
\]
\[
+ (S^*_jT_{1k} + S^*_jT_{j2} + \cdots + S^*_jT_{nk})f_j
\]
\[
\vdots
\]
\[
+ (\text{error term} < \varepsilon).
\]

We claim that \( (T^*_jf_j) \) is concentrated in the \( j^{th} \) row. In fact,

\[
(T^*_jf_j)E_{mk} = T(f^*_jE_{mk}) = 0 \text{ if } m \neq j,
\]

whereas \( (T^*_jf_j)E_{jk} = \text{dot product of } k^{th} \text{ row of } T \text{ with } j^{th} \text{ row of } f_j \).

Then,

\[
S(T^*_jf_j) = (T_{11}f_{j1} + T_{12}f_{j2} + \cdots + )S_{j1}
\]
\[
\vdots
\]
\[
+ (T_{n1}f_{j1} + T_{n2}f_{j2} + \cdots + )S_{jn}
\]
\[
+ (\text{error term} < \varepsilon).
\]

Hence \( |(ST)f_j - (S^*_jT)f_j| < 2\varepsilon \), since for a finite collection of convergent series
\[ \sum_{k=1}^{\infty} (a_k^1 + \cdots + a_k^n) = \sum_{k=1}^{\infty} a_k^1 + \cdots + \sum_{k=1}^{\infty} a_k^n. \]

**Definition 4.6.** A shrinking basis \((e_j)\) for a Banach space is called boundedly growing if there exists an \(\varepsilon > 0\) and an integer \(n\), such that \(x_1 + \cdots + x_n < n - \varepsilon\) whenever the \(x_j\)'s have norm 1 and are linear combinations of distinct basic vectors. For example the canonical bases for \(c_0\) or \(l^p, p > 1\) are boundedly growing. Finite direct sums of boundedly growing Banach spaces are boundedly growing. Also \(l^p(X_i)\) for \(p > 1\) is boundedly growing if the \(X_i\) have a common \(n\) and \(\varepsilon\).

**Corollary 4.2.** If a Banach space \(X\) has an unconditionally monotone, boundedly growing basis then \(B(X)\) is the largest subset in \(c_{\mathcal{E}}\) in which \(c_{\mathcal{E}}\) is a two sided ideal.

**Proof.** In proving Proposition 3.3, we showed that if the finite matrices are not dense in \(c_{\mathcal{E}}\) then the basis is not boundedly growing. Similarly, if the matrices with a finite number of rows are not dense in \(c_{\mathcal{E}}\), then the basis is not boundedly growing.

**Bibliography**


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