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DONALD STEVEN PASSMAN

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EXCEPTIONAL 3/2-TRANSITIVE PERMUTATION GROUPS

D. S. PASSMAN

Solvable 3/2-transitive permutation groups were previously classified to within a finite number of exceptions. In this paper the exceptional groups are determined. They have degrees 3^2, 5^2, 7^2, 11^2, 17^2 and 3^4. In addition, these groups are shown to have no transitive extensions.

There are three families of groups which play a special role here. Let q be a prime. We let $\mathscr{S}(q^n)$ denote the set of all semilinear transformations on the finite field $GF(q^n)$. Thus $\mathscr{S}(q^n)$ consists of all transformations

$$x \longrightarrow ax^\sigma + b$$

with $a, b \in GF(q^n)$, $a \neq 0$ and σ a field automorphism. Clearly this is a solvable group, doubly transitive on $GF(q^n)$.

We let $\mathscr{S}_0(q^n)$ be the group acting on a 2-dimensional space over $GF(q^n)$ which contains the transformations

$$(x, y) \longrightarrow (x, y)\begin{pmatrix} a & 0 \\ 0 & \pm a^{-1} \end{pmatrix} + (b, c)$$

and

$$(x, y) \longrightarrow (x, y)\begin{pmatrix} 0 & a \\ \pm a^{-1} & 0 \end{pmatrix} + (b, c)$$

with $a, b, c \in GF(q^n)$ and $a \neq 0$. We see easily that $\mathscr{S}_0(q^n)$ is solvable and if $q \neq 2$ then it acts 3/2-transitively on the 2-dimensional space.

Finally we let $\Gamma(q^n)$ denote the set of all functions of the form

$$x \longrightarrow \frac{ax^\sigma + b}{cx^\sigma + d}$$

with $a, b, c, d \in GF(q^n)$, $ad - bc \neq 0$ and σ a field automorphism. These functions permute the set $GF(q^n) \cup \{\infty\}$ and $\Gamma(q^n)$ is triply transitive. Clearly $\Gamma(q^n)_\infty = \mathscr{S}(q^n)$ is solvable. Let $\bar{\Gamma}(q^n)$ denote the subgroup of $\Gamma(q^n)$ consisting of these functions of the form

$$x \longrightarrow \frac{ax + b}{cx + d}$$

with $ad - bc$ a nonzero square in $GF(q^n)$.

The following results are proved here.

THEOREM A. *Let $\mathfrak{G}$ be a linear group acting on vector space $\mathfrak{V}$ of order q^n. Suppose that $\mathfrak{G}$ acts half-transitively but not semi-regularly on $\mathfrak{V}^\#$. If $\mathfrak{G}$ is primitive as a linear group then*

(i) *$O_p(\mathfrak{G})$ is cyclic for $p > 2$.*

(ii) *The Frattini subgroup $\Phi(O_2(\mathfrak{G}))$ is cyclic and*

$$[O_2(\mathfrak{G}) : \Phi(O_2(\mathfrak{G}))] \leqq 2^6 .$$

THEOREM B. *Let $\mathfrak{G}$ be a solvable 3/2-transitive permutation group. Then with suitable identification, $\mathfrak{G}$ satisfies one of the following.*

(i) *$\mathfrak{G}$ is a Frobenius group.*

(ii) *$\mathfrak{G} \subseteqq \mathscr{S}(q^n)$*

(iii) *$\mathfrak{G} = \mathscr{S}_0(q^n)$ or*

(iv) *$\mathfrak{G}$ has degree $3^2, 5^2, 7^2, 11^2, 17^2$ or 3^4.*

The exceptions of (iv) above do in fact exist. If deg $\mathfrak{G} \neq 17^2$ then we can take $\mathfrak{G}$ to be an exceptional solvable doubly transitive group, while if deg $\mathfrak{G} = 17^2$ then we construct this group explicitly and show that it has order $96 \cdot 17^2$.

THEOREM C. *Let $\mathfrak{G}$ be a 5/2-transitive permutation group and suppose that the stabilizer of a point is solvable. Then with suitable identification we have one of the following*

(i) *$\mathfrak{G}$ is a Zassenhaus group or*

(ii) *$\Gamma(q^n) \supseteqq \mathfrak{G} > \bar{\Gamma}(q^n)$.*

The main result here is Theorem B. Theorem A isolates that part of the proof in which solvability is not assumed. Theorem C follows immediately from the results of [8] and the fact that these exceptional groups have no transitive extensions.

1. Preliminaries. We will be concerned here with linear groups $\mathfrak{G}$ which act half-transitively but not semiregularly on the set $\mathfrak{V}^\#$ of nonzero vectors. This implies (see [11], Th. 10.4) that $\mathfrak{G}$ acts irreducibly on $\mathfrak{V}$. There are thus two possibilities according to whether $\mathfrak{G}$ is primitive or imprimitive as a linear group. The latter case is completely classified in Theorem 4.2 of [7] which we restate below for convenience.

THEOREM 1.1. *Let $\mathfrak{G}$ act faithfully on vector space $\mathfrak{V}$ over $GF(q)$ and let $\mathfrak{G}$ act half-transitively but not semiregularly on $\mathfrak{V}^\#$. If $\mathfrak{G}$ is imprimitive as a linear group, then $\mathfrak{G}$ satisfies one of the following*

(i) *$\mathfrak{G} = \mathscr{T}_0(q^n)$ with $q \neq 2$ and n an integer.*

(ii) $|\mathfrak{V}| = 3^4$ *and* $\mathfrak{G}$ *is isomorphic to a central product of the dihedral and quaternion groups of order* 8.

(iii) $|\mathfrak{V}| = 2^6$ *and* $\mathfrak{G}$ *is isomorphic to the dihedral group of order* 18 *with cyclic Sylow* 3-*subgroup.*

Here $\mathscr{T}_0(q^n)$ is the stabilizer in $\mathscr{S}_0(q^n)$ of the zero vector and hence we know all these groups explicitly. Thus we need only consider the primitive case here.

Let $\mathfrak{G}$ be a primitive linear group and let $\mathfrak{P}$ be a normal p-subgroup of $\mathfrak{G}$. Since every normal abelian subgroup of $\mathfrak{G}$ is cyclic (see for example [9], Lemma 1) it follows that every characteristic abelian subgroup of $\mathfrak{P}$ is cyclic. Hence by definition $\mathfrak{P}$ is a group of symplectic type. A characterization of these groups can be found in [1]. In particular for $p > 2$, $\mathfrak{P}$ is a central product of one cyclic p-group and any number of nonabelian groups of order p^3 and period p. If $p = 2$, then $\mathfrak{P}$ is a central product of either a cyclic 2-group or a 2-group of maximal class (that is, a dihedral, semidihedral or quaternion group) and any number of nonabelian groups of order 8. A special case of these are groups of type $E(p, m)$.

We say $\mathfrak{E}$ is a group of type $E(p, m)$ with $m \neq 0$ if $\mathfrak{E}$ has the following structure. If $p > 2$, then $\mathfrak{E}$ is a central product of m nonabelian groups of order p^3 and period p. If $p = 2$, then $\mathfrak{E}$ is a central product of a cyclic group of order 2 or 4, and m nonabelian groups of order 8. Thus in both cases $|\mathfrak{E}'| = p$, $\boldsymbol{Z}(\mathfrak{E})$, the center of $\mathfrak{E}$, is cyclic and $[\mathfrak{E} : \boldsymbol{Z}(\mathfrak{E})] = p^{2m}$. Moreover $|\boldsymbol{Z}(\mathfrak{E})| = p$ for $p > 2$ and $|\boldsymbol{Z}(\mathfrak{E})| = 2$ or 4 for $p = 2$. We call m the width of $\mathfrak{E}$.

Again let $\mathfrak{P}$ be of symplectic type. If $p > 2$, then $\Omega_1(\mathfrak{P})$, the subgroup generated by all elements of order p, is either cyclic (if $\mathfrak{P}$ is) or of type $E(p, m)$. If $p = 2$, then the Frattini subgroup $\Phi(\mathfrak{P})$ is cyclic, and $\Omega_2(C_{\mathfrak{P}}\Phi(\mathfrak{P}))$ is either cyclic or of type $E(2, m)$. The latter group is cyclic only if $\mathfrak{P}$ is cyclic or $|\mathfrak{P}| \geqq 16$ and $\mathfrak{P}$ is maximal class. Thus modulo the above mentioned exceptions $\mathfrak{P}$ contains a characteristic subgroup $\mathfrak{E}$ of type $E(p, m)$ with $m \neq 0$.

If $p > 2$, then for each (p, m) there is precisely one group of type $E(p, m)$. On the other hand, if $p = 2$, then for each m there are three isomorphism classes for $E(2, m)$ and we describe these now. For convenience we will use the following notation throughout this paper: $\mathfrak{D}$ denotes the dihedral group of order 8, $\mathfrak{Q}$ denotes the quaternion group of order 8, and $\mathfrak{Z}$ denotes a cyclic group of order 4. Furthermore any product of these written as $\mathfrak{D}\mathfrak{Q}$, $\mathfrak{Z}\mathfrak{D}\mathfrak{D}$, etc. will indicate a central product. Now we have easily $\mathfrak{D}\mathfrak{D} \cong \mathfrak{Q}\mathfrak{Q}$ and $\mathfrak{Z}\mathfrak{D} \cong \mathfrak{Z}\mathfrak{Q}$. Hence if $\mathfrak{E}$ is type $E(2, m)$ then $\mathfrak{E}$ is isomorphic to one of the following three groups.

$$\text{iso I}: \quad \mathfrak{E} \cong \underbrace{\mathfrak{Q}\mathfrak{Q}\cdots\mathfrak{Q}}_{m-1}$$
$$\text{iso II}: \quad \mathfrak{E} \cong \mathfrak{D}\underbrace{\mathfrak{Q}\cdots\mathfrak{Q}}_{m-1}$$
$$\text{iso III}: \quad \mathfrak{E} \cong \mathfrak{Z}\underbrace{\mathfrak{Q}\mathfrak{Q}\cdots\mathfrak{Q}}_{m-1}\,.$$

We will see below that these three groups are nonisomorphic.

For any group $\mathfrak{G}$ we let $I(\mathfrak{G})$ denote the number of its noncentral involutions.

LEMMA 1.2. *Let $\mathfrak{E}$ be a group of type $E(2, m)$. Then*

$$\begin{aligned} I(\mathfrak{E}) &= 2^{2m} + (-2)^m - 2 \quad \textit{if } \mathfrak{E} = \text{iso I} \\ &= 2^{2m} - (-2)^m - 2 \quad \textit{if } \mathfrak{E} = \text{iso II} \\ &= 2^{2m+1} - 2 \qquad\qquad \textit{if } \mathfrak{E} = \text{iso III}\,. \end{aligned}$$

In particular these three groups are nonisomorphic. Moreover with the exception of $\mathfrak{E} = \mathfrak{Q}$, $\mathfrak{E}$ is generated by all its noncentral involutions.

Proof. Let $I^*(\mathfrak{G})$ denote the number of elements $G \in \mathfrak{G}$ with $G^2 = 1$. Then $I(\mathfrak{E}) = I^*(\mathfrak{E}) - 2$. Suppose $\mathfrak{E}$ is iso I or II and write $\mathfrak{E} = \mathfrak{E}_1\mathfrak{Q}$ where $\mathfrak{E}_1$ is type $E(2, m-1)$. Clearly

$$I^*(\mathfrak{E}) = 3(|\,\mathfrak{E}_1\,| - I^*(\mathfrak{E}_1)) + I^*(\mathfrak{E}_1)\,.$$

Thus if $I^*(\mathfrak{E}_1) = 2^{2(m-1)} + \delta(-2)^{m-1}$ then $I^*(\mathfrak{E}) = 2^{2m} + \delta(-2)^m$. Hence the first two results follow easily. If $\mathfrak{E} = \text{iso III}$, let $Z(\mathfrak{E}) = \langle Z \rangle$. Then the map $X \to XZ$ yields a one to one correspondence between the elements of $\mathfrak{E}$ with square 1 and those of order 4. Hence clearly $I^*(\mathfrak{E}) = 1/2\,|\,\mathfrak{E}\,| = 2^{2m+1}$.

Now any such $\mathfrak{E}$ can be written as $\mathfrak{E}_1\mathfrak{D}\mathfrak{D}\cdots\mathfrak{D}$ and of course $\mathfrak{D}$ is generated by its noncentral involutions. Since the same is easily seen to be true for $\mathfrak{E}_1 = \mathfrak{D}, \mathfrak{Q}\mathfrak{D}$ or $\mathfrak{Z}\mathfrak{D}$, the result follows.

Let $\mathfrak{E}$ be type $E(p, m)$ and let $\mathfrak{W} = \mathfrak{E}/Z(\mathfrak{E})$. Then $\mathfrak{W}$ is elementary abelian of order p^{2m} and we view this additively as a $2m$-dimensional vector space over $GF(p)$. If $p = 2$ we say $W \in \mathfrak{W}$ is an involution vector if the coset corresponding to W in $\mathfrak{E}$ contains an involution of $\mathfrak{E}$. Here we let $i(\mathfrak{W})$ denote the number of such involution vectors.

LEMMA 1.3. *Let $\mathfrak{H}$ be a group of automorphisms of group $\mathfrak{E}$ of type $E(p, m)$ which centralizes $Z(\mathfrak{E})$ and let $\mathfrak{K}$ be the subgroup of $\mathfrak{H}$ consisting of those elements which centralize $\mathfrak{W}$. Then*

(i) *$\mathfrak{K}$ is isomorphic to a subgroup of the direct product of*

$\boldsymbol{Z}(\mathfrak{E})$ taken $2m$ times.

(ii) *The commutator map (,) of $\mathfrak{E}$ induces a nonsingular skew-symmetric bilinear form on $\mathfrak{W}$. As such $\mathfrak{H}/\mathfrak{K}$ is contained isomorphically in the sympletic group $Sp(2m, p)$.*

(iii) *If $p = 2$, then in addition $\mathfrak{H}/\mathfrak{K}$ permutes the $i(\mathfrak{W})$ involution vectors of $\mathfrak{W}$. Here*

$$\begin{aligned} i(\mathfrak{W}) &= 2^{2m-1} - (-2)^{m-1} - 1 \quad \textit{if } \mathfrak{E} = \text{iso} \quad \mathrm{I} \\ &= 2^{2m-1} + (-2)^{m-1} - 1 \quad \textit{if } \mathfrak{E} = \text{iso} \quad \mathrm{II} \\ &= 2^{2m} - 1 \qquad\qquad\qquad \textit{if } \mathfrak{E} = \text{iso} \quad \mathrm{III}\,. \end{aligned}$$

Proof. (i) Let $E_1, \cdots, E_{2m}$ be a set of coset representatives of $\boldsymbol{Z}(\mathfrak{E})$ in $\mathfrak{E}$. We define $\theta : \mathfrak{K} \rightarrow \dot{\prod} \boldsymbol{Z}(\mathfrak{E})$ ($2m$ times) by $\theta(K) = \dot{\prod} E_i^K E_i^{-1}$. This is easily seen to be a monomorphism.

(ii) and (iii) If W is an involution vector then we see easily that the coset of W contains precisely two noncentral involutions of $\mathfrak{E}$. Hence $i(\mathfrak{W}) = 1/2I(\mathfrak{E})$. The result now follows easily.

We now consider the action of $\mathfrak{E}$ on a vector space $\mathfrak{V}$.

LEMMA 1.4. *Let group $\mathfrak{E}$ of type $E(p, m)$ act on vector space $\mathfrak{V}$ of order q^n. Suppose further that $\mathfrak{E}'$ acts without fixed points on $\mathfrak{V}^\#$. Then*

(i) *$sp^m \mid n$ where s is the smallest positive integer with $|\boldsymbol{Z}(\mathfrak{E})| \mid q^s - 1$.*

(ii) *If $T \in \mathfrak{E} - \boldsymbol{Z}(\mathfrak{E})$ has order p then $|\boldsymbol{C}_{\mathfrak{V}}(T)| = q^{n/p}$.*

(iii) *If $x \in \mathfrak{V}^\#$, then $\mathfrak{E}_x$, the stabilizer of x in $\mathfrak{E}$ is elementary abelian.*

Proof. (i) Since $\mathfrak{E}'$ acts without fixed points $q \neq p$. By complete reducibility we can assume that $\mathfrak{E}$ acts irreducibly on $\mathfrak{V}$. Let χ be the character of an absolutely irreducible constituent of $\mathfrak{E}$. From the representation of $\mathfrak{E}$ as a homomorphic image of a direct product of nonabelian group of order p^3 (and possibly a cyclic group of order 4 if $p = 2$) we see easily that deg $\chi = p^m$ and χ vanishes off $\boldsymbol{Z}(\mathfrak{E})$. Hence by definition of s, $GF(q)(\chi) = GF(q^s)$ and $\mathfrak{V}$ contains as absolutely irreducible constituents the s algebraic conjugates of the representation affording χ. Thus (i) follows.

(ii) We wish to show here that dim $\boldsymbol{C}_{\mathfrak{V}}(T) = n/p$. This dimension is clearly invariant under field extension so by complete reducibility we can assume $\mathfrak{V}$ is absolutely irreducible. If θ is the corresponding complex character then $\theta(T)$ is a sum of pth roots of unity (including 1) and $\theta(T) = 0$. Hence all eigenvalues occur with the same multiplicity n/p and (ii) follows.

(iii) This is clear since $\Phi(\mathfrak{E})$ acts semiregularly on $\mathfrak{V}^\#$.

LEMMA 1.5. *Let group $\mathfrak{E}$ of type $E(p, m)$ act on vector space $\mathfrak{V}$ of order q^n and let $T \in \mathfrak{E} - \boldsymbol{Z}(\mathfrak{E})$ have order p. Suppose further that $\mathfrak{E}$ acts without fixed points on $\mathfrak{V}$. Then*

(i) There exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle 1 \rangle$ with the following exceptions which occur for $p = 2$: (a) $q^n = 3^2$, $\mathfrak{E} = \mathfrak{D}$, (b) $q^n = 5^2$, $\mathfrak{E} = \mathfrak{Z}\mathfrak{Q}$, (c) $q^n = 3^4$, $\mathfrak{E} = \mathfrak{D}\mathfrak{Q}$. *In each of these exceptions $|\mathfrak{E}_x| = 2$ for all $x \in \mathfrak{V}^\#$.*

(ii) There exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle T \rangle$ with the following exceptions which occur for $p = 2$: (a) $q^n = 3^4$, $\mathfrak{E} = \mathfrak{Q}\mathfrak{Q}$, (b) $q^n = 5^4$, $\mathfrak{E} = \mathfrak{Z}\mathfrak{Q}\mathfrak{Q}$, (c) $q^n = 3^8$, $\mathfrak{E} = \mathfrak{Q}\mathfrak{Q}\mathfrak{Q}$. *In each of these exceptions $|\mathfrak{E}_x| = 4$ or 1 for all $x \in \mathfrak{V}^\#$.*

Proof. (i) We first note that by [4] Theorem II (a), (b) and (c) are in fact exceptions. Suppose now that $\mathfrak{E}_x \neq \langle 1 \rangle$ for all $x \in \mathfrak{V}^\#$. Then every element of $\mathfrak{V}^\#$ is centralized by a noncentral element $P \in \mathfrak{E}$ of order p. Thus

$$\mathfrak{V} = \bigcup_P C_{\mathfrak{V}}(P)$$

where the union is over respesentatives of the N noncentral subgroups of $\mathfrak{E}$ of order p. By Lemma 1.4 we have

$$q^n = |\mathfrak{V}| \leqq N q^{n/p}$$

and $q^{n(1-1/p)} \leqq N$.

