

# Pacific Journal of Mathematics

**MATRICES WITH PRESCRIBED CHARACTERISTIC  
POLYNOMIAL AND A PRESCRIBED SUBMATRIX.**

GRACIANO DE OLIVEIRA

## MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND A PRESCRIBED SUBMATRIX

GRACIANO N. DE OLIVEIRA

**Let  $A$  be an arbitrary (complex)  $n \times n$  matrix and let  $f(\lambda)$  be a polynomial (with complex coefficients) of degree  $n+1$  with leading coefficient  $(-1)^{n+1}$ . In this paper we solve the problem: under what conditions does there exist an  $(n+1) \times (n+1)$  (complex) matrix  $B$  of which  $A$  is the submatrix standing in the top left-hand corner and such that  $f(\lambda)$  is its characteristic polynomial?**

In [1] Farahat and Ledermann proved that if  $A$  is a nonderogatory matrix over a field  $\Phi$  and  $f(\lambda)$  is a monic polynomial over  $\Phi$ , then there exists an  $(n+1) \times (n+1)$  matrix  $B$  over  $\Phi$  with  $A$  standing in its top left-hand corner and such that  $f(\lambda) = \det(\lambda E_{n+1} - B)$ . Now, our main results are:

**THEOREM 1.** *Let  $A$  be an  $n \times n$  complex matrix whose distinct characteristic roots are  $w_\alpha$  ( $\alpha = 1, \dots, t$ ). Let us suppose that in the Jordan normal form of  $A$ ,  $w_\alpha$  appears in  $r_\alpha$  distinct diagonal blocks of orders  $v_1^{(\alpha)}, \dots, v_{r_\alpha}^{(\alpha)}$  respectively. We assume that*

$$v_1^{(\alpha)} \leq \dots \leq v_{r_\alpha}^{(\alpha)}.$$

*Let  $\theta_\alpha = \sum_{j=1}^{r_\alpha} v_j^{(\alpha)}$ . There exists an  $(n+1) \times (n+1)$  complex matrix  $B$  having  $A$  in the top left-hand corner and with  $f(\lambda)$  as characteristic polynomial (i.e.,  $f(\lambda) = \det(B - \lambda E_{n+1})$ ) if and only if  $f(\lambda)$  is divisible by  $\prod_{\alpha=1}^t (w_\alpha - \lambda)^{\theta_\alpha}$ .*

**THEOREM 2.** *Let  $A$  be a real  $n \times n$  symmetric matrix whose distinct characteristic roots are  $w_\alpha$  ( $\alpha = 1, \dots, t$ ). Let  $r_\alpha$  be the multiplicity of  $w_\alpha$ . There exists a real  $(n+1) \times (n+1)$  symmetric matrix  $B$  having  $A$  in the top left-hand corner and with  $f(\lambda)$  (now with real coefficients) as characteristic polynomial if and only if*

$$(a) \quad f(\lambda) \text{ is divisible by } \prod_{\alpha=1}^t (w_\alpha - \lambda)^{r_\alpha-1}$$

and

$$(b) \quad \left[ \frac{f(\lambda)}{(w_\beta - \lambda)^{r_\beta-1}} \right]_{\lambda=w_\beta} \cdot \prod_{\substack{\alpha=1 \\ \alpha \neq \beta}}^t (w_\alpha - w_\beta)^{r_\alpha} \quad (\beta = 1, \dots, t)$$

*is real and nonpositive.*

**REMARK.** There is no difficulty in seeing that the conditions (a)

and (b) imposed on  $f(\lambda)$  are equivalent to the following:  $f(\lambda)$  has only real roots which are interlaced by the  $n$  characteristic roots of  $A$ .

2. We start with the following

LEMMA. Let  $A$  be any  $n \times n$  complex matrix with normal Jordan form  $J$ . In order that the matrix  $B$  referred in Theorem 1 exists, it is necessary and sufficient that there should exist a column  $X_1$  (with  $n$  elements), a row  $Y_1$  (with  $n$  elements) and a number  $q_1$  such that

$$\begin{bmatrix} J & X_1 \\ Y_1 & q_1 \end{bmatrix}$$

has  $f(\lambda)$  as characteristic polynomial.

