

# Pacific Journal of Mathematics

**MATRICES WITH PRESCRIBED CHARACTERISTIC  
POLYNOMIAL AND A PRESCRIBED SUBMATRIX. II**

GRACIANO DE OLIVEIRA

## MATRICES WITH PRESCRIBED CHARACTERISTIC POLYNOMIAL AND A PRESCRIBED SUBMATRIX-II

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Let  $A = [a_{ij}]$  be an  $n \times n$  complex matrix and  $f(\lambda)$  be a polynomial with complex coefficients of degree  $n + k$  and leading coefficient  $(-1)^{n+k}$ . In the present paper we solve the following problem: under what conditions does there exist an  $(n + k) \times (n + k)$  complex matrix  $B$  of which  $A$  is the submatrix standing in the top left-hand corner and such that  $f(\lambda)$  is its characteristic polynomial?

In [3] we solved this problem for  $k = 1$ . It can be seen that from our Theorem 2 in [3] the solution of the general case ( $k > 1$ ) comes out very easily when  $A$  is real symmetric (hermitian) and  $B$  is required to be of the same kind. This last problem had actually already been solved by Ky Fan and G. Pall (see [1]). Now we will prove the following

**THEOREM.** Let  $A$  be an  $n \times n$  complex matrix whose distinct characteristic roots are  $w_i$  ( $i = 1, \dots, t$ ). Let us suppose that in the Jordan normal form of  $A$ ,  $w_i$  appears in  $r_i$  distinct diagonal blocks of orders  $v_1^{(i)}, \dots, v_{r_i}^{(i)}$  respectively. Let us assume that  $v_1^{(i)} \leq \dots \leq v_{r_i}^{(i)}$ . Let  $\theta_i = \sum_{j=1}^{r_i-k} v_j^{(i)}$ , with  $\theta_i = 0$  if  $r_i - k < 1$ . There exists an  $(n + k) \times (n + k)$  complex matrix  $B$  having  $A$  in the top left-hand corner and with  $f(\lambda)$  as characteristic polynomial if and only if  $f(\lambda)$  is divisible by  $\prod_{i=1}^t (w_i - \lambda)^{\theta_i}$ .

First we prove that the condition is necessary. Let  $T$  be a nonsingular matrix that transforms  $A$  into its Jordan normal form  $J$ :  $TAT^{-1} = J$ , with  $J = \text{diag}(J_1, \dots, J_m)$ . The block  $J_i$  will be of the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \ddots & \vdots \\ & & \lambda_i \\ 0 & & & \lambda_i \end{bmatrix}$$

and we will suppose that  $J_i$  is of type  $s_i \times s_i$ . Let

$$B = \begin{bmatrix} A & X_1 \\ Y_1 & S_1 \end{bmatrix}$$

where  $X_1, Y_1, S_1$  are blocks of type  $n \times k, k \times n, k \times k$  respectively. Let us assume that  $f(\lambda) = \det(B - \lambda E_{n+k})$  where  $E_{n+k}$  denotes the identity matrix of order  $n + k$ . If

$$T_i = \begin{bmatrix} T & 0 \\ 0 & E_k \end{bmatrix},$$

we will have

$$B_1 = T_1 B T_1^{-1} = \begin{bmatrix} J & X \\ Y & S \end{bmatrix}$$

with  $X = T X_1, Y = Y_1 T^{-1}$  and  $S = S_1$ . As  $i \neq j$  implies  $w_i \neq w_j$  all we need to prove is that  $\det(B_1 - \lambda E_{n+k})$  is divisible by  $(w_i - \lambda)^{\theta_i}$  ( $i = 1, \dots, t$ ). We will do it for  $(w_1 - \lambda)^{\theta_1}$  as the proof is the same for the other cases. We can assume that  $w_1$  appears in the first  $u$  diagonal blocks of  $J$  and that  $s_1 \leq s_2 \leq \dots \leq s_u$ . Let us expand  $\det(B_1 - \lambda E_{n+k})$  by Laplace rule in terms of its first  $\sum_{i=1}^u s_i$  rows. The necessity of the condition of the theorem will be proved if we show that all the nonzero minors contained in the first  $\sum_{i=1}^u s_i$  rows have determinants which are divisible by  $(w_1 - \lambda)^{\theta_1}$ . These minors are  $\text{diag}(J_1 - \lambda E^{(i)}, \dots, J_u - \lambda E^{(u)})(E^{(i)}$  denotes the identity matrix of the same order as  $J_i$ ) and all the minors obtained from this one by replacing no more than  $k$  of its columns by the same number of columns taken from the matrix which remains after suppressing the last  $\sum_{i=u+1}^m s_i$  rows of  $X$ . As  $J_i$  ( $i = 1, \dots, u$ ) are diagonal matrices with  $w_1$  in the principal diagonal our assertion follows.