Let $p > 2$. Then $N < p^{2m+1}/(p-1)$ and $n \geqq sp^m$. Furthermore $p \mid q^s - 1$ so $q^s \geqq p + 1$. Thus

$$\begin{aligned} p^{p^m - p^{m-1}} < (p+1)^{p^m - m - 1} &\leqq q^{s(p^m - p^{m-1})} \\ &\leqq q^{n(1-1/p)} \leqq N < p^{2m+1}/(p-1) < p^{2m+1} . \end{aligned}$$

This yields $p^{m-1}(p-1) < 2m+1$ and since $p > 2$ we have $p = 3$, $m = 1$ here. However with $p = 3$, $m = 1$ the equation

$$(p+1)^{p^m - p^{m-1}} < p^{2m+1}/(p-1)$$

is not satisfied so $p > 2$ cannot occur here.

Now let $p = 2$ so that $N = I(\mathfrak{E})$. Suppose first that $|\boldsymbol{Z}(\mathfrak{E})| = 4$. Then $4 \mid q^s - 1$ and $I(\mathfrak{E}) < 2^{2m+1}$. Thus

$$5^{2^{m-1}} \leqq q^{s2^{m-1}} \leqq q^{n(1-1/p)} \leqq N \leqq 2^{2m+1} .$$

This yields $5^{2^{m-1}} < 2^{2m+1}$ so $m = 1$ or 2. If $m = 1$, then $q^{n/2} < 8$ and $4 \mid q^s - 1$ yields $q^n = 5^2$ and we have exception (b). If $m = 2$, then $q^{n/2} < 32$, $4 \mid q^s - 1$ and $4s \mid n$ yields $q^n = 5^4$. We show now that this possibility does not occur. Let $x \in \mathfrak{V}^\#$ and suppose that $\mathfrak{E}_x \neq \langle 1 \rangle$.

Choose $P \in \mathfrak{E}_x^\#$. Since $\mathfrak{E}_x$ is abelian $\mathfrak{E}_x \subseteqq \boldsymbol{C}_{\mathfrak{E}}(P) = \langle P \rangle \times \bar{\mathfrak{E}}$ where $\bar{\mathfrak{E}} \cong \mathfrak{Z}\mathfrak{Q}$. Now $x \in \boldsymbol{C}_{\mathfrak{V}}(P)$, $|\boldsymbol{C}_{\mathfrak{V}}(P)| = 5^2$ and $\bar{\mathfrak{E}}$ acts on this subspace. Since this action yields the exceptional case (b) we have $|\bar{\mathfrak{E}}_x| = 2$ and hence $|\mathfrak{E}_x| = 4$. Thus for all $x \in \mathfrak{V}^\#$, $|\mathfrak{E}_x| = 1$ or 4. This, by the way, is the exceptional case (b) of part (ii). If $\mathfrak{V} = \bigcup \boldsymbol{C}_{\mathfrak{V}}(P)$ then since each $\mathfrak{E}_x$ is elementary abelian, we see that this union covers $\mathfrak{V}$ three times. Thus

$$5^4 - 1 = |\mathfrak{V}^\#| \leqq \frac{1}{3} I(\mathfrak{E}) \cdot (5^2 - 1) < \frac{1}{3} \cdot 2^5(5^2 - 1)$$

a contradiction.

Now let $|\boldsymbol{Z}(\mathfrak{E})| = 2$ so $I(\mathfrak{E}) \leqq 2^{2m} + 2^m - 2$. Since $q^s \geqq 3$ we have

$$3^{2^{m-1}} \leqq q^{s2^{m-1}} \leqq q^{n(1-1/p)} \leqq N \leqq 2^{2m} + 2^m - 2\,.$$

This yields $3^{2^{m-1}} < 2^{2m} + 2^m$ so $m = 1$ or 2. If $m = 1$ then $q^{n/2} \leqq 4$ so $q^n = 3^2$. Clearly $\mathfrak{E} \not\simeq \mathfrak{Q}$ so we have exception (a) here. If $m = 2$, then $4 \mid n$ and $q^{n/2} \leqq 18$ yields $q^n = 3^4$. If $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}$ we have exception (c). We show finally that $\mathfrak{E} \not\simeq \mathfrak{Q}\mathfrak{Q}$. Let $x \in \mathfrak{V}^\#$ and suppose $\mathfrak{E}_x \neq \langle 1 \rangle$. Choose $P \in \mathfrak{E}_x^\#$ and let $\boldsymbol{C}_{\mathfrak{E}}(P) = \langle P \rangle \times \bar{\mathfrak{E}}$. Here $\bar{\mathfrak{E}}$ is nonabelian of order 8. Since $\boldsymbol{C}_{\mathfrak{E}}(\bar{\mathfrak{E}})$ contains P we see that $\boldsymbol{C}_{\mathfrak{E}}(\bar{\mathfrak{E}}) \cong \mathfrak{D}$ and hence $\mathfrak{E} \cong \mathfrak{D}\bar{\mathfrak{E}}$. Thus $\bar{\mathfrak{E}} \cong \mathfrak{D}$. This implies as above that $|\bar{\mathfrak{E}}_x| = 2$ and $|\mathfrak{E}_x| = 4$, thereby yielding exception (a) of part (ii). Again if $\mathfrak{V} = \bigcup \boldsymbol{C}_{\mathfrak{V}}(P)$, then $\mathfrak{V}$ is triply covered so

$$3^4 - 1 = |\mathfrak{V}^\#| \leqq \frac{1}{3} I(\mathfrak{E})(3^2 - 1) < \frac{1}{3} 20(3^2 - 1)\,,$$

a contradiction. This completes the proof of (i).

(ii) If $m = 1$, then any abelian subgroup of $\mathfrak{E}$ of order 4 meets $\boldsymbol{Z}(\mathfrak{E})$. Since $\boldsymbol{Z}(\mathfrak{E})$ acts semiregularly, we conclude that for all $x \in \mathfrak{V}^\#$, $|\mathfrak{E}_x| = 1$ or 2. Thus the result follows here.

Let $m \geqq 2$. Then $\boldsymbol{C}_{\mathfrak{E}}(T) = \langle T \rangle \times \bar{\mathfrak{E}}$ where $\bar{\mathfrak{E}}$ is type $E(p, m-1)$. Note that $\mathfrak{E} = \bar{\mathfrak{E}}\boldsymbol{C}_{\mathfrak{E}}(\bar{\mathfrak{E}})$ and $T \in \boldsymbol{C}_{\mathfrak{E}}(\bar{\mathfrak{E}})$. Thus if $p = 2$ then $\boldsymbol{C}_{\mathfrak{E}}(\bar{\mathfrak{E}}) \cong \mathfrak{D}$ and the isomorphism class of $\bar{\mathfrak{E}}$ is uniquely determined by $\mathfrak{E} \cong \bar{\mathfrak{E}}\mathfrak{D}$. Now $\bar{\mathfrak{E}}$ acts on $\boldsymbol{C}_{\mathfrak{V}}(T)$ a subspace of size $q^{n/2}$ and hence if this is not one of the exceptions of part (i), then there exists $x \in \boldsymbol{C}_{\mathfrak{V}}(T)^\#$ with $\bar{\mathfrak{E}}_x = \langle 1 \rangle$. Since $T \in \mathfrak{E}_x$ and $\mathfrak{E}_x$ is abelian, it then follows that $\mathfrak{E}_x = \langle T \rangle$. The result now clearly follows.

We now turn to a variant of an argument used in [2] (§ 2.5).

LEMMA 1.6. *Let $\mathfrak{G} = \mathfrak{E}\mathfrak{J}$ where $\mathfrak{E}$ is type $E(p, m)$, $\mathfrak{E} \triangle \mathfrak{G}$ and $\mathfrak{J} = \langle J \rangle$ is cyclic of order j. Suppose $\mathfrak{G}$ acts on F-vector space $\mathfrak{V}$ in such a way that the restriction to $\mathfrak{E}$ is faithful and absolutely* *irreducible. If further the characteristic of F is prime to $|\mathfrak{G}|$, then there exists nonnegative integers $a_0, a_1, \cdots, a_{j-1}$ satisfying*

(i) $a_0 + a_1 + \cdots + a_{j-1} = p^m$

(ii) $a_0^2 + a_1^2 + \cdots + a_{j-1}^2 \leqq N$ *and*

(iii) $a_0 = \dim_F C_{\mathfrak{V}}(J)$

where N is the number of orbits in $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ under the action of $\mathfrak{J}$.

Proof. Since $\dim_F C_{\mathfrak{V}}(J)$ is clearly invariant under field extension, we can assume F is algebraically closed. Let $\varepsilon \in F$ be a primitive jth root of unity and suppose that ε^i occurs as an eigenvalue of J with multiplicity a_i for $i = 0, 1, \cdots, j-1$. If Σ denotes the enveloping algebra of this representation then clearly

$$a_0 + a_1 + \cdots + a_{j-1} = \dim_F \mathfrak{V}$$
$$a_0^2 + a_1^2 + \cdots + a_{j-1}^2 = \dim_F C_{\Sigma}(J)$$
$$a_0 = \dim_F C_{\mathfrak{V}}(J)\ .$$

Now $\mathfrak{V}$ is a faithful absolutely irreducible $\mathfrak{E}$-module so $\dim_F \mathfrak{V} = p^m$. Hence (i) and (iii) follow. In addition the group ring $F(\mathfrak{E})$ maps onto Σ in the obvious manner. Under this map $\boldsymbol{Z}(\mathfrak{E})$ is sent into the field of scalars so the image of $F[\mathfrak{E}]$ is spanned by p^{2m} coset representatives of $\boldsymbol{Z}(\mathfrak{E})$ in $\mathfrak{E}$. But $\dim_F \Sigma = p^{2m}$ so these must in fact form a basis of Σ. With this choice of basis we see clearly that $\dim_F C_{\Sigma}(J)$ is at most equal to the number of orbits of $\mathfrak{J}$ on $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ so the result follows.

The following two results enable us to use inductive methods in our study of half-transitive linear groups.

LEMMA 1.7. *Let $\mathfrak{G}$ be a half-transitive permutation group and let $\mathfrak{N} \triangle \mathfrak{G}$. Suppose that either $\mathfrak{N} = \langle 1 \rangle$ or $\mathfrak{N}$ acts half-transitively. Let $\mathfrak{G} \supseteqq \mathfrak{H} \supseteqq \mathfrak{N}$ where $\mathfrak{H}/\mathfrak{N}$ is a normal Hall subgroup of $\mathfrak{G}/\mathfrak{N}$. Then $\mathfrak{H}$ acts half-transitively.*

Proof. See Lemma 2.1 of [5].

LEMMA 1.8. (*Reduction Lemma*). *Let $\mathfrak{G}$ be a linear group on $GF(q)$-vector space $\mathfrak{V}$ and suppose that $\mathfrak{G}$ acts half-transitively but not semiregularly on $\mathfrak{V}^{\#}$. Let $\mathfrak{E}$ be a group of type $E(p, m)$ with $\mathfrak{E} \triangle \mathfrak{G}$. Then there exists a linear group $\bar{\mathfrak{G}}$ acting on $GF(q)$-vector space $\mathfrak{U}$ and a normal subgroup $\bar{\mathfrak{E}}$ of $\bar{\mathfrak{G}}$ satisfying*

(i) *$\bar{\mathfrak{G}}$ acts half-transitively on $\mathfrak{U}$.*

(ii) *$\bar{\mathfrak{E}} \cong \mathfrak{E}$ and $\bar{\mathfrak{E}}$ acts irreducibly on $\mathfrak{U}$.*

(iii) *If $\mathfrak{G}$ is solvable so is $\bar{\mathfrak{G}}$.*

(iv) *If* $\mathfrak{E} \neq \mathfrak{Q}$, *then* $\bar{\mathfrak{G}}$ *does not act semiregularly on* $\mathfrak{U}$.

(v) *Suppose that either* $p > 2$ *or* $p = 2$ *and* $m \geqq 2$. *Then either* $\bar{\mathfrak{G}} = \bar{\mathfrak{E}} \cong \mathfrak{D}\mathfrak{Q}$ *with* $q = 3$ *or* $\bar{\mathfrak{G}}$ *is primitive as a linear group.*

Proof. Since $\mathfrak{G}$ does not act semiregularly, it acts irreducibly on $\mathfrak{V}$. By Clifford's theorem all irreducible $\mathfrak{E}$ constituents of $\mathfrak{V}$ are conjugate and hence $\mathfrak{E}$ acts faithfully on each. Let $\mathfrak{U}$ be an irreducible $\mathfrak{E}$-submodule of $\mathfrak{V}$ and let $\mathfrak{N} = \{G \in \mathfrak{G} \mid \mathfrak{U}G = \mathfrak{U}\}$. Suppose $x \in \mathfrak{U}^{\#}$. Since $\mathfrak{E} \triangle \mathfrak{G}$

$$(x\mathfrak{E})\mathfrak{G}_x = (x\mathfrak{G}_x)\mathfrak{E} = x\mathfrak{E}$$

and hence $\mathfrak{G}_x$ normalizes $x\mathfrak{E}$. Moreover $\mathfrak{E}$ acts irreducibly on $\mathfrak{U}$ so $\mathfrak{U}$ is the linear span of $x\mathfrak{E}$ and hence $\mathfrak{G}_x \subseteqq \mathfrak{N}$. If $\mathfrak{K}$ is the kernel of the action of $\mathfrak{N}$ on $\mathfrak{U}$, then clearly $\bar{\mathfrak{G}} = \mathfrak{N}/\mathfrak{K}$ acts semiregularly on $\mathfrak{U}^{\#}$. Since $\mathfrak{E}$ acts faithfully on $\mathfrak{U}$, $\bar{\mathfrak{E}} = \mathfrak{E}\mathfrak{K}/\mathfrak{K} \cong \mathfrak{E}$. Also $\bar{\mathfrak{E}} \triangle \bar{\mathfrak{G}}$ and $\bar{\mathfrak{E}}$ acts irreducibly on $\mathfrak{U}$ so (i), (ii) and (iii) follow.

We have $\bar{\mathfrak{E}} \cong \mathfrak{E}$. Thus if $\mathfrak{E} \neq \mathfrak{Q}$ then $\bar{\mathfrak{E}}$, and hence $\bar{\mathfrak{G}}$, cannot act semiregularly. This yields (iv). Finally suppose that either $p > 2$ or $p = 2$ and $m \geqq 2$. Then $\mathfrak{E} \neq \mathfrak{Q}$ so $\bar{\mathfrak{G}}$ does not act semiregularly. Hence if $\bar{\mathfrak{G}}$ is imprimitive as a linear group, then the structure of $\bar{\mathfrak{G}}$ is given in Theorem 1.1. In both (i) and (iii) of that theorem $\bar{\mathfrak{G}}$ has a normal abelian subgroup of index 2 and hence $\bar{\mathfrak{G}}$ could not possibly contain $\bar{\mathfrak{E}}$. Thus only (ii) of that theorem can occur here and since $m \geqq 2$ this yields $\bar{\mathfrak{G}} = \bar{\mathfrak{E}} \cong \mathfrak{D}\mathfrak{Q}$ and $|\mathfrak{U}| = 3^4$. This completes the proof of the lemma.

We close this section by offering a precise statement of Lemma 6 of [4]. The proof is the same and will not be repeated.

LEMMA 1.9. *Let* $\mathfrak{G}$ *act faithfully on vector space* $\mathfrak{V}$ *and half-transitively on* $\mathfrak{V}^{\#}$. *Suppose that for all* $x \in \mathfrak{V}^{\#}$, $|\mathfrak{G}_x| = 2$. *If* $\mathfrak{G}$ *has a central involution, then* $|\mathfrak{V}| = q^{2r}$ *with* $q \neq 2$ *and* $q^r + 1 = I(\mathfrak{G})$.

2. Theorem A.

The following assumptions hold throughout this section.

ASSUMPTIONS. Group $\mathfrak{G}$ acts faithfully on vector space $\mathfrak{V}$ of order q^n and half-transitively but not semiregularly on $\mathfrak{V}^{\#}$. $\mathfrak{E}$ is a group of type $E(p, m)$ with $\mathfrak{E} \triangle \mathfrak{G}$. In addition $\mathfrak{E}$ acts irreducibly on $\mathfrak{V}$ and $\mathfrak{G}$ is primitive as a linear group.

It is convenient to keep track of four separate possibilities.

DEFINITION. We define the type of $\mathfrak{E}$ as follows.

type I: $p > 2$
type II: $p = 2$, $|\boldsymbol{Z}(\mathfrak{E})| = 2$
type III: $p = 2$, $|\boldsymbol{Z}\mathfrak{E})| = 4$, $\boldsymbol{Z}(\mathfrak{E}) \subseteqq \boldsymbol{Z}(\mathfrak{G})$
type IV: $p = 2$, $|\boldsymbol{Z}(\mathfrak{E})| = 4$, $\boldsymbol{Z}(\mathfrak{E}) \not\subseteq \boldsymbol{Z}(\mathfrak{G})$.

LEMMA 2.1. *Let $s \geqq 1$ be minimal with $|\boldsymbol{Z}(\mathfrak{E})| \mid q^s - 1$. Let $\mathfrak{M}$ be any subgroup of $\mathfrak{G}$ with $\mathfrak{E} \subseteqq \mathfrak{M} \subseteqq \boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))$. Then $\mathfrak{M} \subseteqq GL(p^m, q^s)$ and this representation of $\mathfrak{M}$ is absolutely irreducible. Furthermore $n = sp^m$ and we have the following*

type I: $s \mid (p-1)$
type II: $s = 1$
type III: $s = 1$ *or* 2
type IV: $s = 2$, *and if $\bar{\mathfrak{M}}$ is a q'-subgroup of $\mathfrak{G}$ with $\mathfrak{E} \subseteqq \bar{\mathfrak{M}}$ and $\bar{\mathfrak{M}} \not\subseteq \boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))$, then $\bar{\mathfrak{M}} \subseteqq GL(p^{m+1}, q)$ and this is an absolutely irreducible representation.*

Proof. If s is defined as above then $GF(q^s)$ is clearly the minimal splitting field of the representation of $\mathfrak{E}$. Hence $n = sp^m$ since we are dealing with finite fields here and since the absolutely irreducible constituents of $\mathfrak{E}$ have degree p^m.

Now $\mathfrak{G}$ is primitive as a linear group so by Lemma 1.1 of [5], $\boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E})) \subseteqq GL(p^m, q^s)$. Let $\mathfrak{M}$ be a subgroup of $\mathfrak{G}$ with

$$\mathfrak{E} \subseteqq \mathfrak{M} \subseteqq \boldsymbol{C}_{\mathfrak{G}}(z(\mathfrak{E}))$$

so that $\mathfrak{M} \subseteqq GL(p^m, q^s)$. Since $\mathfrak{M} \supseteqq \mathfrak{E}$ and the degree of this representation is p^m, the representation is clearly absolutely irreducible.