*Proof.* Let  $T$  be an  $n \times n$  nonsingular matrix such that  $TAT^{-1} = J$ . Suppose  $B$  exists and is given by

$$B = \begin{bmatrix} A & X \\ Y & q \end{bmatrix}.$$

Let

$$S = \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix}.$$

We have

$$SBS^{-1} = \begin{bmatrix} J & TX \\ YT^{-1} & q \end{bmatrix}.$$

and so we can take  $Y_1 = YT^{-1}$ ,  $X_1 = TX$  and  $q_1 = q$ .

The converse is easily proved in a similar way.

Our next step is to deduce the characteristic polynomial of the matrix:

$$(2.1) \quad C_i = \begin{bmatrix} J_i & 0 & \cdots & 0 & X_i \\ 0 & J_{i+1} & \cdots & 0 & X_{i+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & \cdots & J_m & X_m \\ Y_i & Y_{i+1} & \cdots & Y_m & q \end{bmatrix}$$

where, with obvious notation,



$$P_\rho = x_\rho^i (\lambda_i - \lambda)^{s_i - \rho} - P_{\rho+1}$$

and by induction it can be easily seen that

$$P_\rho = \sum_{\tau=0}^{s_i - \rho} (-1)^\tau x_{\rho+\tau}^i (\lambda_i - \lambda)^{s_i - \rho - \tau};$$

so we can write

$$H_\rho = \sum_{\tau=0}^{s_i - \rho} (-1)^{\tau + s_i - \rho} x_{\rho+\tau}^i (\lambda_i - \lambda)^{s_i - \tau - 1}.$$

Let us now calculate the complementary minor  $\tilde{H}_\rho$  of  $H_\rho$  in  $C_i - \lambda E_i$ . There is no difficulty in seeing that

$$\tilde{H}_\rho = \begin{array}{c} \text{1 column} \\ \left( \begin{array}{cccccc} \overbrace{0}^{J_{i+1} - \lambda E^{(i+1)}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & J_{i+2} - \lambda E^{(i+2)} & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & J_m - \lambda E^{(m)} \\ \text{1 row} \{ y_\rho^i & Y_{i+1} & Y_{i+2} & \cdots & Y_{m-1} & Y_m \end{array} \right) \end{array}$$

We have

$$\tilde{H}_\rho = (-1)^\sigma y_\rho^i \prod_{j=i+1}^m (\lambda_j - \lambda)^{s_j},$$

with

$$\sigma = \sum_{k=i+1}^m s_k.$$

Bearing in mind that  $H_\rho$  was formed from the rows  $1, \dots, s_i$  and columns  $1, \dots, \rho - 1, \rho + 1, \dots, s_i, \sum_{k=i}^m s_k + 1$ , we have

$$\det(C_i - \lambda E_i) = \sum_{\rho=1}^{s_i} \sum_{\tau=0}^{s_i - \rho} (-1)^{\tau+1} y_\rho^i x_{\rho+\tau}^i (\lambda_i - \lambda)^{s_i - \tau - 1} \prod_{j=i+1}^m (\lambda_j - \lambda)^{s_j} + \det(J_i - \lambda E^{(i)}) \det[\text{comp}(J_i - \lambda E^{(i)})],$$

where the symbol  $\text{comp}(J_i - \lambda E^{(i)})$  means the complementary minor of  $J_i - \lambda E^{(i)}$  in the matrix  $C_i - \lambda E_i$ . Interchanging the order of the first two sums, noting that  $\det(J_i - \lambda E^{(i)}) = (\lambda_i - \lambda)^{s_i}$  and that  $\text{comp}(J_i - \lambda E^{(i)}) = \det(C_{i+1} - \lambda E_{i+1})$  we get

$$\det(C_i - \lambda E_i) = \sum_{\tau=0}^{s_i-1} \sum_{\rho=1}^{s_i-\tau} (-1)^{\tau+1} y_\rho^i x_{\rho+\tau}^i (\lambda_i - \lambda)^{s_i - \tau - 1} \prod_{j=i+1}^m (\lambda_j - \lambda)^{s_j} + (\lambda_i - \lambda)^{s_i} \det(C_{i+1} - \lambda E_{i+1}).$$