Let us now prove that the condition is sufficient. For this we need an auxiliary proposition.

**LEMMA.** *Let  $A$  be an  $n \times n$  complex matrix whose distinct characteristic roots are  $w_1, \dots, w_t$ . Let us assume that in the Jordan normal form of  $A$ ,  $w_i$  ( $i = 1, \dots, t$ ) appears in  $r_i$  diagonal blocks of orders  $v_1^{(i)} \leq v_2^{(i)} \leq \dots \leq v_{r_i}^{(i)}$ . Then it is possible to construct a matrix  $A_1$  of type  $(n + 1) \times (n + 1)$  containing  $A$  in its top left-hand corner and such that: ( $\alpha$ ) The characteristic polynomial of  $A_1$  is  $\prod_{i=1}^t (w_i - \lambda)^{\sigma_i} \varphi(\lambda)$ , where  $\sigma_i = \sum_{j=1}^{r_i-1} v_j^{(i)}$  and  $\varphi(\lambda)$  is any polynomial in  $\lambda$  of degree  $\rho = n + 1 - \sum_{i=1}^t \sigma_i$ , leading coefficient  $(-1)^\rho$  and such that  $\varphi(w_i) \neq 0$  ( $i = 1, \dots, t$ ). ( $\beta$ ) In the Jordan normal form of  $A_1$  the characteristic root  $w_i$  appears in exactly  $r_i - 1$  diagonal blocks of orders*

$$v_1^{(i)}, \dots, v_{r_i-1}^{(i)} \quad (i = 1, \dots, t).$$

*Proof.* We can suppose, without loss of generality, that  $A$  is in its Jordan normal form.

The matrix  $A_1$ , if it exists, will have the form

$$A_1 = \begin{bmatrix} J_1 & 0 & \cdots & 0 & X_1 \\ 0 & J_2 & \cdots & 0 & X_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_m & X_m \\ Y_1 & Y_2 & \cdots & Y_m & q \end{bmatrix}$$

with  $X_i = [x_1^i \cdots x_{s_i}^i]^T$  and  $Y_i = [y_1^i \cdots y_{s_i}^i]$ . The  $x_j^i$  and  $y_j^i$  must satisfy

$$\sum_{j=1}^{h+1} (-1)^{s_i-h} y_j^i x_{j+s_i-1-h}^i = b_{ih} \quad (h = 0, \dots, s_i - 1)$$

where the  $b_{ih}$  are calculated by a process we give in [3]. Moreover, we recall that for each  $i$  we can give to the  $x_j^i$  ( $j = 1, \dots, s_i$ ) arbitrary nonzero values. Let us suppose that we have fixed all the matrices  $X_1, \dots, X_m$  with  $x_j^i \neq 0$  ( $i = 1, \dots, m; j = 1, \dots, s_i$ ). We can assume that  $w_i$  appears in the diagonal blocks  $J_{u_{i-1}+1}, \dots, J_{u_i-1}, J_{u_i}$  ( $i = 1, \dots, t; u_0 = 0, u_t = m$ ) of orders  $s_{u_{i-1}+1} \leq \dots \leq s_{u_i-1} \leq s_{u_i}$  respectively. Let us now choose  $Y_{u_{i-1}+1} = 0, \dots, Y_{u_i-1} = 0$  ( $i = 1, \dots, t$ ). Let

$$A_2 = \begin{bmatrix} J_{u_1} & 0 & \cdots & 0 & X_{u_1} \\ 0 & J_{u_2} & \cdots & 0 & X_{u_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & J_{u_t} & X_{u_t} \\ Y_{u_1} & Y_{u_2} & \cdots & Y_{u_t} & q \end{bmatrix}.$$