The results on the value of s for types I, II and III are clear. Let $\mathfrak{E}$ be type IV. Then certainly $s = 1$ or 2. If $s = 1$, then since $\mathfrak{G}$ is primitive, $\boldsymbol{Z}(\mathfrak{E})$ consists of scalar matrices and is therefore central in $\mathfrak{G}$, a contradiction. Thus $s = 2$. Let $\bar{\mathfrak{M}}$ be given with $\mathfrak{E} \subseteqq \bar{\mathfrak{M}}$, $\bar{\mathfrak{M}} \not\subseteq \boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))$. Since $s = 2$, $\bar{\mathfrak{M}} \subseteqq GL(p^{m+1}, q)$. Clearly $\bar{\mathfrak{M}}$ is either absolutely irreducible or it has two absolutely irreducible constituents of degree p^m. In the latter case, $\boldsymbol{Z}(\mathfrak{E})$ would be central in each such constituent and hence in $\bar{\mathfrak{M}}$, a contradiction.

LEMMA 2.2. *Let $\mathfrak{M}$ be a p-group acting faithfully and absolutely irreducibly on F-vector space $\mathfrak{V}$. Let $\dim_F \mathfrak{V} = k$. Then there exists subgroups $\mathfrak{N}$ and $\mathfrak{K}$ of $\mathfrak{M}$ and an $\mathfrak{N}$-subspace $\mathfrak{U}$ of $\mathfrak{V}$ with the representation of $\mathfrak{M}$ on $\mathfrak{V}$ induced from that of $\mathfrak{N}$ on $\mathfrak{U}$. Furthermore $\mathfrak{K} = \boldsymbol{C}_{\mathfrak{N}}(\mathfrak{U})$ and either*

(i) $[\mathfrak{M} : \mathfrak{N}] = k$, $\dim \mathfrak{U} = 1$ *and $\mathfrak{N}/\mathfrak{K}$ is cyclic, or*

(ii) $[\mathfrak{M} : \mathfrak{N}] = k/2$, $\dim \mathfrak{U} = 2$, *$\mathfrak{N}/\mathfrak{K}$ is dihedral, semidihedral or quaternion and $p = 2$.*

Proof. The result is trivial if char $F = p$ so assume this is not the case. Applying Roquette's theorem ([9]) repeatedly we can find $\mathfrak{N}, \mathfrak{K}$ and $\mathfrak{U}$ as above with $\mathfrak{N}/\mathfrak{K}$ cyclic, dihedral, semidihedral or quaternion. Since $\mathfrak{M}$ is absolutely irreducible so is the action of $\mathfrak{N}/\mathfrak{K}$ on $\mathfrak{U}$. Thus $\dim \mathfrak{U} = 1$ if $\mathfrak{N}/\mathfrak{K}$ is cyclic and $\dim \mathfrak{U} = 2$ otherwise.

LEMMA 2.3. *Let w denote the period of a Sylow p-subgroup of $C_{\mathfrak{G}}(Z(\mathfrak{E}))$. Then for all $x \in \mathfrak{B}^{\#}$ we have*

type I: $[\mathfrak{G} : \mathfrak{G}_x]_p \leqq p^m \min\{w, |q^s - 1|_p\}$
type II: $[\mathfrak{G} : \mathfrak{G}_x]_p \leqq p^{m+1} \min\{w, |q^2 - 1|_p\}$
type III: $[\mathfrak{G} : \mathfrak{G}_x]_p \leqq p^m \min\{w, |q^2 - 1|_p\}$
type IV: $[\mathfrak{G} : \mathfrak{G}_x]_p \leqq p^{m+1} \min\{w, |q^2 - 1|_p\}$.

Proof. We consider types I, II and III first. Let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{G}$. Then $\mathfrak{P} \supseteqq \mathfrak{E}$ and $Z(\mathfrak{E})$ is central in $\mathfrak{P}$. By Lemma 2.1 we can view $\mathfrak{P}$ as a subgroup of $GL(p^m, q^s)$ and this representation is absolutely irreducible. Let $\mathfrak{N}$, $\mathfrak{K}$ and $\mathfrak{U}$ be as in the preceding lemma with $\mathfrak{M} = \mathfrak{P}$. Note that for $y \in \mathfrak{U}^{\#}$, $\mathfrak{P}_y \supseteqq \mathfrak{K}$. If $\mathfrak{N}/\mathfrak{K}$ is cyclic, then $[\mathfrak{P} : \mathfrak{N}] = p^m$, $[\mathfrak{N} : \mathfrak{K}] \leqq \min\{w, |q^s - 1|_p\}$ so

$$[\mathfrak{P} : \mathfrak{P}_y] \leqq p^m \min\{\mathrm{w}, |q^s - 1|_p\}\,.$$

Suppose that $\mathfrak{N}/\mathfrak{K}$ is not cyclic. Then $p = 2$. Now it is clear that $Z(\mathfrak{E}) \subseteqq Z(\mathfrak{P}) \subseteqq \mathfrak{N}$ and $Z(\mathfrak{E}) \cap \mathfrak{K} = \langle 1 \rangle$. Thus since 2-groups of maximal class have centers of order 2, $\mathfrak{E}$ must be type II. Here $[\mathfrak{P} : \mathfrak{N}[= p^{m-1}$ and $[\mathfrak{N} : \mathfrak{K}] \leqq p \min\{w, |q^{2s} - 1|_p\}$ since $\mathfrak{N}/\mathfrak{K}$ has a cyclic subgroup of index $p = 2$ which has a faithful irreducible representation in $GF(q^{2s})$. Note that $s = 1$ here. Now by half-transitivity, for all $x \in \mathfrak{B}^{\#}$

$$[\mathfrak{G} : \mathfrak{G}_x]_p = [\mathfrak{G} : \mathfrak{G}_y]_p \leqq [\mathfrak{P} : \mathfrak{P}_y]\,.$$

Thus the first three results follow.

Now let $\mathfrak{E}$ be type IV and again let $\mathfrak{P}$ be a Sylow p-subgroup of $\mathfrak{G}$. Let $\mathfrak{M} = C_{\mathfrak{P}}(Z(\mathfrak{E}))$ so that $\mathfrak{P} > \mathfrak{M} \supseteqq \mathfrak{E}$ and $[\mathfrak{P} : \mathfrak{M}] = 2$. By Lemma 2.1, $\mathfrak{P}$ is absolutely irreducible as a subgroup of $GL(p^{m+1}, q)$. We extend the field now to $GF(q^s) = GF(q^2)$. Thus we let $\mathfrak{P}$ act on $\mathfrak{B} \otimes GF(q^2)$ and this representation is again absolutely irreducible. If the restriction to $\mathfrak{M}$ were irreducible, then since $4 \mid q^s - 1$, $Z(\mathfrak{E})$ which is central in $\mathfrak{M}$ would consist of scalar matrices and hence it would be central in $\mathfrak{P}$, a contradiction. Thus the representation of $\mathfrak{P}$ is induced from one of $\mathfrak{M}$. Let $\mathfrak{N}$, $\mathfrak{K}$ and $\mathfrak{U} \subseteqq \mathfrak{B} \otimes GF(q^s)$ be as in the preceding lemma with $\mathfrak{N} \subseteqq \mathfrak{M}$. Since $Z(\mathfrak{E}) \subseteqq \mathfrak{N}$ and $|Z(\mathfrak{E})| = 4$ we see that $\mathfrak{N}/\mathfrak{K}$ is cyclic. Hence $[\mathfrak{N} : \mathfrak{K}] \leqq \min\{w, |q^s - 1|_p\}$. Moreover $[\mathfrak{P} : \mathfrak{N}] = p^{m+1}$ so

$$[\mathfrak{P} : \mathfrak{K}] \leqq p^{m+1}\{w, |q^s - 1|_p\}\,.$$

Now all elements of $\mathfrak{K}$ have a common nonzero fixed point in $\mathfrak{V} \otimes GF(q^s)$. This means that a certain set of simultaneous linear equations over $GF(q)$ has a nonzero solution over $GF(q^s)$. Thus there is a nonzero solution over $GF(q)$ and hence there exists $y \in \mathfrak{V}^{\#}$ with $\mathfrak{P}_y \supseteqq \mathfrak{K}$. The result now follows as above.

LEMMA 2.4. *Let $\mathfrak{A} = C_{\mathfrak{G}}(\mathfrak{E})$. Then $\mathfrak{A}$ is a normal cyclic subgroup of $\mathfrak{G}$ which is central in $C_{\mathfrak{G}}(Z(\mathfrak{E}))$ and acts semiregularly on $\mathfrak{V}^{\#}$. Suppose that $m \geqq 3$ if $p = 2$. Then there exists $x \in \mathfrak{V}^{\#}$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ and hence $[\mathfrak{G} : \mathfrak{G}_x]_p \geqq |\mathfrak{A}_p| p^{2m}$ where $\mathfrak{A}_p$ is the normal Sylow p-subgroup of $\mathfrak{A}$. This yields*

type I: $w \geqq p^m |\mathfrak{A}_p|$, $|q^s - 1|_p \geqq p^{m+1}$
type II: $w \geqq p^{m-1} |\mathfrak{A}_p|$, $|q^2 - 1|_p \geqq p^m$
type III: $w \geqq p^m |\mathfrak{A}_p|$, $|q^2 - 1|_p \geqq p^{m+2}$
type IV: $w \geqq p^{m-1} |\mathfrak{A}_p|$, $|q^2 - 1|_p \geqq p^{m+1}$.

Proof. Since $\mathfrak{E}$ is irreducible, Schur's lemma guarantees that $\mathfrak{A}$ is cyclic and acts semiregularly. Clearly $\mathfrak{A} \subseteqq C_{\mathfrak{G}}(Z(\mathfrak{E}))$. By Lemma 2.1, $\mathfrak{E} \subseteqq C_{\mathfrak{G}}(Z(\mathfrak{E})) \subseteqq GL(p^m, q^s)$ and this is an absolutely irreducible representation of $\mathfrak{E}$. Since $\mathfrak{A}$ centralizes $\mathfrak{E}$, $\mathfrak{A}$ consists of scalar matrices here and hence $\mathfrak{A}$ is central in $C_{\mathfrak{G}}(Z(\mathfrak{E}))$.

If $p > 2$ set $\mathfrak{E}^* = \mathfrak{E}$ while if $p = 2$ we set $\mathfrak{E}^* = \mathfrak{A}^*\mathfrak{E}$ where $\mathfrak{A}^* = \{A \in \mathfrak{A} \mid A^4 = 1\}$. Then $\mathfrak{E}^*$ is also of type $E(p, m)$ and every subgroup of $\mathfrak{A}\mathfrak{E}$ of order p is in $\mathfrak{E}^*$. With the additional assumption that $m \geqq 3$ if $p = 2$, Lemma 1.5 applied to $\mathfrak{E}^*$ guarantees the existence of a point $x \in \mathfrak{V}^{\#}$ with $\mathfrak{G}_x \cap \mathfrak{E}^* = \langle 1 \rangle$. This clearly yields $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$.

Now $\mathfrak{A}_p\mathfrak{E} \triangle \mathfrak{G}$ and $|\mathfrak{A}_p\mathfrak{E}| = |\mathfrak{A}_p| p^{2m}$. If x is as above then

$$|\mathfrak{G}|_p \geqq |\mathfrak{G}_x\mathfrak{A}_p\mathfrak{E}|_p = |\mathfrak{G}_x|_p |\mathfrak{A}_p\mathfrak{E}| = |\mathfrak{G}_x|_p |\mathfrak{A}_p| p^{2m}$$

and hence $[\mathfrak{G} : \mathfrak{G}_x]_p \geqq |\mathfrak{A}_p| p^{2m}$. By half-transitivety this holds for all $x \in \mathfrak{V}^{\#}$. Combining this with the results of Lemma 2.3 and noting that $|\mathfrak{A}_p| \geqq p$ for type I and II groups and $|\mathfrak{A}_p| \geqq p^2$ for type III and IV groups, we clearly obtain our result.

LEMMA 2.5. *Let $\mathfrak{H} = C_{\mathfrak{G}}(Z(\mathfrak{E}))$. Then $\mathfrak{G}$ has the following structure.*

(i) *$\mathfrak{G}/\mathfrak{H}$ is cyclic*

(ii) *$\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ acts faithfuly on $\mathfrak{W} = \mathfrak{E}/Z(\mathfrak{E})$ and as a linear group on $\mathfrak{W}$ we have $\mathfrak{H}/\mathfrak{A}\mathfrak{E} \subseteqq Sp(2m, p)$*

(iii) *$\mathfrak{A}\mathfrak{E}/\mathfrak{A}$ is elementary abelian of order p^{2m}*

(iv) *$\mathfrak{A}$ is cyclic.*

Proof. All results but (ii) are clear. Let $\mathfrak{B} = C_{\mathfrak{H}}(\mathfrak{W})$. Clearly $\mathfrak{B} \supseteqq \mathfrak{A}\mathfrak{E}$. The result will follow from Lemma 1.3 if we show that

$\mathfrak{B} = \mathfrak{A}\mathfrak{E}$.

Suppose first that $|Z(\mathfrak{E})| = p$ so $\mathfrak{E}$ is type I or II. By Lemma 1.3, $\mathfrak{B}/\mathfrak{A} \subseteqq Z(\mathfrak{E}) \times Z(\mathfrak{E}) \times \cdots \times Z(\mathfrak{E})$ ($2m$ times). Hence $[\mathfrak{B} : \mathfrak{A}] \leqq p^{2m}$. Since $[\mathfrak{A}\mathfrak{E} : \mathfrak{A}] = p^{2m}$ we have $\mathfrak{B} = \mathfrak{A}\mathfrak{E}$ here. Now let $|Z(\mathfrak{E})| = p^2$ so $p = 2$ and $\mathfrak{E}$ is type III or IV. As above $\mathfrak{B}/\mathfrak{A} \subseteqq Z(\mathfrak{E}) \times Z(\mathfrak{E}) \times \cdots \times Z(\mathfrak{E})$ ($2m$ times) so $\mathfrak{B}/\mathfrak{A}$ is a 2-group. Since $\mathfrak{A}$ is central in $\mathfrak{H}$, $\mathfrak{B}$ is nilpotent with Sylow 2-subgroup $\mathfrak{B}_2$. Now $\mathfrak{G}$ is primitive and $\mathfrak{B}_2 \triangle \mathfrak{G}$ so $\mathfrak{B}_2$ is of symplectic type. Clearly $Z(\mathfrak{B}_2) = \mathfrak{A}_2$ and $|\mathfrak{A}_2| \geqq 4$ here. Hence $\mathfrak{B}_2$ is the central product of $\mathfrak{A}_2$ and a group of type $E(2, r)$. Thus $\mathfrak{B}_2/\mathfrak{A}_2$ has period 2 and we can conclude again that $[\mathfrak{B} : \mathfrak{A}] \leqq p^{2m}$. The result follows.

LEMMA 2.6. *We must have one of the following.*

type I: $p = 3,\ m \leqq 2$
type II: $p = 2,\ m \leqq 6$
type III: $p = 2,\ m \leqq 3$
type IV: $p = 2,\ m \leqq 5$.

Proof. We first show the following.

type I: $w \leqq p(2m - 1)|\mathfrak{A}_p|$
$w \leqq |\mathfrak{A}_p|$ for $m = 1,\ p > 3$
type II: $w \leqq p^2(2m - 1)|\mathfrak{A}_p|$
type III: $w \leqq p(2m - 1)|\mathfrak{A}_p|$
type IV: $w \leqq p(2m - 1)|\mathfrak{A}_p|$.

Now the p-period of $Sp(2m, p)$ is clearly at most $(2m - 1)p$. If $\mathfrak{E}$ is type I, III or IV, then the period of $\mathfrak{A}_p\mathfrak{E}$ is $|\mathfrak{A}_p|$. If $\mathfrak{E}$ is type II, then the period of $\mathfrak{A}_p\mathfrak{E}$ is at most $p|\mathfrak{A}_p|$. Combining these facts with the structure given in the preceding lemma yields all the above facts except for the one concerning $p > 3,\ m = 1$.

Now let $m = 1$ and $p > 3$. Let $\mathfrak{P}$ be a Sylow p-subgroup of $C_{\mathfrak{G}}(Z(\mathfrak{E}))$ and hence a Sylow p-subgroup of $\mathfrak{G}$. Thus $\mathfrak{P} \supseteqq \mathfrak{A}_p\mathfrak{E}$ and since $|Sp(2, p)|_p = p$ we have $[\mathfrak{P} : \mathfrak{A}_p\mathfrak{E}] \leqq p$. Since $\mathfrak{E}$ does not act semiregularly we have $p \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{B}^\#$. As we have seen, there exists $x \in \mathfrak{B}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$. Let $\bar{\mathfrak{P}}$ be a subgroup of $\mathfrak{G}_x$ of order p. By taking a suitable conjugate of $\mathfrak{P}$ if necessary we can assume that $\bar{\mathfrak{P}} \subseteqq \mathfrak{P}$. Then $\mathfrak{P} = \mathfrak{A}_p(\mathfrak{E}\bar{\mathfrak{P}})$. Now $|\mathfrak{E}\bar{\mathfrak{P}}| = p^4$ and this group is generated by elements of order p. Hence if $p > 3$, then $\mathfrak{E}\bar{\mathfrak{P}}$ has period p. Since $\mathfrak{A}_p$ is central in $\mathfrak{P}$ we see that $\mathfrak{P}$ has period $|\mathfrak{A}_p|$ and the above follows.

Combining the above with the lower bound for w given in Lemma 2.4 yields the following equations.

type I: $p^m|\mathfrak{A}_p| \leqq p(2m - 1)|\mathfrak{A}_p|$
$p|\mathfrak{A}_p| \leqq |\mathfrak{A}_p|$ for $m = 1,\ p > 3$
type II: $p^{m-1}|\mathfrak{A}_p| \leqq p^2(2m - 1)|\mathfrak{A}_p|$
type III: $p^m|\mathfrak{A}_p| \leqq p(2m - 1)|\mathfrak{A}_p|$

type IV: $p^{m-1}|\mathfrak{A}_p| \leqq p(2m-1)|\mathfrak{A}_p|$.

Note that the equations for types II, III and IV hold only for $m \geqq 3$. The result now follows easily.

We note that the above yields a stronger result than Proposition 2.1 of [6] and the proof is considerably less computational. We now strengthen the above argument to eliminate additional cases. We first eliminate $p = 3$.

LEMMA 2.7. *$p = 3$, $m = 1$ does not occur.*

Proof. Suppose $p = 3$ and $m = 1$. Then $\mathfrak{G}$ has the structure described in Lemma 2.5. In addition, $[\mathfrak{G} : C_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))] = 1$ or 2 and $Sp(2m, p) = SL(2, p)$. By Lemma 2.4, $\mathfrak{A}$ is central in $C_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E})) = \mathfrak{H}$.

Suppose that $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ has a normal Sylow 3-subgroup $\mathfrak{B}/\mathfrak{A}\mathfrak{E}$. Then $\mathfrak{B}/\mathfrak{A}$ is a normal Sylow 3-subgroup of $\mathfrak{G}/\mathfrak{A}$. Now both $\mathfrak{G}$ and $\mathfrak{A}$ act half-transitively so by Lemma 1.7 $\mathfrak{B}$ acts half-transitively on $\mathfrak{V}^{\#}$. Since $\mathfrak{A}$ is central in $\mathfrak{B}$, $\mathfrak{B}$ is nilpotent and hence its normal Sylow 3-subgroup $\mathfrak{B}_3$ acts half-transitively. By Theorem II of [4], $\mathfrak{B}_3$ is cyclic, a contradiction since $\mathfrak{B}_3 \supseteqq \mathfrak{E}$. Hence $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ is a subgroup of $SL(2, 3)$ which does not have a normal Sylow 3-subgroup. This implies that $\mathfrak{H}/\mathfrak{A}\mathfrak{E} \cong SL(2, 3)$, a group of order 24.