Putting here successively  $i = 1, 2, \dots, m$  and writing for the sake of simplicity

$$(2.3) \quad b_{k\mu} = \sum_{\rho=1}^{\mu+1} (-1)^{s_k-\mu} y_{\rho}^k w_{\rho+s_k-1-\mu}^k \quad (\mu = 0, \dots, s_k - 1),$$

we get after some manipulation

$$(2.4) \quad \det(C_1 - \lambda E_1) = \sum_{k=1}^m \left\{ \left[ \sum_{\mu=0}^{s_k-1} b_{k\mu} (\lambda_k - \lambda)^\mu \right] \left[ \prod_{\substack{j=1 \\ j \neq k}}^m (\lambda_j - \lambda)^{s_j} \right] \right\} + (q - \lambda) \prod_{j=1}^m (\lambda_j - \lambda)^{s_j}.$$

We are now ready for the proof of Theorem 1. Because of the lemma it is sufficient to prove the theorem assuming that  $A$  is in the Jordan normal form  $J = \text{diag}(J_1, \dots, J_m)$  with  $J_j$  ( $j = 1, \dots, m$ ) given by (2.2). So what we have to do is to find out under what conditions it is possible to find columns  $X_1, \dots, X_m$ , rows  $Y_1, \dots, Y_m$  and a number  $q$  such that the characteristic polynomial (2.4) of the matrix  $C_1$  be  $f(\lambda)$ .

As in the Jordan normal form the order in which the diagonal blocks occur is arbitrary, we can suppose without loss of generality that

$$\begin{aligned} \lambda_1 &= \dots = \lambda_{u_1} \quad (= w_1) \\ &\dots \dots \dots \dots \dots \dots \\ \lambda_{u_{\alpha-1}+1} &= \dots = \lambda_{u_\alpha} \quad (= w_\alpha) \\ &\dots \dots \dots \dots \dots \dots \\ \lambda_{u_{t-1}+1} &= \dots = \lambda_{u_t} \quad (= w_t) \end{aligned}$$

$(u_\alpha = \sum_{\beta=1}^{\alpha} r_\beta; r_\beta \text{ defined in Theorem 1})$

with  $w_\alpha \neq w_\beta$  if  $\alpha \neq \beta$ . With this notation, in  $J$  the characteristic root  $w_\alpha$  appears in the diagonal blocks  $J_{u_{\alpha-1}+1} \dots, J_{u_\alpha}$  which are of orders  $s_{u_{\alpha-1}+1} \dots, s_{u_\alpha}$  respectively. We will assume that

$$s_{u_{\alpha-1}+1} \leq \dots \leq s_{u_\alpha}$$

for every  $\alpha$ .

Let

$$\theta_\alpha = \sum_{\mu=u_{\alpha-1}+1}^{u_\alpha-1} s_\mu.$$

From (2.4) we have

$$\det(C_1 - \lambda E_1) = (w_\alpha - \lambda)^{\theta_\alpha} \varphi_\alpha(\lambda)$$

where  $\varphi_\alpha(\lambda)$  is a polynomial in  $\lambda$  which is not necessarily divisible by  $w_\alpha - \lambda$ . As  $\alpha \neq \beta$  implies  $w_\alpha \neq w_\beta$  we will have

$$(2.5) \quad \det(C_1 - \lambda E_1) = \prod_{\alpha=1}^t (w_\alpha - \lambda)^{\theta_\alpha \psi(\lambda)}$$

where  $\psi(\lambda)$  is a polynomial in  $\lambda$  not necessarily divisible by any factor of  $h(\lambda) = \prod_{\alpha=1}^t (w_\alpha - \lambda)^{\theta_\alpha}$ . Therefore, if  $f(\lambda)$  is not divisible by  $h(\lambda)$  it is impossible to find  $X_i, Y_i (i = 1, \dots, m)$  and  $q$  such that  $f(\lambda) = \det(C_1 - \lambda E_1)$ . Let us now suppose that  $f(\lambda) = h(\lambda)f_1(\lambda)$ . All we have to prove is that it is possible to find  $X_i, Y_i (i = 1, \dots, m)$  and  $q$  such that  $\psi(\lambda) = f_1(\lambda)$ .