We have

$$\det(A_1 - \lambda E_1) = \prod_{i=1}^t (w_i - \lambda)^{\sigma_i} \det(A_2 - \lambda E_2)$$

where  $\sigma_i = \sum_{j=u_{i-1}+1}^{u_i-1} s_j$  and  $E_j$  is the identity matrix of the same order as  $A_j$  ( $j = 1, 2$ ). The matrix  $\text{diag}(J_{u_1}, J_{u_2}, \dots, J_{u_t})$  is obviously a nonderogatory matrix and so according to the corollary to Theorem 1 in [3] we can choose  $Y_{u_1}, \dots, Y_{u_t}$  and  $q$  such that

$$\det(A_2 - \lambda E_2) = \varphi(\lambda).$$

With this choice  $A_1$  has the required characteristic polynomial. Let us find the diagonal blocks of the Jordan normal form of  $A_1$  corresponding to  $w_i$  ( $i = 1, \dots, t$ ). This amounts to finding all the elementary divisors of  $A$  of the form  $(\lambda - w_i)^r$  ( $i = 1, \dots, t$ ). Let us consider, for example, the case  $i = 1$  as the other cases can be treated in the same fashion.  $A_1$  can be written in the form

$$A_1 = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

where  $A_{11} = \text{diag}(J_1, \dots, J_{u-1})$  and the matrix  $A_{22}$  has not the characteristic root  $w_1$ . Therefore (see [2], p. 85) the elementary divisors of  $A_1$  of the form  $(\lambda - w_1)^r$  are exactly

$$(\lambda - w_1)^{s_1}, (\lambda - w_1)^{s_2}, \dots, (\lambda - w_1)^{s_{u-1}}$$

and the proof of the lemma is concluded.

Let us now complete the proof of the theorem.

Let

$$\theta_{ih} = \sum_{j=1}^{r_i-h} v_j^{(s)} \quad (h = 1, \dots, k-1; \theta_{ih} = 0 \text{ if } r_i - h < 1).$$

Let

$$f_j(\lambda) = \prod_{i=1}^t (w_i - \lambda)^{\theta_{ij}} \varphi_j(\lambda) \quad (j = 1, \dots, k-1),$$

where the  $\varphi_j(\lambda)$  are polynomials in  $\lambda$  chosen arbitrarily but with the following properties :

( $\alpha$ ) The leading coefficient and the degree of  $\varphi_j(\lambda)$  ( $j = 1, \dots, k-1$ ) are such that  $f_j(\lambda)$  has degree  $n + j$  and leading coefficient  $(-1)^{n+j}$

( $\beta$ ) For  $j = 1, \dots, k-1$  the roots of  $\varphi_j(\lambda)$  are distinct,  $\varphi_j(w_i) \neq 0$  ( $i = 1, \dots, t$ ) and if  $\varphi_j(\xi) = 0$  then  $\varphi_{j+1}(\xi) \neq 0$ .

Obviously there are infinitely many possibilities of choice for the  $\varphi_j(\lambda)$  ( $j = 1, \dots, k-1$ ).

Because of the lemma we can border  $A$  with a row (below) and a column (on the right hand side) to obtain a matrix  $A_1$  with characteristic polynomial  $f_1(\lambda)$  and such that in its Jordan normal form  $w_i$  ( $i = 1, \dots, t$ ) appears in exactly  $r_i - 1$  diagonal blocks whose orders are  $v_1^{(i)}, \dots, v_{r_i-1}^{(i)}$ . Now we can border  $A_1$  with another row (below) and a column (on the right hand side) in such a way that we obtain a matrix  $A_2$  with  $f_2(\lambda)$  as characteristic polynomial and such that in the Jordan normal form of  $A_2$  the characteristic root  $w_i$  ( $i = 1, \dots, t$ ) appears in exactly  $r_i - 2$  diagonal blocks of orders  $v_1^{(i)}, \dots, v_{r_i-2}^{(i)}$ . We can continue in this fashion up to the matrix  $A_{k-1}$ . Using now Theorem 1 of [3] with  $A_{k-1}$ , the proof is complete.

In an  $(n + k) \times (n + k)$  matrix any principal minor of type  $n \times n$  can be brought to the top left-hand corner by a permutation of rows and the same permutation of columns. This remark combined with the Theorem above solves the following problem: under what conditions does there exist an  $(n + k) \times (n + k)$  complex matrix  $B$  of which  $A$  is the principal minor contained in the rows of orders  $i_1, \dots, i_n$

( $1 \leq i_1 < \dots < i_n \leq n + k$ ) and such that  $f(\lambda)$  is its characteristic polynomial?

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