We show now that we cannot have $8 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{V}^{\#}$. Assume by way of contradiction that this is the case. Let $\mathfrak{P}$ be a subgroup of $\mathfrak{E}$ of order 3 having a fixed point $y \neq 0$. Since $\mathfrak{A}_y = \langle 1 \rangle$ we see that $8 \mid |\mathfrak{A}\mathfrak{G}_y/\mathfrak{A}|$ so $4 \mid |\mathfrak{A}\mathfrak{H}_y/\mathfrak{A}|$. Now a Sylow 2-subgroup of $\mathfrak{H}/\mathfrak{A}$ is quaternion of order 8 so $\mathfrak{H}_y$ has an element B of order 4. Since $B^2 \notin \mathfrak{A}\mathfrak{E}$, B does not normalize $\mathfrak{P}\boldsymbol{Z}(\mathfrak{E})/\boldsymbol{Z}(\mathfrak{E})$. Thus $\mathfrak{E} = \langle \mathfrak{P}, \mathfrak{P}^B \rangle \subseteqq \mathfrak{G}_y$, a contradiction since $\boldsymbol{Z}(\mathfrak{E})$ acts semiregularly.

Let $\mathfrak{P}$ be a subgroup of $\mathfrak{G}$ of order 3. We show that $\dim C_{\mathfrak{V}}(\mathfrak{P}) = 0$ or s. Since $\mathfrak{P} \subseteqq GL(3, q^s)$ we see that $\dim C_{\mathfrak{V}}(\mathfrak{P}) = 0, s$ or $2s$. Suppose the dimension is $2s$. By Lemma 1.4, $\mathfrak{P} \not\subseteqq \mathfrak{A}\mathfrak{E}$. Since $\mathfrak{H}/\mathfrak{A}\mathfrak{E} \cong SL(2, 3)$ there exists $G \in \mathfrak{G}$ such that $\mathfrak{P}$ and $\mathfrak{P}^G$ generate this quotient. Now $\mathfrak{V}$ is 3-dimensional over $GF(q^s)$ and $C_{\mathfrak{V}}(\mathfrak{P})$ and $C_{\mathfrak{V}}(\mathfrak{P}^G)$ are 2-dimensional subspaces. Thus there exists $x \in \mathfrak{V}^{\#}$ with $\mathfrak{P}, \mathfrak{P}^G \subseteqq \mathfrak{G}_x$. This implies that $24 \mid |\mathfrak{G}_x|$ and this contradicts the comments of the preceding paragraph.

We now proceed to count. The group $\mathfrak{G}/\mathfrak{A}$ is easily seen to contain at most 40 subgroups of order 3. If $\mathfrak{P}$ is a group of order 3 in $\mathfrak{G}$, then $\mathfrak{P}\mathfrak{A}$ being abelian has at most 3 subgroups of order 3 other than $\boldsymbol{Z}(\mathfrak{E})$. Hence $\mathfrak{G}$ has at most $3.40 = 120$ subgroups of order 3 other than $\boldsymbol{Z}(\mathfrak{E})$. Each such $\mathfrak{P}$ fixes at most $q^s - 1$ points of $\mathfrak{V}^{\#}$ so since clearly $3 \mid |\mathfrak{G}_x|$ we have

$$120(q^s - 1) \geqq |\mathfrak{V}^{\#}| = q^{3s} - 1$$

and

$$120 \geqq q^{2s} + q^s + 1 .$$

Thus $q^s \leqq 10$. However by Lemma 2.4, $3^2 \mid (q^s - 1)$ so q^s being a prime power is at least 19, a contradiction. Thus $p = 3$, $m = 1$ does not occur.

LEMMA 2.8. *$p = 3$, $m = 2$ does not occur.*

Proof. The equation obtained in the proof of Lemma 2.6 is an equality at $p = 3$, $m = 2$. Thus all inequalities used in obtaining it must also be equalities. Thus from Lemma 2.4 we must have $w = p^m \mid \mathfrak{A}_p \mid$. Furthermore if $x \in \mathfrak{V}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$, then $\mid \mathfrak{G} \mid_p = \mid \mathfrak{G}_x \mid_p \mid \mathfrak{A}_p\mathfrak{E} \mid$.

The latter fact implies that $\mathfrak{A}_p\mathfrak{E}$ has a complement $\mathfrak{L}$ in $\mathfrak{P}$ a Sylow p-subgroup of $\mathfrak{G}$. Since $\mathfrak{L} \subseteqq Sp(4, 3)$, $\mathfrak{L}$ has period at most $(2m - 1)p = 9$ and thus $\mathfrak{E}\mathfrak{L}$ has period at most $3.9 = 27$. Since $\mathfrak{P} = \mathfrak{A}_p(\mathfrak{E}\mathfrak{L})$ and $\mathfrak{A}_p$ is central here, we have clearly $w \leqq \max \{\mid \mathfrak{A}_p \mid, p^3\}$. But $w = p^2 \mid \mathfrak{A}_p \mid$ so we must have $\mid \mathfrak{A}_p \mid = p$ and $\mathfrak{L}$ has period 9.

Let $\mathfrak{J} = \langle J \rangle$ be a subgroup of order 9 with $\mathfrak{J} \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$. We see clearly that the Jorden form of the matrix of J with respect to its action on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Thus $\mathfrak{J}$ has

$$(3^4 - 3^3)/3^2 + (3^3 - 3)/3 + 3 = 17$$

orbits on $\mathfrak{W}$. Note that $\mathfrak{E}\mathfrak{J} \subseteqq GL(p^m, q^s)$ and the restriction to $\mathfrak{E}$ is absolutely irreducible. Thus if $a_0 = \dim_{GF(q^s)} C_{\mathfrak{V}}(J)$ then by Lemma 1.6

$$a_0 + a_1 + \cdots + a_8 = p^m = 9 .$$
$$a_0^2 + a_1^2 + \cdots + a_8^2 \leqq 17 .$$

These yield easily $a_0 \leqq 3$ and hence $\dim_{GF(q)} C_{\mathfrak{V}}(J) = sa_0 \leqq 3s$.

Let $\mathscr{N}$ denote the set of subgroups of $\mathfrak{E}$ of order 3 together with the set of cyclic subgroup $\mathfrak{J}$ of order 9 with $\mathfrak{J} \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$. By the above and Lemma 1.4, if $\mathfrak{N} \in \mathscr{N}$ then $\dim C_{\mathfrak{V}}(\mathfrak{N}) \leqq 3s$. We have also shown above that for all $y \in \mathfrak{V}$ there exists $\mathfrak{N} \in \mathscr{N}$ with $y \in C_{\mathfrak{V}}(\mathfrak{N})$, since in that argument, if $\mathfrak{G}_x \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$ then $\mathfrak{J} = \mathfrak{L} \subseteqq \mathfrak{G}_x$. Hence $\mathfrak{V} = \bigcup_{\mathscr{N}} C_{\mathfrak{V}}(\mathfrak{N})$. If $\mid \mathscr{N} \mid = N$, then this yields

$$q^{9s} = \mid \mathfrak{V} \mid \leqq Nq^{3s}$$

or $q^{6s} \leqq N$. On the other hand by Lemma 2.5,

$$\begin{aligned} |\mathfrak{G}| &\leqq 2\,|\mathfrak{A}|p^4|Sp(4, p)| \\ &\leqq 2\,|\mathfrak{A}|p^4p^4(p^4-1)(p^2-1) \leqq 2\,|\mathfrak{A}|p^{14} . \end{aligned}$$

Since $\mathfrak{A}$ is central in the absolutely irreducible representation $\mathfrak{A}\mathfrak{E} \subseteqq GL(p^m, q^s)$ we have $|\mathfrak{A}| < q^s$. Thus

$$N \leqq |\mathfrak{G}|/2 < q^s p^{14} .$$

Combining this with the lower bound we previously obtained for N yields $q^{5s} < p^{14}$. Finally by Lemma 2.4, $q^s \geqq p^3$ so $p^{15} < q^{5s} < p^{14}$, a contradiction. Thus $p = 3$, $m = 2$ does not occur.

We now consider special cases with $p = 2$.

LEMMA 2.9. *The cases* type II, $m = 6$, type III, $m = 3$ *and* type IV, $m = 5$ *do not occur.*

Proof. If we consider the inequalities obtained in the proof of Lemma 2.6, we see that any of the above mentioned cases would be eliminated if a strengthening of the inequalities by a factor of $p = 2$ could be obtained. Let us suppose that one of the above occurs.

The results of Lemma 2.4 concerning $[\mathfrak{G} : \mathfrak{G}_x]_p$ and w must be equalities. In particular this implies that for given $x \in \mathfrak{B}^\sharp$ with $\mathfrak{G}_x \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$ we must have $|\mathfrak{G}|_p = |\mathfrak{G}_x\mathfrak{A}_p\mathfrak{E}|_p$. Thus $\mathfrak{A}_p\mathfrak{E}$ has a complement in a Sylow p-subgroup of $\mathfrak{G}$ and thus also in $\mathfrak{P}$, a Sylow p-subgroup of $\mathfrak{H}$. Let $\mathfrak{P} = \mathfrak{A}_p\mathfrak{E}\mathfrak{L}$ where $\mathfrak{L} \cap \mathfrak{A}_p\mathfrak{E} = \langle 1 \rangle$ and let w^* denote the period of the group $\mathfrak{E}\mathfrak{L}/\mathfrak{E}'$. Since $\mathfrak{A}_p$ is central in $\mathfrak{P}$ we have

$$w \leqq \max\{|\mathfrak{A}_p|, 2w^*\} \leqq \begin{cases} |\mathfrak{A}_p|\,w^* & \text{type II} \\ \dfrac{1}{2}|\mathfrak{A}_p|\,w^* & \text{types III, IV} . \end{cases}$$

We consider w^*. Let $\mathfrak{W}^* = \mathfrak{E}/\mathfrak{E}'$ so $\mathfrak{W}^*$ is elementary abelian of order p^{2m} or p^{2m+1}. Since $\mathfrak{L}$ acts faithfully on $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$, it also acts faithfully on $\mathfrak{W}^*$. If $\mathfrak{L}$ has period p^d, then $w^* = p^d$ or p^{d+1}. Note that $\mathfrak{L} \subseteqq GL(2m+1, p)$. If $w^* = p^d$, then since $p^d \leqq p(2m)$ we have $w^* \leqq p(2m)$. If $w^* = p^{d+1}$, then there must exist an element $L \in \mathfrak{L}$ of order p^d whose minimal polynomial in $GL(2m+1, p)$ has degree p^d. Thus we must have $p^d \leqq 2m+1$ and $w^* \leqq p(2m+1)$. The latter bound being the larger of the two holds in all cases. Now w^* is a power of 2 and in the three cases we are considering neither $2m+1$ nor $2m$ is a power of 2. Hence we have $w^* \leqq p(2m-1)$ and

$$\begin{aligned} &w \leqq p(2m-1)\,|\mathfrak{A}_p| && \text{for type II} \\ &w \leqq (2m-1)\,|\mathfrak{A}_p| && \text{for types III, IV} . \end{aligned}$$

This therefore improves the bounds on w given in the proof of Lemma 2.6 by a factor of $p = 2$ and, as we mentioned above, this yields a contradiction.

LEMMA 2.10. *The case* type IV, $m = 4$ *does not occur.*

Proof. We see that in the inequalities obtained in the proof of Lemma 2.6, a strengthening by a factor of $p = 2$ would eliminate this possibility. Hence if this case occurs, then we must have the following. If $x \in \mathfrak{V}^{\#}$, then either x is fixed by a subgroup of $\mathfrak{E}$ of order 2 or a cyclic subgroup $\mathfrak{J} \subseteqq \mathfrak{H}$ of order 8 with $\mathfrak{J} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Let $\mathscr{N}$ denote collection of such subgroups of both types.

We show now that if $\mathfrak{J} \in \mathscr{N}$ then $\dim C_{\mathfrak{V}}(\mathfrak{J}) \leqq n/2$. We know this to be the case if $\mathfrak{J} \subseteqq \mathfrak{E}$ so suppose $\mathfrak{J} = \langle J \rangle$ has order 8. Then $\mathfrak{J}$ acts faithfully on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. Since $|\mathfrak{J}| = 8$ we see that in its action on $\mathfrak{W}$, J must have one Jordan block of rank at least 5. This implies easily that $\mathfrak{J}$ has at most

$$\frac{2^8 - 2^7}{8} + \frac{2^7 - 2^5}{4} + \frac{2^5 - 2^4}{2} + 2^4 = 2^6$$

orbits on $\mathfrak{W}$. We apply Lemma 1.6 to each of the two absolutely irreducible constituents of $\mathfrak{E}\mathfrak{J}$ on $\mathfrak{V} \otimes GF(q^2)$. Hence

$$a_0^2 + a_1^2 + \cdots + a_7^2 \leqq 2^6 .$$

Thus $a_0 \leqq 8$ and since $\dim C_{\mathfrak{V}}(\mathfrak{J})$ is invariant under field extension we have $\dim C_{\mathfrak{V}}(\mathfrak{J}) \leqq 2a_0 \leqq n/2$. Now

$$\mathfrak{V} = \bigcup_{\mathfrak{J} \in \mathscr{N}} C_{\mathfrak{V}}(\mathfrak{J})$$

and if $N = |\mathscr{N}|$, then $q^n = |\mathfrak{V}| \leqq Nq^{n/2}$ and $q^{n/2} \leqq N$. By Lemma 2.5

$$|\mathfrak{H}| \leqq |\mathfrak{A}|\, 2^{2m}\, |Sp(2m, 2)| .$$

Since $|\mathfrak{A}| \leqq q^s$ and $|Sp(8, 2)| \leqq 2^{36}$ we have $N \leqq |\mathfrak{G}| \leqq q^s \cdot 2^{44}$. With $n = 2^m s = 16s$ this yields

$$q^{8s} = q^{n/2} \leqq N \leqq q^s \cdot 2^{44}$$

or $q^{7s} \leqq 2^{44}$. Now $s = 2$ and by Lemma 2.4, $2^5 = 2^{m+1}$ divides $q^2 - 1 = q^s - 1$. Since $s = 2$ and $q^s > 9$ it follows (see for example Lemma 4 of [4]) that $q^s - 1$ cannot be a power of 2. Hence $q^s > q^s - 1 \geqq 3 \cdot 2^5$ so $q^{7s} > 3^7 \cdot 2^{35}$. Combining this with the above yields $3^7 \cdot 2^{35} < q^{7s} \leqq 2^{44}$ or $3^7 < 2^9$, a contradiction. Therefore this case does not occur.

LRMMA 2.11. *The case* type II, $m = 5$ *does not occur.*

Proof. In the inequality in the proof of Lemma 2.6 for type II, $m = 5$ we see that a strengthening by a factor of $p^2 = 4$ will yield a contradiction. Hence if $x \in \mathfrak{V}^{\#}$ is such that $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ and if $\mathfrak{P}$ is a Sylow 2-subgroup of $\mathfrak{G}$ extending one of $\mathfrak{G}_x$, then either (a) $\mathfrak{P}_x\mathfrak{E} = \mathfrak{P}$ and $w \geqq 32$ or (b) $[\mathfrak{P} : \mathfrak{P}_x\mathfrak{E}] = 2$ and $w \geqq 64$. In the latter case $\mathfrak{P}_x\mathfrak{E} \triangle \mathfrak{P}$ so in both cases $\mathfrak{P}_x\mathfrak{E}$ has period $\geqq 32$ and $\mathfrak{P}_x$ has period $\geqq 8$. Note $|\mathfrak{A}_2| = 2$ here by Lemma 2.6.

Let $\mathfrak{J} = \langle J \rangle$ be a cyclic subgroup of $\mathfrak{P}_x$ of order 8 and let $a_0 = \dim_{GF(q)} C_{\mathfrak{V}}(J)$. Since $\mathfrak{J}$ acts faithfully on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ and $|\mathfrak{J}| = 8$ we see that J must have one Jordan block of rank at least 5. This implies easily that $\mathfrak{J}$ has at most

$$\frac{2^{10} - 2^9}{8} + \frac{2^9 - 2^7}{4} + \frac{2^7 - 2^6}{2} + 2^6 = 2^8$$

orbits on $\mathfrak{W}$. Hence by Lemma 1.6

$$a_0 + a_1 + \cdots + a_7 = p^m = 32$$
$$a_0^2 + a_1^2 + \cdots + a_7^2 \leqq 2^8 .$$

Thus $a_0 < 2^4 = 16$ and $|C_{\mathfrak{V}}(J)| = q^{a_0} \leqq q^{15}$.

Now if $\mathfrak{T}$ is a subgroup of $\mathfrak{A}\mathfrak{E}$ of order 2 then $|C_{\mathfrak{V}}(\mathfrak{T})| \leqq q^{n/2} = q^{16}$. We have shown that with the above notation

$$\mathfrak{V} = \bigcup_{\mathfrak{J}} C_{\mathfrak{V}}(\mathfrak{J}) \cup \bigcup_{\mathfrak{T}} C_{\mathfrak{V}}(\mathfrak{T}) .$$

Now $\mathfrak{A}$ is cyclic and central and by Lemma 2.6, $4 \nmid |\mathfrak{A}|$. Hence the number of choices for $\mathfrak{T}$ is at most $|\mathfrak{E}| = 2^{11}$ and the number of choices for $\mathfrak{J}$ is at most $1/4\,|\mathfrak{G}/\mathfrak{A}_{2'}|$. Here $\mathfrak{A}_{2'}$ is the normal 2-complement of $\mathfrak{A}$ and the 1/4 factor comes from the fact that $\mathfrak{J}$ has four distinct generators. Since $|\mathfrak{G}/\mathfrak{A}_{2'}| \leqq |\mathfrak{E}|\ |Sp(10, 2)| \leqq 2^{66}$, the above union yields

$$q^{32} = |\mathfrak{V}| \leqq 2^{66}q^{15}/4 + 2^{11}q^{16} .$$

Putting $q^{15} < q^{16}/2$ in the above we have

$$q^{32} < (2^{63} + 2^{11})q^{16} < 2^{64}q^{16}$$

so $q^{16} < 2^{64}$ and $q < 2^4 = 16$. On the other hand by Lemma 2.4 $2^5 \mid q^2 - 1$ so $16 \mid q \pm 1$. This yields $q \geqq 17$, a contradiction.

The following partial result will be completed later under the additional assumption of solvability.

LEMMA 2.12. *In the case* type II, $m = 4$ *we have* $q \geqq 7$ *and* $|\mathfrak{G}/\mathfrak{A}\mathfrak{E}| > 10^4$.

Proof. In the inequalities of the proof of Lemma 2.6 for type II, $m = 4$, we sée that a strengthening by a factor of $p^2 = 4$ will yield a contradiction. Suppose $x \in \mathfrak{V}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ and let $\mathfrak{P}$ be a Sylow 2-subgroup of $\mathfrak{G}$ extending one of $\mathfrak{G}_x$. Using the same argument as in the preceding lemma we conclude that $\mathfrak{P}_x\mathfrak{E}$ has period $\geqq 16$ and hence $\mathfrak{P}_x\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ has period $\geqq 8$.

Suppose first that $\mathfrak{P}_x$ has a cyclic subgroup $\mathfrak{J} = \langle J \rangle$ of order 8. Then in its action on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$, J has a Jordan block of rank at least 5 so $\mathfrak{J}$ has at most

$$\frac{2^8 - 2^7}{8} + \frac{2^7 - 2^5}{4} + \frac{2^5 - 2^4}{2} + 2^4 = 64$$

orbits on $\mathfrak{W}$. By Lemma 1.6 if $a_0 = \dim \boldsymbol{C}_{\mathfrak{V}}(J)$ then

$$a_0^2 + a_1^2 + \cdots + a_7^2 \leqq 64$$

and $a_0 \leqq 8$.