Setting

$$(2.6) \quad S_k(\lambda) = \sum_{\mu=0}^{s_k-1} b_{k\mu} (\lambda_k - \lambda)^\mu$$

and

$$\xi_\alpha = \sum_{\mu=u_{\alpha-1}+1}^{u_\alpha} s_\mu,$$

(2.4) gives

$$(2.7) \quad \det(C_1 - \lambda E_1) = \sum_{\beta=0}^{t-1} \sum_{k=u_{\beta+1}}^{u_{\beta+1}} S_k(\lambda) \frac{\prod_{\alpha=1}^t (w_\alpha - \lambda)^{\xi_\alpha}}{(w_{\beta+1} - \lambda)^{s_\beta}} + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda)^{\xi_\alpha} \quad (u_0 = 0).$$

Let us choose  $b_{k\mu} = 0$  for every  $k \neq u_{\beta+1} (\beta = 0, \dots, t - 1; \mu = 0, \dots, s_k - 1)$ . With this choice (2.7) gives

$$\det(C_1 - \lambda E_1) = \prod_{\gamma=1}^t (w_\gamma - \lambda)^{\theta_\gamma} \left[ \sum_{\beta=0}^{t-1} S_{u_{\beta+1}}(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - \lambda)^{s_{u_\alpha}} + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda)^{\xi_\alpha - \theta_\alpha} \right]$$

and so by (2.5)

$$\psi(\lambda) = \sum_{\beta=0}^{t-1} S_{u_{\beta+1}}(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - \lambda)^{s_{u_\alpha}} + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda)^{s_{u_\alpha}}.$$

By (2.6)  $S_{u_{\beta+1}}(\lambda)$  is a polynomial in  $(w_{\beta+1} - \lambda)$  of degree  $s_{u_{\beta+1}} - 1$ . For the sake of simplicity we now change the notation (in an obvious way) writing

$$\psi(\lambda) = \sum_{\beta=0}^{t-1} R_\beta(\lambda) \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - \lambda)^{t_\alpha} + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda)^{t_\alpha}.$$

Let

$$R_\beta(\lambda) = \sum_{\mu=0}^{t_{\beta+1}-1} \delta_{\beta\mu} (w_{\beta+1} - \lambda)^\mu.$$

We can write

$$(2.8) \quad \frac{\psi(\lambda)}{\prod_{\alpha=1}^t (w_\alpha - \lambda)^{t_\alpha}} = \sum_{\beta=0}^{t-1} \sum_{\mu=0}^{t_{\beta+1}-1} \frac{\delta_{\beta\mu}}{(w_{\beta+1} - \lambda)^{t_{\beta+1}-\mu}} + q - \lambda .$$

Let us resolve  $f_1(\lambda)/\prod_{\alpha=1}^t (w_\alpha - \lambda)^{t_\alpha}$  into partial fractions. We will get

$$\frac{f_1(\lambda)}{\prod_{\alpha=1}^t (w_\alpha - \lambda)^{t_\alpha}} = \sum_{\beta=0}^{t-1} \sum_{\mu=0}^{t_{\beta+1}-1} \frac{A_{\beta\mu}}{(w_{\beta+1} - \lambda)^{t_{\beta+1}-\mu}} + Q - \lambda .$$

If now in (2.8) we take  $\delta_{\beta\mu} = A_{\beta\mu}$  and  $q = Q$  we will have  $\psi(\lambda) = f_1(\lambda)$  as required. So we have given a process to choose all the  $b_{k\mu}$  appearing in (2.6). To conclude the proof we show that it is always possible to find values  $x_\sigma^i, y_\sigma^i$  satisfying (2.3), no matter what values we have given to the  $b_{k\mu}$ . In fact, let us give to the  $x_\sigma^i$  arbitrary nonzero values ( $x_\sigma^i = 1$ , for example). Then, for each  $k$ , (2.3) becomes a system of linear equations in the  $y_\sigma^k$  with a triangular matrix whose principal elements are different from zero. This means that the system is compatible. The proof of Theorem 1 is now complete.

**COROLLARY.** *If  $A$  is a complex nonderogatory matrix, then the matrix  $B$  of Theorem 1 always exists.*

*Proof.* If  $A$  is nonderogatory in its Jordan normal form there are no two diagonal blocks corresponding to the same characteristic root. So in Theorem 1 we have  $r_\alpha = 1$  and so  $\theta_\alpha = 0$ . This means that  $B$  exists.