Now suppose $\mathfrak{P}_x$ has period 4. Then since $\mathfrak{P}_x\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ has period 8, $\mathfrak{P}_x$ must contain an element J of order 4 with a Jordan block of rank 4. If $\mathfrak{J} = \langle J \rangle$, then $\mathfrak{J}$ has at most

$$\frac{2^8 - 2^6}{4} + \frac{2^6 - 2^5}{2} + 2^5 = 96$$

orbits on $\mathfrak{W}$. By Lemma 1.6 if $a_0 = \dim \boldsymbol{C}_{\mathfrak{V}}(J)$ then

$$a_0 + a_1 + a_2 + a_3 = p^m = 16$$
$$a_0^2 + a_1^2 + a_2^2 + a_3^2 \leqq 96 \; .$$

It is easy to see that the possibility $a_0 = 9$ is excluded and hence in both cases $a_0 \leqq 8$.

We have

$$\mathfrak{V} = \bigcup \boldsymbol{C}_{\mathfrak{V}}(\mathfrak{J}) \cup \bigcup \boldsymbol{C}_{\mathfrak{V}}(\mathfrak{T})$$

where the subgroups $\mathfrak{J}$ are as above and the subgroups $\mathfrak{T}$ have order 2 and are contained in $\mathfrak{E}$. This follows since $4 \nmid |\mathfrak{A}|$ by Lemma 2.6. The number of choices for $\mathfrak{J}$ or $\mathfrak{T}$ is clearly at most $|\mathfrak{G}/\mathfrak{A}_{2'}|$ where $\mathfrak{A}_{2'}$ is the normal 2-complement of $\mathfrak{A}$. Since $4 \nmid |\mathfrak{A}|$ we have $|\mathfrak{G}/\mathfrak{A}_{2'}| = 2^9 |\mathfrak{G}/\mathfrak{A}\mathfrak{E}|$. Therefore the above union yields

$$q^{16} = |\mathfrak{V}| \leqq q^8 2^9 |\mathfrak{G}/\mathfrak{A}\mathfrak{E}|$$

since $|\boldsymbol{C}_{\mathfrak{V}}(\mathfrak{J})|$ and $|\boldsymbol{C}_{\mathfrak{V}}(\mathfrak{T})|$ are both at most q^8. Thus $|\mathfrak{G}/\mathfrak{A}\mathfrak{E}| \geqq q^8/2^9$. By Lemma 2.4, $2^4 \mid q^2 - 1$ so $q \geqq 7$. This yields

$$|\mathfrak{G}/\mathfrak{A}\mathfrak{E}| \geqq 7^8/2^9 = (2401)^2/2^9 > 10^4$$

and the result follows.

We now temporarily drop the assumptions stated at the beginning of this section and prove the first of our three theorems.

Proof of Theorem A. Let $\mathfrak{G}$ be a linear group acting on vector space $\mathfrak{V}$ of order q^n and suppose that $\mathfrak{G}$ acts half-transitively but not semiregularly on $\mathfrak{V}^{\#}$. Let $\mathfrak{P} = \boldsymbol{O}_p(\mathfrak{G})$ be the maximal normal p-subgroup of $\mathfrak{G}$. By assumption $\mathfrak{G}$ is primitive so $\mathfrak{P}$ is of symplectic type. Suppose first that $p > 2$. If $\mathfrak{P}$ is not cyclic, then $\mathfrak{P}$ contains a characteristic subgroup $\mathfrak{E}$ of type $E(p, m)$. By the Reduction Lemma (Lemma 1.8) and Lemmas 2.6, 2.7 and 2.8 we have a contradiction.

Now let $p = 2$ so that $\Phi(\mathfrak{P})$ is cyclic. Suppose $[\mathfrak{P} : \Phi(\mathfrak{P})] > 2^8$. Then $\mathfrak{P}$ has a characteristic subgroup $\mathfrak{E}$ of type $E(2, m)$ with $m > 3$. Thus by the Reduction Lemma and Lemmas 2.6 through 2.11 we see that $m = 4$ and $|\boldsymbol{Z}(\mathfrak{E})| = 2$. But then $|\Phi(\mathfrak{P})| = 2$ so $\mathfrak{P} = \mathfrak{E}$ and $[\mathfrak{P} : \Phi(\mathfrak{P})] \leqq 2^8$ here also. This completes the proof.

3. **Solvable cases,** $m = 1$. We have seen in the preceding section that if $\mathfrak{E}$ is a group of type $E(p, m)$ normal in a half-transitive linear group $\mathfrak{G}$, then $p = 2$ and $m \leqq 4$. We will consider these cases in the next few sections under the additional assumption that $\mathfrak{G}$ is solvable.

For convenience we restate Lemmas 1.3 and 1.4 of [5].

LEMMA 3.1. *Suppose $\mathfrak{G}$ is an irreducible linear group of degree n over $GF(q)$ and $\mathfrak{A} = \langle \mathfrak{A} \rangle$ is a cyclic normal subgroup all of whose irreducible constituents are similar. Let ζ be an eigenvalue of A with $GF(q)(\zeta) = GF(q^r)$ and $n/r = k$. Let p be a prime and suppose that for all vectors x, $p \mid |\mathfrak{G}_x|$. Consider those subgroups $\mathfrak{P}/\mathfrak{A}$ of $\mathfrak{G}/\mathfrak{A}$ of order p for which there exists an $x \neq 0$ with $\mathfrak{P} \cap \mathfrak{G}_x \neq \langle 1 \rangle$. If λ_1 of the $\mathfrak{P}$ are contained in $\boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A})$ and λ_2 are not, then*

(i) $\dfrac{q^{kr} - 1}{q^r - 1} \leqq \lambda_1\left\{1 + \dfrac{q^{r(k-1)} - 1}{q^r - 1}\right\} + \lambda_2\left\{\dfrac{q^{rk/p} - 1}{q^{r/p} - 1}\right\}$

(ii) $q^r + 1 \leqq 2\lambda_1 + \lambda_2(q^{r/p} + 1)$ *for* $k = 2$

(iii) $q^r < 2(\lambda_1 + \lambda_2)$ *for* $k > 2$.

This is a very coarse statement which we will have to strengthen at times. The following assumptions hold throughout the remainder of this section.

ASSUMPTIONS. Group $\mathfrak{G}$ acts faithfully on vector space $\mathfrak{V}$ of order q^n and half-transitively but not semiregularly on $\mathfrak{V}^{\#}$. $\mathfrak{E}$ is a group of type $E(2, 1)$ which is normal in $\mathfrak{G}$ and acts irreducibly on $\mathfrak{V}$.

Note that we do not assume that $\mathfrak{G}$ is primitive here. The reason for this, is that part (v) of the Reduction Lemma does not guarantee primitivity in this case.

LEMMA 3.2. *Let* $\mathfrak{E} \cong \mathfrak{Q}$ *(that is,* $\mathfrak{E} = \mathrm{iso}\, I$*). Then* $q^n = 3^2, 5^2, 7^2, 11^2$ *or* 17^2.

Proof. Clearly $q^n = q^2$ and hence $C_{\mathfrak{G}}(\mathfrak{E})$ consists of scalar matrices so $C_{\mathfrak{G}}(\mathfrak{E}) = Z(\mathfrak{G})$. Note that Aut $\mathfrak{E} \cong \mathrm{Sym}_4$, the symmetric group of degree 4.

Suppose first that $3 \nmid |\mathfrak{G}/Z(\mathfrak{G})|$. Then $|\mathfrak{G}/Z(\mathfrak{G})| = 4$ or 8 and hence $\mathfrak{G}$ is nilpotent. Thus $\mathfrak{G}_2 = O_2(\mathfrak{G})$ is half-transitive. Since $O_{2'}(\mathfrak{G}) \subseteq Z(\mathfrak{G})$ acts semiregularly, we conclude that $\mathfrak{G}_2$ is not semiregular. Hence $\mathfrak{G}_2 > \mathfrak{E}$ and since $[\mathfrak{G}_2 : Z(\mathfrak{G}_2)] = 4$ or 8 we have by Theorem II of [4], $q^n = 3^2$, 5^2 or 7^2.

We assume now that $3 \mid |\mathfrak{G}/Z(\mathfrak{E})|$. We consider the possibility $3 \mid |\mathfrak{G}_x|$ first. If $\mathfrak{L}$ is a subgroup of $\mathfrak{G}$ of order 3 fixing a vector x, then $|C_{\mathfrak{B}}(\mathfrak{L})| = q$ clearly. Also either $q = 3$ or by complete reducibility $3 \mid q - 1$. Now $\mathfrak{G}/Z(\mathfrak{G})$ has at most 4 subgroups of order 3 and since $Z(\mathfrak{G})$ is cyclic, we see that $\mathfrak{G}$ contains at most $4 \cdot 3 = 12$ subgroups of order 3 not contained in $Z(\mathfrak{G})$. From $\mathfrak{B} = \bigcup C_{\mathfrak{B}}(\mathfrak{L})$ we obtain easily

$$q^2 - 1 = |\mathfrak{B}^{\#}| \leqq 12(q - 1)$$

so $q + 1 \leqq 12$. Since either $q = 3$ or $3 \mid q - 1$ we have $q = 3$ or 7 here.

We now assume that $3 \nmid |\mathfrak{G}_x|$. If $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{E} \neq \langle 1 \rangle$ for all $x \in \mathfrak{B}^{\#}$, then by Lemma 1.5 $q^n = 3^2$ or 5^2. Thus we can suppose that some $x \in \mathfrak{B}^{\#}$, $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{E} = \langle 1 \rangle$. This yields $|\mathfrak{G}_x| = 2$ and by Lemma 1.9, $I(\mathfrak{G}) = q + 1$. We have actually shown above that $|\mathfrak{G}/Z(\mathfrak{G})|$ is divisible by $3 \cdot 8$ so $\mathfrak{G}/Z(\mathfrak{G}) \cong \mathrm{Sym}_4$ and this group has two conjugacy classes of involutions, $\mathfrak{C}_1$ of size 3 and $\mathfrak{C}_2$ of size 6. If $\bar{T} \in \mathfrak{G}/Z(\mathfrak{G})$ is an involution then since $Z(\mathfrak{G})$ is cyclic of even order and central, the coset corresponding to $\bar{T}$ will contain either 0 or 2 noncentral involutions of $\mathfrak{G}$ and this number is the same for all conjugates of $\bar{T}$. Thus we have

$$q + 1 = I(\mathfrak{G}) = \delta_1 \cdot 2 \cdot 3 + \delta_2 \cdot 2 \cdot 6$$

where $\delta_1, \delta_2 = 0$ or 1. Moreover since for some $x \in \mathfrak{B}^{\#}$, $\mathfrak{G}_x \cap Z(\mathfrak{G})\mathfrak{E} = \langle 1 \rangle$ we have $\delta_2 = 1$. Thus $q + 1 = 6\delta_1 + 12$ and $q = 11$ or 17. This completes the proof.

LEMMA 3.3. *Let* $\mathfrak{E} \cong \mathfrak{D}$ *(that is,* $\mathfrak{E} = \mathrm{iso\, II}$*). Then* $q^n = 3^2$, 5^2 *or* 7^2.

Proof. Clearly $q^n = q^2$ so $C_{\mathfrak{G}}(\mathfrak{E}) = Z(\mathfrak{G})$ consists of scalar matrices.

Now $|\operatorname{Aut}\mathfrak{E}| = 8$ so $[\mathfrak{G} : \boldsymbol{Z}(\mathfrak{G})] = 4$ or 8 and hence $\mathfrak{G}$ is nilpotent. Then $\mathfrak{G}_2 = \boldsymbol{O}_2(\mathfrak{G})$ is half-transitive but not semiregular and $[\mathfrak{G}_2 : \boldsymbol{Z}(\mathfrak{G}_2] = 4$ or 8. By Theorem II of [4], $q^n = q^2 = 3^2$, 5^2 or 7^2.

LEMMA 3.4. *Let* $\mathfrak{E} \cong \mathfrak{Z}\mathfrak{Q}$ *(that is,* $\mathfrak{E} =$ iso III*). Then* $q^n = 5^2, 17^2$ *or* $\mathfrak{G}$ *is imprimitive and* $q^n = 3^4$.

Proof. Here $q^n = q^2$ if $q \equiv 1 \bmod 4$ and $q^n = q^4$ if $q \equiv -1 \bmod 4$. Say $q^n = q^{2r}$.

Suppose first that $\mathfrak{G}$ is imprimitive. Here we can apply Theorem 1.1. Note that if $q^r - 1$ is not a power of 2 then $\boldsymbol{O}_2(\mathscr{T}_0(q^r))$ is abelian. Hence by Theorem 1.1 either $q^n = 3^4$ or $\mathfrak{G} = \mathscr{T}_0(q)$ for Fermat prime q. Here $q \equiv 1 \bmod 4$ so $q \geqq 5$. Let $\mathfrak{B}$ be the diagonalized subgroup of $\mathfrak{G}$ of index 2 so $\mathfrak{B}$ is abelian. Then $\mathfrak{G} = \mathfrak{B}\mathfrak{E}$ and $\mathfrak{G}' = (\mathfrak{B}, \mathfrak{E}) \subseteqq \mathfrak{E}$. Since $\mathfrak{G}'$ is cyclic of order $(q-1)/2$ we have $(q-1)/2 \leqq 4$ so $q \leqq 9$ and hence since q is a Fermat prime $q = 5$ and $q^n = 5^2$.

Now we assume that $\mathfrak{G}$ is primitive and we use the notation of Lemma 2.5. Then $[\mathfrak{G} : \mathfrak{H}] = 1$ or 2 where $\mathfrak{H} = \boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))$ and $\mathfrak{H}/\mathfrak{A}\mathfrak{E} \subseteqq Sp(2, 2) = SL(2, 2)$, a group of order 6. Now $\mathfrak{E}$ has precisely 3 abelian subgroups of order 8 and these are not cyclic. Since $\mathfrak{G}$ is primitive none of these groups is normal. Hence $\mathfrak{G}$ permutes these transitively so $3 \mid |\mathfrak{H}/\mathfrak{A}\mathfrak{E}|$. Now $\mathfrak{G}/\mathfrak{A}\mathfrak{E}$ also acts on $\mathfrak{E}/\mathfrak{E}'$ and this action is clearly faithful on $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$. If $\bar{\mathfrak{F}} = \mathfrak{F}/\mathfrak{A}\mathfrak{E}$ is the normal 3-subgroup of $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ then $\bar{\mathfrak{F}}$ centralizes $\boldsymbol{Z}(\mathfrak{E})/\mathfrak{E}'$ and acts faithfully on the commutator $\mathfrak{E}_0/\mathfrak{E}'$, a 2-dimensional complement. Clearly $\mathfrak{E}_0 \cong \mathfrak{Q}$ and $\mathfrak{E}_0 \triangle \mathfrak{G}$. If $n = 2$, then by Lemma 3.2 and the fact that $q \equiv 1 \bmod 4$ we have $q^n = 5^2$ or 17^2.

Let $n = 4$ so $q \equiv -1 \bmod 4$. $\mathfrak{G}/\mathfrak{A}$ acts on $\mathfrak{E}_0$ and the kernel acts faithfully on $\boldsymbol{Z}(\mathfrak{E})$. Thus we see that either $\mathfrak{G}/\mathfrak{A} \subseteqq \operatorname{Aut}\mathfrak{E}_0 = \mathrm{Sym}_4$ or $\mathfrak{G}/\mathfrak{A} \subseteqq \mathfrak{H}/\mathfrak{A} \times \mathfrak{J}/\mathfrak{A} \subseteqq \mathrm{Sym}\, 4 \times \bar{\mathfrak{J}}$ where $|\bar{\mathfrak{J}}| = |\mathfrak{J}/\mathfrak{A}| = 2$. We apply Lemma 3.1 with $p = 2$. We have clearly $\lambda_1 \leqq 9$, $\lambda_2 \leqq 10$ and since $r = 2$, $k = 2$, $n = 4$ we obtain

$$q^2 + 1 \leqq 18 + 10(q + 1)$$

or $q(q - 10) \leqq 27$ so $q < 13$. Since $q \equiv 3 \bmod 4$ we have $q \equiv 3$, 7 or 11. Suppose $3 \mid |\mathfrak{G}_x|$. Let T be a noncentral involution of $\mathfrak{E}$. By Lemma 1.5 there exists a point $x \in \mathfrak{B}^*$ with $\mathfrak{E}_x = \langle T \rangle$. Let $\mathfrak{L}$ be a subgroup of $\mathfrak{G}_x$ of order 3. Then $\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$, $\mathfrak{L} \subseteqq \mathfrak{H}$ and $\mathfrak{L}$ normalizes $\mathfrak{G}_x \cap \mathfrak{E} = \mathfrak{E}_x$, a contradiction since $\mathfrak{L}$ acts irreducibly on $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. Hence $3 \nmid |\mathfrak{G}_x|$ and since $3 \mid |\mathfrak{G}|$ we conclude that $q \neq 3$.

Let $q = 7$ or 11. By Lemma 1.5 there exists a point $x \in \mathfrak{B}^{\#}$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Since $3 \nmid |\mathfrak{G}_x|$ we see that $|\mathfrak{G}_x| = 2$ or 4. Suppose $|\mathfrak{G}_x| = 4$. Then certainly $2 \mid |\mathfrak{H}_x|$ for all $x \in \mathfrak{B}^{\#}$ and Lemma 3.1

applies to $\mathfrak{H}$. Here $\lambda_1 \leqq 9$, $\lambda_2 = 0$, $r = 2$, $n = 4$, $k = 2$ so

$$q^2 + 1 \leqq 2\cdot 9 + 0 ,$$

a contradiction. Thus $|\mathfrak{G}_x| = 2$ and by Lemma 1.9, $I(\mathfrak{G}) = q^2 + 1$. Let $\mathfrak{L}$ be a Sylow 3-subgroup of $\mathfrak{G}$. Then $\mathfrak{L}$ permutes by conjugation the noncentral involutions of $\mathfrak{G}$. Since $3 \nmid (q^2 + 1)$, $\mathfrak{L}$ must centralize such an involution. Now subgroups of Sym_4 of order 3 are self-centralizing so this implies that $\mathfrak{G}/\mathfrak{A} \not\subseteq \mathrm{Sym}_4$. Hence $\mathfrak{G}/\mathfrak{A} \subseteq \mathfrak{H}/\mathfrak{A} \times \mathfrak{Z}/\mathfrak{A}$ where $\mathfrak{H}/\mathfrak{A} \subseteq \mathrm{Sym}_4$ and $|\mathfrak{Z}/\mathfrak{A}| = 2$. Clearly $\mathfrak{H}/\mathfrak{A} \supseteq \mathrm{Alt}_4$ and if $\mathfrak{H}/\mathfrak{A} \cong \mathrm{Alt}_4$ then in the notation of Lemma 3.1 with $p = 2$, $\lambda_1 \leqq 3$, $\lambda_2 \leqq 4$ and

$$(q^2 + 1) \leqq 2\lambda_1 + (q + 1)\lambda_2 \leqq 6 + 4(q + 1)$$

a contradiction for $q = 7,11$. Hence $\mathfrak{H}/\mathfrak{A} \cong \mathrm{Sym}_4$ and $\mathfrak{G}/\mathfrak{A}$ has five classes $\mathfrak{C}_i$ of involutions. These satisfy $\mathfrak{C}_1, \mathfrak{C}_2 \subseteq \mathfrak{H}/\mathfrak{A}$ with $|\mathfrak{C}_1| = 3$, $|\mathfrak{C}_2| = 6$ and $C_3, \mathfrak{C}_4, \mathfrak{C}_5 \not\subseteq \mathfrak{H}/\mathfrak{A}$ with $|\mathfrak{C}_3| = 1$, $|\mathfrak{C}_4| = 3$, $|\mathfrak{C}_5| = 6$.