*Proof of Theorem 2.* If  $A$  is real and symmetric, the matrix  $T$  such that  $TAT^{-1} = J$  can be chosen orthogonal and  $J$  will be a diagonal matrix. So using Theorem 1 we have  $v_1^{(\alpha)} = \dots = v_{r_\alpha}^{(\alpha)} = 1$  and  $\theta_\alpha = r_\alpha - 1$ . It follows that (a) is necessary and sufficient for the existence of a matrix  $B$  (not necessarily real and symmetric) of type  $(n + 1) \times (n + 1)$  having  $A$  in the top left-hand corner and with  $f(\lambda)$  as characteristic polynomial. Let us now find out the conditions for  $B$  to be real and symmetric. Choosing  $T$  orthogonal for  $B$  to fulfill this condition it is necessary and sufficient that there exist real  $X_j, Y_j, q$  ( $j = 1, \dots, m$ ) with  $X_j = Y_j$ . Let us write  $x_\rho^i = y_\rho^i$ . We have now  $\xi_\alpha = r_\alpha, \theta_\alpha = \xi_\alpha - 1$  and  $S_k(\lambda) = b_{k0}$ . Let

$$(2.9) \quad c_{\beta 0} = \sum_{k=u_{\beta+1}}^{u_{\beta+1}} b_{k0} .$$

The formula (2.7) gives



$$\det(C_1 - \lambda E_1) = \prod_{\gamma=1}^t (w_\gamma - \lambda)^{r_\gamma - 1} \left[ \sum_{\beta=0}^{t-1} c_{\beta 0} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - \lambda) + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda) \right]$$

and so

$$(2.10) \quad \psi(\lambda) = \sum_{\beta=0}^{t-1} c_{\beta 0} \prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - \lambda) + (q - \lambda) \prod_{\alpha=1}^t (w_\alpha - \lambda).$$

We are assuming that  $f(\lambda)$  is divisible by

$$h(\lambda) = \prod_{\alpha=1}^t (w_\alpha - \lambda)^{r_{\alpha-1}}.$$

Let  $f(\lambda)/h(\lambda) = f_1(\lambda)$ . Resolving  $f_1(\lambda)/\prod_{\alpha=1}^t (w_\alpha - \lambda)$  into partial fractions we get

$$\frac{f_1(\lambda)}{\prod_{\alpha=1}^t (w_\alpha - \lambda)} = \sum_{\beta=0}^{t-1} \frac{B_\beta}{w_{\beta+1} - \lambda} + Q_1 - \lambda$$

with

$$B_\beta = \frac{f_1(w_{\beta+1})}{\prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - w_{\beta+1})}.$$

From (2.10) we have

$$\frac{\psi(\lambda)}{\prod_{\alpha=1}^t (w_\alpha - \lambda)} = \sum_{\beta=0}^{t-1} \frac{c_{\beta 0}}{w_{\beta+1} - \lambda} + q - \lambda.$$

So we must take

$$c_{\beta 0} = \frac{f_1(w_{\beta+1})}{\prod_{\substack{\alpha=1 \\ \alpha \neq \beta+1}}^t (w_\alpha - w_{\beta+1})}, \quad q = Q_1.$$

The equations (2.3) now take the form

$$b_{k0} = -[x_1^k]^2$$

or, by (2.9)

$$c_{\beta 0} = - \sum_{k=u_{\beta+1}}^{u_{\beta+1}} [x_1^k]^2.$$

So  $B$  can be real and symmetric if and only if  $c_{\beta 0} \leq 0$  and  $Q_1$  is real. The condition  $c_{\beta 0} \leq 0$  is equivalent to (b). Bearing in mind that  $\sum_{\alpha=1}^t w_\alpha$  is real we can see easily that  $Q_1$  is always real. With this the proof is complete.

In a similar way we could prove a theorem analogous to Theorem 2 but with 'real symmetric' substituted by 'hermitian'.

*Note.* After I had written this paper I noticed that Theorem 2 is not new. It is essentially equivalent to Theorem 1 in Fan and Pall, *Imbedding Conditions for Hermitian and Normal Matrices*, *Canad. J. Math.* 9 (1957), 298-304. However, the proof I have given here is a bit different from the proof of Fan and Pall. For further details see my forthcoming paper *Matrices with prescribed characteristic polynomial and a prescribed submatrix-II* (submitted to *Pacific J. Math.*).

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