Let $\bar{T}$ be an involution of $\mathfrak{G}/\mathfrak{A}$. If the coset of $\bar{T}$ contains α involutions, then the same is true for all conjugates of $\bar{T}$. If $\bar{T} \in \mathfrak{H}/\mathfrak{A}$ then certainly $\alpha = 0$ or 2. If $T \notin \mathfrak{H}/\mathfrak{A}$, then by Lemma 1.1 of [5] $\bar{T}$ acts on $\mathfrak{A}$ like a field automorphism of $GF(q^2)$ of order 2 (that is, the map $x \rightarrow x^q$). Suppose the coset contains an involution T. Then for $B \in \mathfrak{A}$, BT is an involution if and only if $B^q = B^T = B^{-1}$. Hence $\alpha = 0$ or the number N of elements of $\mathfrak{A}$ of order dividing $q + 1$. Note that since $|\boldsymbol{Z}(\mathfrak{E})| = 4$ we have $N = 4$ or 8 for $q = 7$ and $N = 4$ or 12 for $q = 11$. Now if $\delta_i = 1$ or 0 according to whether the coset of $\bar{T} \in \mathfrak{C}_i$ contains an involution of $\mathfrak{G}$ then we obtain

$$q^2 + 1 = I(\mathfrak{G}) = 6\delta_1 + 12\delta_2 + N(\delta_3 + 3\delta_4 + 6\delta_5) .$$

Considering this modulo 3 we have

$$2 \equiv q^2 + 1 \equiv \mathrm{N}\delta_3 \bmod 3 .$$

This shows that $q \neq 11$. If $q = 7$ then $N = 8$ so $8 \mid |\mathfrak{A}|$ and $\delta_3 = 1$. Furthermore $\delta_5 = 0$ and then $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1$.

Since $\delta_2 = 1$ we can find an involution $T \in \mathfrak{H}$ corresponding to a transposition in $\mathfrak{H}/\mathfrak{A} \cong \mathrm{Sym}_4$. Now T normalized $\mathfrak{E}_0 \cong \mathfrak{Q}$ as mentioned before and T does not fix $\mathfrak{E}_0/\mathfrak{E}_0'$ since T does not fix $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. Thus $\langle \mathfrak{E}_0, T \rangle$ is a maximal class group of order 16 and hence this group has a cyclic subgroup $\mathfrak{B}$ of order 8. The group $\mathfrak{A}_2\mathfrak{B}$ is abelian and has period $|\mathfrak{A}_2|$ since $\mathfrak{B} \subseteq \mathfrak{H}$ and $|\mathfrak{A}_2| \geqq |\mathfrak{B}|$. Also $|\mathfrak{B} \cap \mathfrak{A}_2| = 2$ so $|\mathfrak{A}_2\mathfrak{B}| = 4|\mathfrak{A}_2|$. Let $\mathfrak{U} \subseteq \mathfrak{V}$ be an irreducible $\mathfrak{A}_2\mathfrak{B}$-submodule and let $\mathfrak{K} \subseteq \mathfrak{A}_2\mathfrak{B}$ be the kernel. Then $\mathfrak{A}_2\mathfrak{B}/\mathfrak{K}$ is cyclic so $|\mathfrak{A}_2\mathfrak{B}/\mathfrak{K}| \leqq |\mathfrak{A}_2|$ and hence $|\mathfrak{K}| \geqq 4$. If $x \in \mathfrak{U}^\#$, then $\mathfrak{G}_x \supseteqq \mathfrak{K}$ and $|\mathfrak{G}_x| \geqq 4$, a contradiction. This completes the proof.

EXAMPLES. The examples with $q^n = 3^2, 5^2, 7^2$ and 11^2 can occur as transitive groups and these are given in [3]. We consider the case $q^n = 17^2$. Let $SL(2, 17)^*$ denote the subgroup of $GL(2, 17)$ consisting of those matrices with determinant ± 1. Let $\mathfrak{H} = \mathfrak{Q}\mathfrak{W}$ where $\mathfrak{Q}$ is the quaternion group of order 8, $\mathfrak{Q} \triangle \mathfrak{H}$ and $\mathfrak{W} \cong \mathrm{Sym}_3$ acts faithfully on $\mathfrak{Q}/\mathfrak{Q}'$. Clearly $\mathfrak{H}' = \mathfrak{Q}\mathfrak{W}' \cong SL(2, 3)$. This group has a unique faithful irreducible rational character of degree 2. Hence $\mathfrak{H}$ has a faithful character χ of degree 2 with $\chi \mid \mathfrak{H}'$ rational. Now all elements of $\mathfrak{H} - \mathfrak{H}'$ are 2-elements and a Sylow 2-subgroup of $\mathfrak{H}$ has period 8. Thus $Q(\chi) \subseteqq Q(\varepsilon)$ where ε is a primitive 8th root of unity. Since $8 \mid |GF(17)^\#|$, this representation of $\mathfrak{H}$ is realizable over $GF(17)$ and hence we can assume $\mathfrak{H} \subseteqq GL(2, 17)$. All subgroups of $\mathfrak{H}$ of order 3 are contained in $SL(2, 17)$ since $3 \nmid |GF(17)^\#|$ so $\mathfrak{H}' \subseteqq SL(2, 17)$ and $\mathfrak{H} \subseteqq SL(2, 17)^*$. Let $i = \sqrt{-1} \in GF(17)$ and let $\mathfrak{Z} = \left\langle \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \right\rangle$. Then $\mathfrak{Z}$ is cyclic of order 4, $\mathfrak{Z} \subseteqq SL(2, 17)^*$ and $\mathfrak{Z}$ is central in $GL(2, 17)$. Set $\mathfrak{G} = \mathfrak{Z}\mathfrak{H}$ so $\mathfrak{G} \subseteqq SL(2, 17)^*$.

We show first that $\mathfrak{G}$ has precisely $17 + 1 = 18$ noncentral involutions. Now $|\mathfrak{Z}| = 4$ and $\mathfrak{G}/\mathfrak{Z} \cong \mathrm{Sym}_4$. This quotient group has two classes of involutions $\mathfrak{C}_1, \mathfrak{C}_2$ with $|\mathfrak{C}_1| = 3$, $|\mathfrak{C}_2| = 6$. If $\bar{T} \in \mathfrak{C}_i$ and the coset of $\bar{T}$ contains an involution of $\mathfrak{G}$, then the same is true for all conjugates of $\bar{T}$. Moreover the coset would then clearly contain precisely two such involutions. Thus if $\delta_i = 0,1$ has the obvious meaning, then

$$I(\mathfrak{G}) = 2\delta_1 |\mathfrak{C}_1| + 2\delta_2 |\mathfrak{C}_2| = 6\delta_1 + 12\delta_2 .$$

Let $W \subseteqq \mathfrak{W}$ have order 2. Then $\bar{W} \in \mathfrak{C}_2$ so $\delta_2 = 1$. Let $Q \in \mathfrak{Q}$ have order 4 and let $\mathfrak{Z} = \langle Z \rangle$. Then QZ has order 2 and $\overline{QZ} \in \mathfrak{C}_1$. Hence $\delta_1 = 1$ and $I(\mathfrak{G}) = 18$.

Let $\mathfrak{V}$ be a 2-dimensional $GF(17)$-vector space and let $x \in \mathfrak{V}^\#$. Since $|\mathfrak{G}|$ is prime to 17 we can write $\mathfrak{G}_x \subseteqq \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in GF(17)^\# \right\}$ by taking a suitable basis. Now $\mathfrak{G} \subseteqq SL(2, 17)^*$ and $\det \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} = a$ so we see that $|\mathfrak{G}_x| = 1$ or 2. If T is a noncentral involution of $\mathfrak{G}$, then $\mathfrak{V} > C_{\mathfrak{V}}(T) > \{0\}$ and hence $|C_{\mathfrak{V}}(T)^\#| = 17 - 1$. By the above the centralizer spaces for the involutions are disjoint. Hence

$$\begin{aligned} |\textstyle\bigcup_T C_{\mathfrak{V}}(T)^\#| &= I(\mathfrak{G})(17 - 1) = (17 + 1)(17 - 1) \\ &= 17^2 - 1 = |\mathfrak{V}^\#| . \end{aligned}$$

Thus $\bigcup_T C_{\mathfrak{V}}(T) = \mathfrak{V}$ and so for all $x \in \mathfrak{V}^\#$, $|\mathfrak{G}_x| \geqq 2$. This yields $|\mathfrak{G}_x| = 2$ and $\mathfrak{G}$ is half-transitive but not semiregular. Finally

$\mathfrak{G} \not\subseteq \mathscr{T}(17^2)$, the semilinear transformations, since $\mathfrak{G}$ does not have a cyclic subgroup of index 2.

We close this section with some additional information about the degree 17^2 group.

LEMMA 3.5. *If $q^n = 17^2$, then $|\mathfrak{G}| = 96$.*

Proof. These groups occur in Lemmas 3.2 and 3.4. However the latter case was deduced from the former so we can assume $\mathfrak{G}$ is as described in the proof of Lemma 3.2. We showed there that $|\mathfrak{G}_x| = 2$, $\mathfrak{G}/\boldsymbol{Z}(\mathfrak{G}) \cong \mathrm{Sym}_4$ and $\delta_1 = \delta_2 = 1$. The latter says that if $\bar{T}$ is any involution of $\mathfrak{G}/\boldsymbol{Z}(\mathfrak{G})$, then its coset contains an involution of $\mathfrak{G}$.

Now $\mathfrak{A} = \boldsymbol{Z}(\mathfrak{G})$ has order dividing $|GF(17)^\#| = 16$. If $|\mathfrak{A}| = 2$, then an involution T in the four groups of Sym_4 would not have an involution of $\mathfrak{G}$ in its coset. We assume that $|\mathfrak{A}| \geqq 8$ and derive a contradiction. Let T be an involution of $\mathfrak{G}$ corresponding to a transposition of Sym_4. Then $\langle \mathfrak{E}, T \rangle$ is a maximal class group of order 16 and this group has a cyclic subgroup $\mathfrak{B}$ of order 8. We see that $|\mathfrak{A} \cap \mathfrak{B}| = 2$ so $|\mathfrak{A}\mathfrak{B}| = 4|\mathfrak{A}|$ and $\mathfrak{A}\mathfrak{B}$ has period $|\mathfrak{A}|$ since $|\mathfrak{A}| \geqq |\mathfrak{B}| = 8$. As in the last paragraph of the proof of the preceding lemma, this implies that $|\mathfrak{G}_x| \geqq 4$, a contradiction. Thus $|\mathfrak{A}| = 4$ and since $\mathfrak{G}/\mathfrak{A} \cong \mathrm{Sym}_4$ we have $|\mathfrak{G}| = 4 \cdot 24 = 96$. This completes the proof of the lemma.

4. Solvable case, $m = 2$. In this and the next section the following assumptions hold.

ASSUMPTIONS. Group $\mathfrak{G}$ acts faithfully on vector space $\mathfrak{B}$ of order q^n and half-transitively but not semiregularly on $\mathfrak{B}^\#$. $\mathfrak{E}$ is a group of type $E(2, m)$ with $\mathfrak{E} \triangle \mathfrak{G}$. In addition $\mathfrak{E}$ acts irreducibly on $\mathfrak{B}$, $\mathfrak{G}$ is primitive as a linear group and $\mathfrak{G}$ is solvable.

We will use the notation of Lemma 2.5. Moreover set $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{A}\mathfrak{E}$ so that $\bar{\mathfrak{H}}$ is a solvable subgroup of $Sp(2m, 2)$. We let $\bar{\mathfrak{F}} = F(\bar{\mathfrak{H}})$, the Fitting subgroup of $\bar{\mathfrak{H}}$, and for each prime p we let $\bar{\mathfrak{F}}_p$ be the normal Sylow p-subgroup $\bar{\mathfrak{F}}$. By Fitting's theorem, $\boldsymbol{C}_{\bar{\mathfrak{H}}}(\bar{\mathfrak{F}}) \subseteq \bar{\mathfrak{F}}$. Recall the possible isomorphism classes for $\mathfrak{E}$ namely: iso I if $\mathfrak{E} \cong \mathfrak{Q}\mathfrak{Q} \cdots \mathfrak{Q}$, iso II if $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}\mathfrak{Q} \cdots \mathfrak{Q}$ and iso III if $\mathfrak{E} \simeq \mathfrak{Z}\mathfrak{Q}\mathfrak{Q}\mathfrak{Q} \cdots \mathfrak{Q}$.

LEMMA 4.1. *Suppose $\bar{\mathfrak{F}}_2 \neq \langle 1 \rangle$. Then $|\bar{\mathfrak{F}}_2| = 2$, $\mathfrak{E} =$ iso I or II and $\mathfrak{G}$ has a normal subgroup $\mathfrak{E}_0$ of type $E(2, m-1)$ with $\mathfrak{E}_0 =$ iso III.*

Proof. Let $\mathfrak{S}$ be the complete inverse image of $\bar{\mathfrak{F}}_2$ in $\mathfrak{H}$ so $\mathfrak{S}/\mathfrak{A}\mathfrak{E} = \bar{\mathfrak{F}}_2$. Then $\mathfrak{S}/\mathfrak{A}$ is a 2-group and since $\mathfrak{A}$ is central in $\mathfrak{H}$, $\mathfrak{S}$

is nilpotent. If $\mathfrak{S}_2$ is the normal Sylow 2-subgroup of $\mathfrak{S}$, then $\mathfrak{S}_2 \supseteqq \mathfrak{E}$ and $\mathfrak{S}_2 \triangle \mathfrak{G}$. Since $\mathfrak{G}$ is primitive, $\mathfrak{S}_2$ is of symplectic type. Suppose $4 \mid |\mathfrak{A}_2|$. Then since $\mathfrak{A}_2$ is central in $\mathfrak{S}_2$, $\mathfrak{S}_2$ has a center of order at least 4 and hence $\mathfrak{S}_2$ is the central product of $\boldsymbol{Z}(\mathfrak{S}_2)$ with a number of nonabelian groups of order 8. Note that since $\mathfrak{E} \subseteqq \mathfrak{S}_2$, $\boldsymbol{Z}(\mathfrak{S}_2) \subseteqq \boldsymbol{C}_{\mathfrak{G}}(\mathfrak{E}) = \mathfrak{A}$ so that $\boldsymbol{Z}(\mathfrak{S}_2) = \mathfrak{A}_2$. Since $|\bar{\mathfrak{F}}_2| > 1$, $\mathfrak{S}_2 \neq \mathfrak{A}_2\mathfrak{E}$ and thus $\mathfrak{S}_2 \supseteqq \mathfrak{A}_2\mathfrak{E}\mathfrak{B}$ where $|\mathfrak{B}| = 8$, $\mathfrak{B} \not\subseteq \mathfrak{A}_2$ and $\mathfrak{B} \subseteqq \boldsymbol{C}_{\mathfrak{G}}(\mathfrak{E})$, a contradiction. Thus $|\mathfrak{A}_2| = 2$ and hence $|\boldsymbol{Z}(\mathfrak{E})| = 2$. This implies that $\dim \mathfrak{V} = 2^m$ and since $\mathfrak{S}_2$ acts faithfully on $\mathfrak{V}$, $\mathfrak{S}_2$ has at most m nonabelian factors. Since $|\boldsymbol{Z}(\mathfrak{S}_2)| = |\mathfrak{A}_2| = 2$ we see that $\mathfrak{S}_2 = \mathfrak{B}_0\mathfrak{B}_1 \cdots \mathfrak{B}_{m-1}$, a central product of nonabelian groups with $|\mathfrak{B}_i| = 8$ if $i > 0$ and $\mathfrak{B}_0$ a maximal class group. Now $\mathfrak{B}_0 \cap \mathfrak{E}$ is a 2-generator subgroup of $\mathfrak{E}$ so $|\mathfrak{B}_0 \cap \mathfrak{E}| \leqq 8$. Thus $|\mathfrak{B}_0\mathfrak{E}| \geqq |\mathfrak{B}_0|\ |\mathfrak{E}|/8 = |\mathfrak{B}_0|\, 2^{2(m-1)} = |\mathfrak{S}_2|$. Hence we have equality throughout and $|\mathfrak{B}_0 \cap \mathfrak{E}| = 8$. Now $\mathfrak{B}_0 \cap \mathfrak{E} \triangle \mathfrak{B}_0$ and $\mathfrak{B}_0 \cap \mathfrak{E}$ is noncyclic. As is well known this implies that $[\mathfrak{B}_0 : \mathfrak{B}_0 \cap \mathfrak{E}] \leqq 2$ so $|\mathfrak{B}_0| \leqq 16$ and $[\mathfrak{S}_2 : \mathfrak{E}] \leqq 2$. If $|\bar{\mathfrak{F}}_2| \geqq 1$, then $|\bar{\mathfrak{F}}_2| = 2$. Finally $\Phi(\mathfrak{S}_2)$ is cyclic of order 4 and from $\mathfrak{S}_2 = \mathfrak{B}_0\mathfrak{E}$ we see that $\mathfrak{E}_0 = \boldsymbol{C}_{\mathfrak{E}}(\Phi(\mathfrak{S}_2))$ has the appropriate properties. Thus the result follows.

We assume throughout the remainder of this section that $m = 2$. Since $\bar{\mathfrak{H}} \subseteqq Sp(4, 2)$ here, we make some comments about this latter group. Suppose $Sp(4, 2)$ acts on symplectic space $\mathfrak{W}$. If $\mathfrak{U}$ is an isotropic subspace of $\mathfrak{W}$ of dimension 2, then the symplectic form restricted to $\mathfrak{U}$ is trivial. We see easily that $\mathfrak{W}$ contains 15 such subspaces. Note that $|Sp(4, 2)| = 2^4 \cdot 3^2 \cdot 5$.

Let $\bar{\mathfrak{K}}$ be a Sylow 3-subgroup of $Sp(4, 2)$. Then $\bar{\mathfrak{K}}$ is abelian of type $(3, 3)$ and contains the four subgroups $\bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2, \bar{\mathfrak{L}}_3, \bar{\mathfrak{L}}_4$ of order 3. We can take (see [10]) the following concrete realization for $\bar{\mathfrak{K}}$. Write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$, a direct sum of two nonisotropic 2-dimensional subspaces and then let $\bar{\mathfrak{L}}_1$ centralize $\mathfrak{W}_2$ and act irreducibly on $\mathfrak{W}_1$ and $\bar{\mathfrak{L}}_2$ centralize $\mathfrak{W}_1$ and act irreducibly on $\mathfrak{W}_2$.

Let $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_1$ or $\bar{\mathfrak{L}}_2$. Then $\mathfrak{W} = \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{L}}) \oplus (\mathfrak{W}, \bar{\mathfrak{L}})$ a direct sum of 2-dimensional subspaces. Let $\mathfrak{U}$ be a 2-dimensional $\bar{\mathfrak{L}}$-subspace of $\mathfrak{W}$. If $\mathfrak{U} \cap (\mathfrak{W}, \bar{\mathfrak{L}}) = \{0\}$, then certainly $\mathfrak{U} \subseteqq \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{L}})$ so $\mathfrak{U} = \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{L}})$. If $\mathfrak{U} \cap (\mathfrak{W}, \bar{\mathfrak{L}}) \neq \{0\}$, then since $\bar{\mathfrak{L}}$ acts irreducibly on $(\mathfrak{W}, \bar{\mathfrak{L}})$ we have $\mathfrak{U} \supseteqq (\mathfrak{W}, \bar{\mathfrak{L}})$ so $\mathfrak{U} = (\mathfrak{W}, \bar{\mathfrak{L}})$. Thus $\mathfrak{U} = \mathfrak{W}_1$ or $\mathfrak{W}_2$. In particular $\bar{\mathfrak{L}}_1$ and $\bar{\mathfrak{L}}_2$ do not normalize a 2-dimensional isotropic subspace of $\mathfrak{W}$. If $\mathfrak{U}$ is a 1-dimensional $\bar{\mathfrak{L}}$-subspace, then certainly $\mathfrak{U} \subseteqq \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{L}})$ so $\mathfrak{U} \subseteqq \mathfrak{W}_1$ or $\mathfrak{W}_2$.

Now let $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_3$ or $\bar{\mathfrak{L}}_4$. Then $\bar{\mathfrak{L}}$ acts irreducibly on both $\mathfrak{W}_1$ and $\mathfrak{W}_2$ so $\bar{\mathfrak{L}}$ has no 1-dimensional invariant subspace. Let $\mathfrak{U}$ be a 2-dimensional $\bar{\mathfrak{L}}$-invariant subspace. If $\mathfrak{U} = \mathfrak{W}_1$ or $\mathfrak{W}_2$, then $\mathfrak{U}$ is nonisotropic.

Suppose $\mathfrak{U} \neq \mathfrak{W}_1$ or $\mathfrak{W}_2$ and $w_1 + w_2 \in \mathfrak{U}$ with $w_i \in \mathfrak{W}_i$. Clearly $w_1, w_2 \neq 0$. It is now easy to see that we get precisely three subspaces $\mathfrak{U}$ and since $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are orthogonal each such $\mathfrak{U}$ is isotropic. Thus $\bar{\mathfrak{L}}$ normalizes two nonisotropic 2-dimensional subspaces and three isotropic ones.

If $\bar{\mathfrak{F}}$ is a subgroup of $Sp(4, 2)$ of order 5, then $\bar{\mathfrak{F}}$ acts irreducibly on $\mathfrak{W}$. Then $|C(\bar{\mathfrak{F}})| \mid 2^4 - 1$ so $|C(\bar{\mathfrak{F}})| = 5$ or 15. In the latter case let $\bar{\mathfrak{L}}$ be a subgroup of order 3 centralizing $\bar{\mathfrak{F}}$. Then $\bar{\mathfrak{F}}$ permutes the two 2-dimensional nonisotropic subspaces normalized by $\bar{\mathfrak{L}}$ and hence $\bar{\mathfrak{F}}$ normalizes each, a contradiction. Thus $Sp(4, 2)$ has no elements of order 10 or 15.

LEMMA 4.2. *$\bar{\mathfrak{F}}$ is a normal 2-complement of $\bar{\mathfrak{H}}$ and $|\bar{\mathfrak{F}}| = 3,5$ or $\bar{\mathfrak{F}}$ is abelian of type* (3,3).

Proof. Suppose first that $\bar{\mathfrak{F}}_2 \neq \langle 1 \rangle$. By Lemma 4.1, $\mathfrak{G}$ has a normal subgroup $\mathfrak{E}_0 \cong \mathfrak{Z}\mathfrak{Q}$ and moreover $4 \nmid |Z(\mathfrak{G})|$. By the Reduction Lemma and Lemma 3.4 we have $q = 3,5$ or 17. Suppose $q = 3$. Since $|Z(\mathfrak{E})| = 2$ and $\mathfrak{E}$ acts irreducibly, $q^n = 3^4$ and thus $\mathfrak{E}_0$ also acts irreducibly. By Lemma 4.1 $\mathfrak{G}$ is imprimitive, a contradiction. Let $q = 5$ or 17. Then $4 \mid q - 1$ and since $\mathfrak{G}$ is primitive and $Z(\mathfrak{E}_0) \triangle \mathfrak{G}$ with $|Z(\mathfrak{E}_0)| = 4$ we conclude that $Z(\mathfrak{E}_0)$ consists of scalar matrices and $4 \mid |Z(\mathfrak{G})|$, a contradiction.

Now suppose $\bar{\mathfrak{F}} = \langle 1 \rangle$. Then $\bar{\mathfrak{H}} = \langle 1 \rangle$. If $Z(\mathfrak{E})$ is central then $\mathfrak{G} = \mathfrak{A}\mathfrak{E}$ is nilpotent so $\mathfrak{G}_2 \supseteq \mathfrak{E}$ is half-transitive. By Theorem II of [4], $\mathfrak{G}_2 \cong \mathfrak{D}\mathfrak{Q}$ and $q^n = 3^4$. Then $|\mathfrak{A}| \mid q - 1$ so $\mathfrak{G} = \mathfrak{G}_2 \cong \mathfrak{D}\mathfrak{Q}$ and this group is imprimitive, a contradiction. Thus $Z(\mathfrak{E})$ is not central and in particular $|Z(\mathfrak{E})| = 4$. Since $|\mathfrak{G}/\mathfrak{H}| = 2$ we see that $\mathfrak{G}$ normalizes a hyperplane in $\mathfrak{W} = \mathfrak{E}/Z(\mathfrak{E})$, say $\mathfrak{W}_0 = \mathfrak{E}_0/Z(\mathfrak{E})$. Then $\mathfrak{E}_0 \triangle \mathfrak{G}$ and $\mathfrak{E}_0$ has period 4. Since $\mathfrak{G}$ is primitive $Z(\mathfrak{E}_0)$ is cyclic so $Z(\mathfrak{E}_0) = Z(\mathfrak{E})$ and then $\mathfrak{E}_0/Z(\mathfrak{E}_0)$ has odd dimension, a contradiction.

Using the fact that $Sp(4, 2)$ has no elements of order 15 we conclude that $\bar{\mathfrak{F}}$ is one of the three possibilities mentioned in the statement of the lemma. Since $\bar{\mathfrak{F}}$ is abelian, $\bar{\mathfrak{H}}/\bar{\mathfrak{F}} \subseteq \operatorname{Aut} \bar{\mathfrak{F}}$ and from this we see easily that $\bar{\mathfrak{F}}$ is a normal 2-complement.

LEMMA 4.3. $\mathfrak{E} = \mathrm{iso}\,\mathrm{I}$ *does not occur*.

Proof. Suppose $\mathfrak{E} \cong \mathfrak{Q}\mathfrak{Q} \cong \mathfrak{D}\mathfrak{D}$. Then $\mathfrak{H} = \mathfrak{G}$ and $\bar{\mathfrak{H}}$ permutes the involution vectors of $\mathfrak{W} = \mathfrak{E}/Z(\mathfrak{E})$. By Lemma 1.3, $i(\mathfrak{W}) = 9$ and this clearly implies that $|\bar{\mathfrak{F}}| \neq 5$. Thus $\bar{\mathfrak{F}}$ is abelian of type (3) or (3, 3). Since $\mathfrak{E} \cong \mathfrak{D}\mathfrak{D}$ we see easily that $\mathfrak{E}$ contains an abelian subgroup $\mathfrak{B}$ of type (2, 2, 2). If $\mathfrak{B}_1$ is an irreducible $\mathfrak{B}$-submodule of $\mathfrak{B}$ then by Schur's lemma, $[\mathfrak{B} : C_{\mathfrak{B}}(\mathfrak{B}_1)] \leqq 2$ so for $x \in \mathfrak{B}_1^{\#}$, $4 \mid |\mathfrak{B}_x|$ and hence $4 \mid |\mathfrak{G}_x|$. Moreover since $\mathfrak{E}_x$ is abelian and $\mathfrak{E}_x \cap Z(\mathfrak{E}) = \langle 1 \rangle$ we see easily that $\mathfrak{E}_x = \mathfrak{B}_x$. Suppose $|\bar{\mathfrak{F}}| = 3$. By Lemma 1.5 there exists $y \in \mathfrak{V}^\#$ with $\mathfrak{G}_y \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Since $\mathfrak{G} = \mathfrak{H}$ and $|\bar{\mathfrak{H}}| = 3$ or 6 by Fitting's theorem, we have $|\mathfrak{G}_y| \mid 6$, a contradiction. Thus $\bar{\mathfrak{F}}$ is abelian of type (3, 3).

First suppose $q = 3$. Then a Sylow 3-subgroup of $\mathfrak{G}$ has a fixed point in $\mathfrak{V}^\#$ and thus by half-transitively $\mathfrak{G}_x \supseteqq \mathfrak{K}$ where x is the above mentioned point and $\mathfrak{K}$ is a Sylow 3-subgroup of $\mathfrak{G}$. Note that if $\bar{\mathfrak{K}}$ is the image of $\mathfrak{K}$ in $\bar{\mathfrak{H}}$ then $\bar{\mathfrak{K}} = \bar{\mathfrak{F}}$. Since $\mathfrak{E}_x = \mathfrak{E} \cap \mathfrak{G}_x \triangle \mathfrak{G}_x$ we see that $\bar{\mathfrak{K}}$ normalizes $\boldsymbol{Z}(\mathfrak{E})\mathfrak{E}_x/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{B}/\boldsymbol{Z}(\mathfrak{E})$ a 2-dimensional isotropic subspace of symplectic space $\mathfrak{W}$. This contradicts our preceding remarks about $Sp(4, 2)$ since the subgroup $\bar{\mathfrak{L}}_1$ of $\bar{\mathfrak{K}}$ normalizes no such subspaces. Thus $q \neq 3$.

Now $\bar{\mathfrak{F}}$ acts on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ and let $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$ be the decomposition of $\mathfrak{W}$ given in our earlier discussion of $Sp(4, 2)$. If $\boldsymbol{Z}(\mathfrak{E}) \subseteqq \mathfrak{E}_i \subseteqq \mathfrak{E}$ with $\mathfrak{E}_i/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}_i$, then $\mathfrak{E}_i$ is nonabelian since $\mathfrak{W}_i$ is nonisotropic, and since $\mathfrak{E}_i$ admits an automorphism of order 3 we have $\mathfrak{E}_i \cong \mathfrak{Q}$. Hence we can find a noncentral involution $T \in \mathfrak{E} - (\mathfrak{E}_1 \cup \mathfrak{E}_2)$. By Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle T \rangle$. Now a Sylow 3-subgroup of $\mathfrak{G}$ is not cyclic, since $\bar{\mathfrak{F}}$ is not cyclic and hence it cannot act semiregularly. By half-transitivety $\mathfrak{G}_x$ contains a subgroup $\mathfrak{L}$ of order 3. Then $\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ so if $\bar{\mathfrak{L}}$ denotes the image of $\mathfrak{L}$ in $\bar{\mathfrak{H}}$, then $|\bar{\mathfrak{L}}| = 3$. Since $\langle T \rangle = \mathfrak{E}_x = \mathfrak{E} \cap \mathfrak{G}_x \triangle \mathfrak{G}_x$ we see that $\bar{\mathfrak{L}}$ normalizes the 1-dimensional subspace $\mathfrak{E}_x\boldsymbol{Z}(\mathfrak{E})/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{U}$. Now T was chosen in such a way that $\mathfrak{U} \not\subseteqq \mathfrak{W}_1$ or $\mathfrak{W}_2$. Hence in the notation of our discussion of $Sp(4, 2)$ we see that $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1$ or $\bar{\mathfrak{L}}_2$. On the other hand $\bar{\mathfrak{L}}_3$ and $\bar{\mathfrak{L}}_4$ do not normalize 1-dimensional subspaces. Hence $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2, \bar{\mathfrak{L}}_3$ or $\bar{\mathfrak{L}}_4$, a contradiction.

LEMMA 4.4. *If* $\mathfrak{E} = \mathrm{iso\,II}$, *then* $q^n = 3^4$.

Proof. Let us assume that $q^n \neq 3^4$. Since $\mathfrak{E}$ acts irreducibly on $\mathfrak{V}$ we have $|\mathfrak{V}| = q^n = q^4$ so $q \geqq 5$. We consider the possibilities for $\bar{\mathfrak{F}}$. Suppose $\bar{\mathfrak{F}}$ is abelian of type (3, 3). Then $\bar{\mathfrak{F}}$ is a Sylow 3-subgroup of $Sp(4, 2)$ and we can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$, the corresponding decomposition of $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}$. If $\mathfrak{E}_i/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}_i$, then since $\mathfrak{W}_i$ is nonisotropic, $\mathfrak{E}_i$ is nonabelian of order 8. Now $\mathfrak{E}_i$ admits an automorphism of order 3 so $\mathfrak{E}_i \cong \mathfrak{Q}$ and $\mathfrak{E} \cong \mathfrak{Q}\mathfrak{Q}$, a contradiction. Thus $|\bar{\mathfrak{F}}| = p$ for $p = 3$ or 5.

Note that $\mathfrak{H} = \mathfrak{G}$ and $|\bar{\mathfrak{H}}/\bar{\mathfrak{F}}| \mid (p-1)$. Thus $\bar{\mathfrak{H}}/\bar{\mathfrak{F}}$ is a cyclic 2-group. Suppose $p \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{V}^\#$. Let T be a noncentral involution of $\mathfrak{E}$. Since $q \neq 3$ there exists by Lemma 1.5 an $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle T \rangle$. Let $\mathfrak{L}$ be a subgroup of $\mathfrak{G}_x$ of order p. Since

$\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$, $\bar{\mathfrak{L}}$, the image of $\mathfrak{L}$ in $\bar{\mathfrak{H}}$, has order p so $\bar{\mathfrak{L}} = \bar{\mathfrak{F}}$. Since $\langle T \rangle = \mathfrak{E}_x = \mathfrak{G}_x \cap \mathfrak{E}$ we see that $\bar{\mathfrak{F}}$ centralizes the involution vector in $\mathfrak{W}$ corresponding to T. By Lemma 1.2, $\bar{\mathfrak{F}}$ centralizes $\mathfrak{W}$, a contradiction. Thus $p \nmid |\mathfrak{G}_x|$ and in particular $p \neq q$.

Suppose $p = 3$. By Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Hence $|\mathfrak{G}_x| \mid |\bar{\mathfrak{H}}|$. Since $|\bar{\mathfrak{H}}| = 6$ we conclude that $|\mathfrak{G}_x| = 2$. We note now that $4 \nmid |\mathfrak{A}|$. Otherwise $\mathfrak{A}\mathfrak{E}$ contains $\mathfrak{E}^* \cong \mathfrak{Z}\mathfrak{D}\mathfrak{D}$ and this group contains an abelian subgroup of type $(2, 2, 2)$. This easily implies that $4 \mid |\mathfrak{G}_x|$, a contradiction. Let $\mathfrak{L}$ be a Sylow 3-subgroup of $\mathfrak{G}$. Since $\bar{\mathfrak{H}}/\bar{\mathfrak{F}}$ acts faithfully on $\bar{\mathfrak{F}}$ we see by the above that if T is a noncentral involution of $\mathfrak{G}$ and $T \subseteq C_{\mathfrak{H}}(\mathfrak{L})$ then $T \in \mathfrak{E}$. Now $\bar{\mathfrak{L}} = \bar{\mathfrak{F}}$ permutes faithfully the $i(\mathfrak{W}) = 5$ involution vectors of $\mathfrak{W}$. Thus $\bar{\mathfrak{L}}$ moves 3 such and fixes 2 such. Since each involution vector corresponds to two noncentral involutions of $\mathfrak{E}$ we see that $\mathfrak{L}$ centralizes precisely four noncentral involutions of $\mathfrak{G}$. Thus clearly $I(\mathfrak{G}) \equiv 4 \bmod 3$. On the other hand by Lemma 1.9 we have $I(\mathfrak{G}) = 1 + q^2$. Thus $q^2 \equiv 0 \bmod 3$, a contradiction since $q \neq 3$.

We consider $p = 5$ so $q \geqq 7$. Let $\bar{I}$ denote the number of involutions of $\mathfrak{G}/\mathfrak{A}$. Since $\mathfrak{A}$ is cyclic and central in $\mathfrak{G}$, each involution of $\mathfrak{G}/\mathfrak{A}$ corresponds to at most two noncentral involutions of $\mathfrak{G}$ so $I(\mathfrak{G}) \leqq 2\bar{I}$. Now $\mathfrak{W}\mathfrak{F} \triangle \mathfrak{G}/\mathfrak{A}$ where $\mathfrak{W}$ is elementary abelian of order 2^4, $|\mathfrak{F}| = 5$ and $\mathfrak{F}$ acts irreducibly on $\mathfrak{W}$. Furthermore $(\mathfrak{G}/\mathfrak{A})/(\mathfrak{W}\mathfrak{F})$ is a cyclic 2-group which acts faithfully on $(\mathfrak{W}\mathfrak{F})/\mathfrak{W}$. Hence we see easily that $\bar{I} \leqq 15 + 5 \cdot 4 = 35$ and $I(\mathfrak{G}) \leqq 70$.

Let T be a noncentral involution of $\mathfrak{G}$. If $T \in \mathfrak{A}\mathfrak{E}$ then certainly $|C_{\mathfrak{V}}(T)| = q^2$. Suppose $T \not\subseteq \mathfrak{A}\mathfrak{E}$. From the structure of $\bar{\mathfrak{H}}$ we see that for some $F \in \mathfrak{G}$, $\langle \overline{T, T^F} \rangle \supseteq \bar{\mathfrak{F}}$. Since $5 \nmid |\mathfrak{G}_x|$ we see that $C_{\mathfrak{V}}(T) \cap C_{\mathfrak{V}}(T^F) = \{0\}$. Hence $|C_{\mathfrak{V}}(T)| \leqq q^2$ here also. Now every element of $\mathfrak{V}^\#$ is fixed by some noncentral involution of $\mathfrak{G}$ so $\mathfrak{V}^\# = \bigcup_T C_{\mathfrak{V}}(T)^\#$ and hence

$$q^4 - 1 = |\mathfrak{V}| \leqq I(\mathfrak{G})(q^2 - 1)$$

or $q^2 + 1 \leqq I(\mathfrak{G}) \leqq 70$. Since $q > 5$, we have $q = 7$.

For $q = 7$ the argument is somewhat involved. Since $|\mathfrak{A}| \mid q - 1$ we have $|\mathfrak{A}| = 2$ or 6. Now $O_3(\mathfrak{A})$ is central in $\mathfrak{G}$ and is a Sylow 3-subgroup of $\mathfrak{G}$. Thus $\mathfrak{G}$ has a normal 3-complement. Since this group is also half-transitive we see that it suffices to assume that $O_3(\mathfrak{A}) = \langle 1 \rangle$ and hence $|\mathfrak{A}| = 2$, $\mathfrak{A}\mathfrak{E} = \mathfrak{E}$.

We can now get a tighter count on $I(\mathfrak{E})$. Let $\bar{I} = \bar{I}_1 + \bar{I}_2$ where $\bar{I}_1$ counts the number of involutions of $\mathfrak{E}/\mathfrak{A}$ and $\bar{I}_2$ counts those of $\mathfrak{G}/\mathfrak{A}$ not in $\mathfrak{E}/\mathfrak{A}$. We have as before $\bar{I}_1 = 15$, $\bar{I}_2 \leqq 20$. If $I(\mathfrak{E}) = I_1 + I_2$ is the corresponding break up of $I(\mathfrak{E})$, then $I_2 \leqq 2\bar{I}_2 \leqq 40$ and

$I_1 = I(\mathfrak{E}) = 10$. Hence $I(\mathfrak{G}) \leqq 50$ here. As above $\mathfrak{B}^\# = \bigcup_T C_{\mathfrak{B}}(T)$ yields $50 = q^2 + 1 \leqq I(\mathfrak{G}) \leqq 50$. Thus we must have equality throughout and hence $\bigcup_T C_{\mathfrak{B}}(T)$ is a disjoint union. This implies that every element $x \in \mathfrak{B}^\#$ is centralized by precisely one involution so $\mathfrak{G}_x$ has a unique involution.

Let $\mathfrak{R}$ be the subgroup of $\mathfrak{G}$ with $\mathfrak{R} \supseteqq \mathfrak{E}$ and $[\bar{\mathfrak{R}} : \bar{\mathfrak{F}}] = 2$. Since $\bar{\mathfrak{G}}/\bar{\mathfrak{F}}$ is cyclic, $\mathfrak{R}$ contains all the involutions of $\mathfrak{E}$. We study the group $\mathfrak{R}$. Note that $\bar{\mathfrak{R}}$ is dihedral of order 10 and $\bar{\mathfrak{F}}$ acts irreducibly on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. Let $\mathfrak{L}$ be a Sylow 5-subgroup of $\mathfrak{R}$ so that $|\mathfrak{L}| = 5$ and let $\mathfrak{N} = N_{\mathfrak{R}}(\mathfrak{L})$. From the above we see that $\mathfrak{N}/\boldsymbol{Z}(\mathfrak{E})$ is dihedral of order 10. Let $\mathfrak{L} = \langle L \rangle$ and let $N \in \mathfrak{N} - \boldsymbol{Z}(\mathfrak{E})$ be a 2-element. Then $L^N = L^{-1}$.

Now $\mathfrak{N}$ permutes the 10 noncentral involutions of $\mathfrak{E}$ and the corresponding five involution vectors of $\mathfrak{W}$. Using (( )) to denote cyclic permutations, it is clear that we can label the involutions by X_i, Y_i, $i = 1, 2, \cdots, 5$ such that $Y_i = -X_i$ and as a permutation

$$L = ((X_1, X_2, X_3, X_4, X_5))((Y_1, Y_2, Y_3, Y_4, Y_5)) .$$

Here for convenience we denoted the central involution of $\mathfrak{E}$ by -1. We consider N. As a permutation, it has order 2. Since N acts on the five involution vectors of $\mathfrak{W}$, N must fix at least one such, say the one corresponding to $\{X_1, Y_1\}$. Then either N fixes both X_1 and Y_1 or N interchanges the two. Since $L^N = L^{-1}$ this completely determines the cycle structure of N and we have either

(a) $N = ((X_1))((X_2, X_5))((X_3, X_4))((Y_1))((Y_2, Y_5))((Y_3, Y_4))$ or

(b) $N = ((X_1, Y_1))((X_2, Y_5))((X_3, Y_4))((X_4, Y_3))((X_5, Y_2))$.

Note that it is easy to see that for $i \neq j$, $(X_i, X_j) = (Y_i, Y_j) = -1$. Now the sum of the five involution vectors of $\mathfrak{W}$ is L invariant and hence must be 0. Thus $Z = X_1X_2X_3X_4X_5 \in \boldsymbol{Z}(\mathfrak{E})$. If N acts like (b) above, then

$$\begin{aligned} Z = Z^N = (X_1X_2X_3X_4X_5)^N &= Y_1Y_5Y_4Y_3Y_2 \\ &= -X_1(X_5X_4X_3X_2) = -Z^{-1} . \end{aligned}$$

Thus $Z^2 = -1$, a contradiction and hence N must act like (a) above.

Suppose N has order 2. Then $\langle N, X_1, Y_1 \rangle$ is elementary abelian of order 8. This yields as usual an element $x \in \mathfrak{B}^\#$ such that $\mathfrak{G}_x$ contains a subgroup of type (2, 2) and this contradicts our preceding remarks. Hence $N^2 = -1$.

Now $\mathfrak{S} = \langle \mathfrak{E}, N \rangle$ is a Sylow 2-subgroup of $\mathfrak{N}$. We show that every involution of $\mathfrak{S}$ is contained in $\mathfrak{E}$. This will imply that $\mathfrak{G}$ contains only 10 noncentral involutions and this will yield the required contradiction. Suppose $T \in \mathfrak{S} - \mathfrak{E}$ is an involution. Then $T = NE$ for some $E \in \mathfrak{E}$. Since $N^2 = -1$ we have

$$1 = T^2 = NENE = -E^N E$$

so $E^N = -E^{-1}$. In particular the image of E in $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ is centralized by N. Now $\boldsymbol{C}_{\mathfrak{W}}(N)$ is a 2-dimensional subspace which is clearly spanned by the images in $\mathfrak{W}$ of X_1 and X_2X_5. Note that X_1 and X_2X_5 commute and X_2X_5 has order 4. Hence $E \in \langle X_1, X_2X_5\rangle = \mathfrak{B}$. We have $X_1^N = X_1 = X_1^{-1}$ and $(X_2X_5)^N = X_5X_2 = (X_2X_5)^{-1}$ so since $\mathfrak{B}$ is abelian, N acts in a dihedral manner on $\mathfrak{B}$. Thus $E^N = E^{-1}$ which contradicts the previous relation $E^N = -E^{-1}$. This implies that T does not exist and the proof is complete.

If $q^n = 3^4$ above then $\mathfrak{E} = F(\mathfrak{G})$ is half-transitive. Thus these groups are given in [5] where uniqueness was proved. Since $\mathfrak{G}$ is primitive, we see that $\mathfrak{G}$ is transitive and hence it is one of the groups given in [3].

LEMMA 4.5. $\mathfrak{E} = \text{iso III}$ *does not occur.*

Proof. Suppose $\mathfrak{E} \cong \mathfrak{Z}\mathfrak{Q}\mathfrak{Q}$. Since $|\boldsymbol{Z}(\mathfrak{E})| = 4$ and $\mathfrak{E}$ acts irreducibly we see that $|\mathfrak{B}| = q^4$ if $q \equiv 1 \bmod 4$ and $|\mathfrak{B}| = q^8$ if $q \equiv -1 \bmod 4$. If $\mathfrak{H} = \boldsymbol{C}_{\mathfrak{G}}(\boldsymbol{Z}(\mathfrak{E}))$, then $[\mathfrak{G} : \mathfrak{H}] = 1$ or 2. Moreover if $[\mathfrak{G} : \mathfrak{H}] = 2$ then $q \equiv -1$.

We consider $\bar{\mathfrak{F}}$. Suppose $|\bar{\mathfrak{F}}| = 5$ or 9 so that $\boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{F}}) = \langle 1\rangle$. Clearly $\bar{\mathfrak{F}}$ acts faithfully on $\mathfrak{E}/\mathfrak{E}'$ and centralizes $\boldsymbol{Z}(\mathfrak{E})/\mathfrak{E}'$. Let $\mathfrak{E}_0$ be the commutator subgroup of $\mathfrak{E}\bar{\mathfrak{F}}$. Then clearly $|\mathfrak{E}_0/\mathfrak{E}'| = 2^4$, $\mathfrak{E}_0 \triangle \mathfrak{G}$ and $\mathfrak{E}_0 = \text{iso I}$ or II. By the Reduction Lemma and the previous two lemmas, $\mathfrak{E}_0 = \text{iso II}$ and $q = 3$. Since as we have seen, this group does not admit an automorphism group of type $(3, 3)$ we must have $|\bar{\mathfrak{F}}| = 5$. Since $q = 3$, $q^n = 3^8$.

Now $\mathfrak{E}$ has an abelian subgroup of type $(2, 2, 2)$ so it follows that $4 \mid |\mathfrak{G}_x|$ and hence $2 \mid |\mathfrak{H}_x|$ for all $x \in \mathfrak{B}^\#$. As in the proof of the previous lemma we see that $5 \nmid |\mathfrak{G}_x|$ and hence if T is a noncentral involution of $\mathfrak{H}$, then $|\boldsymbol{C}_{\mathfrak{B}}(T)| \leqq 3^4$. Now $\mathfrak{H}/\mathfrak{A}$ contains at most $15 + 5\cdot 4 = 35$ involutions and hence since $\mathfrak{A}$ is central and cyclic we have $I(\mathfrak{H}) \leqq 2\cdot 35 = 70$. Since $\mathfrak{B} = \bigcup_T \boldsymbol{C}_{\mathfrak{B}}(T)$ we have

$$3^8 = |\mathfrak{B}| \leqq 3^4 I(\mathfrak{H}) \leqq 3^4\cdot 70$$

or $3^4 \leqq 70$, a contradiction.

Finally let $|\bar{\mathfrak{F}}| = 3$. As above we see that $4 \mid |\mathfrak{G}_x|$. Since by Lemma 1.5 there exists $x \in \mathfrak{B}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1\rangle$, we conclude that $4 \mid |\mathfrak{G}/\mathfrak{A}\mathfrak{E}|$. Hence $|\bar{\mathfrak{H}}| = 6$ and $[\mathfrak{G} : \mathfrak{H}] = 2$ so $q \equiv -1 \bmod 4$, $q^n = q^8$ and $q \neq 5$. By Lemma 1.5, if T is a noncentral involution of $\mathfrak{E}$ then for some $x \in \mathfrak{B}^\#$, $\mathfrak{E}_x = \langle T\rangle$. Hence if $3 \mid |\mathfrak{G}_x|$, then $\bar{\mathfrak{F}}$ fixes all involution vectors of $\mathfrak{W}$ and $\bar{\mathfrak{F}}$ centralizes $\mathfrak{W}$, a contradiction. Thus

$3 \nmid |\mathfrak{G}_x|$ and this implies easily that if T is an involution of $\mathfrak{H}$, then $|C_{\mathfrak{V}}(T)| \leqq q^4$. Also $q \neq 3$ so $q \geqq 7$. We have clearly $I(\mathfrak{H}) \leqq 2\cdot 2\cdot 16\cdot 3 = 192$ and since $\mathfrak{V} = \bigcup_T C_{\mathfrak{V}}(T)$ we have

$$q^8 = |\mathfrak{V}| \leqq q^4 I(\mathfrak{H}) \leqq 192 q^4 .$$

Thus $7^4 \leqq q^4 \leqq 192$, a contradiction. This completes the proof of the lemma.

5. Solvable case, $m = 3$ and 4.

We continue with the assumptions of the preceding section except that $m = 3$ or 4 here. First let $m = 3$. Now $|Sp(2m, 2)| = 2^9\cdot 3^4\cdot 5\cdot 7$. We consider the possibilities for $\bar{\mathfrak{F}}$.

LEMMA 5.1. *$\bar{\mathfrak{F}}$ is a 3-group.*

Proof. If p is a prime, we let $\bar{\mathfrak{F}}_p$ denote the normal Sylow p-subgroup of $\bar{\mathfrak{F}}$. We show here that $\bar{\mathfrak{F}}_2 = \bar{\mathfrak{F}}_5 = \bar{\mathfrak{F}}_7 = \langle 1\rangle$.

Suppose $\bar{\mathfrak{F}}_2 \neq \langle 1\rangle$. By Lemma 4.1 $\mathfrak{G}$ has a normal subgroup $\mathfrak{E}_0 \cong \mathfrak{Z}\mathfrak{Q}\mathfrak{Q}$. By the Reduction Lemma and Lemma 4.5 this does not occur.

Suppose $\bar{\mathfrak{F}}_7 \neq \langle 1\rangle$. Then $|\bar{\mathfrak{F}}_7| = 7$ and $\bar{\mathfrak{F}}_7$ acts irreducibly on $\mathfrak{W}$. By Schur's lemma, $C_{\bar{\mathfrak{H}}}(\bar{\mathfrak{F}}_7)$ is a cyclic group of odd order and $[\bar{\mathfrak{H}} : C_{\bar{\mathfrak{H}}}(\bar{\mathfrak{F}}_7)] \mid 6$. Hence if $\mathfrak{E} = \text{iso I}$ or II then $4 \nmid [\mathfrak{G} : \mathfrak{A}\mathfrak{E}]$ while if $\mathfrak{E} = \text{iso III}$, then $8 \nmid [\mathfrak{G} : \mathfrak{A}\mathfrak{E}]$. Now if $\mathfrak{E} = \text{iso I}$ or II then $\mathfrak{E}$ has an abelian subgroup of type $(2, 2, 2)$ so for some $y \in \mathfrak{V}^\#$, $4 \mid |\mathfrak{G}_y|$. If $\mathfrak{E} = \text{iso III}$, then $\mathfrak{E}$ has an abelian subgroup of type $(2, 2, 2, 2)$ so $8 \mid |\mathfrak{G}_y|$. Finally by Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1\rangle$ so $|\mathfrak{G}_x| \mid [\mathfrak{G} : \mathfrak{A}\mathfrak{E}]$. Since $|\mathfrak{G}_x| = |\mathfrak{G}_y|$ we have a contradiction.

Suppose $\bar{\mathfrak{F}}_5 \neq \langle 1\rangle$. Then $|\bar{\mathfrak{F}}_5| = 5$ and we can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$ where $|\mathfrak{W}_1| = 2^2$, $|\mathfrak{W}_2| = 2^4$, both these spaces are $\bar{\mathfrak{F}}_5$ invariant and $\mathfrak{W}_1 = C_{\mathfrak{W}}(\bar{\mathfrak{F}}_5)$. Let $\mathfrak{E} \supseteqq \mathfrak{E}_i \supseteqq Z(\mathfrak{E})$ with $\mathfrak{E}_i/Z(\mathfrak{E}) = \mathfrak{W}_i$. Clearly $\mathfrak{E}_i \triangle \mathfrak{G}$ and since $\mathfrak{G}$ is primitive each $\mathfrak{E}_i$ is of symplectic type. By the Reduction Lemma applied to $\mathfrak{E}_2$ and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$ and $\mathfrak{E}_2 \cong \mathfrak{D}\mathfrak{Q}$. Hence $|Z(\mathfrak{E})| = 2$ so $\mathfrak{E} \neq \text{iso III}$.

Now $\mathfrak{W}_1$ and $\mathfrak{W}_2$ are nonisotropic and we know that $\bar{\mathfrak{F}}_5$ is self-centralizing in its action on $\mathfrak{W}_2$. Write $C_{\bar{\mathfrak{H}}}(\bar{\mathfrak{F}}_5) = \bar{\mathfrak{B}} \times \bar{\mathfrak{F}}_5$ where $\bar{\mathfrak{B}} \triangle \bar{\mathfrak{H}}$. Then $\bar{\mathfrak{B}}$ acts faithfully on $\mathfrak{W}_1$ so since $\bar{\mathfrak{F}}_2 = \langle 1\rangle$, either $\bar{\mathfrak{B}} = \langle 1\rangle$ or $\bar{\mathfrak{B}}$ has a normal 3-subgroup of order 3 which is clearly $\bar{\mathfrak{F}}_3$. Suppose $\bar{\mathfrak{B}} = \langle 1\rangle$. Then $\bar{\mathfrak{H}}/\bar{\mathfrak{F}}_5$ is a 2-group which acts on $\mathfrak{W}_1$ and hence there is a 1-dimensional $\bar{\mathfrak{H}}$-invariant subspace $\mathfrak{W}_0$ of $\mathfrak{W}_1$. Note that $\bar{\mathfrak{H}} = \bar{\mathfrak{G}}$ since $\mathfrak{E} \neq \text{iso III}$ and thus if $\mathfrak{E} \supseteqq \mathfrak{E}_3 \supseteqq Z(\mathfrak{E})$ with $\mathfrak{E}_3/Z(\mathfrak{E}) = \mathfrak{W}_0 \oplus \mathfrak{W}_2$ then $\mathfrak{E}_3 \triangle \mathfrak{G}$. By the Reduction Lemma and Lemma 4.5 we have a contradiction since clearly $\mathfrak{E}_3 \cong \mathfrak{Z}\mathfrak{Q}\mathfrak{Q}$.

Thus $\bar{\mathfrak{B}} \supseteq \bar{\mathfrak{F}}_3$ and $|\bar{\mathfrak{F}}_3| = 3$. Since $q = 3$ we see that the Sylow 3-subgroups of $\mathfrak{G}$ have order 3. Now $\bar{\mathfrak{F}}_3$ centralizes $\mathfrak{W}_2$ so clearly $\mathfrak{G}$ contains precisely four Sylow 3-subgroups say $\mathfrak{L}_i$ for $i = 1, 2, 3, 4$. Since $q = 3$ each $\mathfrak{L}_i$ has a fixed point on $\mathfrak{V}^\#$ so by half-transitivety $\mathfrak{V} = \bigcup_1^4 C_{\mathfrak{V}}(\mathfrak{L}_i)$. Hence since the $\mathfrak{L}_i$ are all conjugate in $\mathfrak{G}$ we see that each $C_{\mathfrak{V}}(\mathfrak{L}_i)$ has codimension 1 in $\mathfrak{V}$. But $|\mathfrak{V}| = 3^8$ so $\mathfrak{V}_0 = \bigcap C_{\mathfrak{V}}(\mathfrak{L}_i) \neq \{0\}$. Since $\mathfrak{V}_0$ is clearly a proper $\mathfrak{G}$-invariant subspace of $\mathfrak{V}$ we have a contradiction.

LEMMA 5.2. *$\bar{\mathfrak{F}}$ is not cyclic* and $q \neq 3$.

Proof. We have shown that $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_3$. If $\bar{\mathfrak{F}}$ is cyclic (including the possibility that $\bar{\mathfrak{F}} = \langle 1 \rangle$) then clearly $4 \nmid |\bar{\mathfrak{H}}|$. If $\mathfrak{E} =$ iso I or II then $\mathfrak{G} = \mathfrak{H}$ so $4 \nmid |\mathfrak{G}/\mathfrak{A}\mathfrak{E}|$. If $\mathfrak{E} =$ iso III then $8 \nmid |\mathfrak{G}/\mathfrak{A}\mathfrak{E}|$. If $\mathfrak{E} =$ iso I or II, then $\mathfrak{E}$ has an abelian subgroup of type $(2, 2, 2)$ so we see that $4 \mid |\mathfrak{G}_x|$. If $\mathfrak{E} =$ iso III, then $\mathfrak{E}$ has an abelian subgroup of type $(2, 2, 2, 2)$ so $8 \mid |\mathfrak{G}_x|$. Now by Lemma 1.5 there exists $y \in \mathfrak{V}^\#$ with $\mathfrak{G}_y \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Hence $|\mathfrak{G}_y| \mid [\mathfrak{G} : \mathfrak{A}\mathfrak{E}]$, a contradiction.

Let $q = 3$ so that for all $x \in \mathfrak{V}^\#$, $\mathfrak{G}_x$ contains a Sylow 3-subgroup of $\mathfrak{G}$. Let $\mathfrak{F}$ be the complete inverse image of $\bar{\mathfrak{F}}$ in $\mathfrak{H}$. For any $x \in \mathfrak{V}^\#$, let $\mathfrak{L}$ be a Sylow 3-subgroup of $\mathfrak{F}_x$. Then clearly $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E} = \bar{\mathfrak{F}}$ and since $\mathfrak{E}_x = \mathfrak{E} \cap \mathfrak{G}_x \triangle \mathfrak{G}_x$ we see that $\bar{\mathfrak{F}}$ normalizes $\mathfrak{E}_x Z(\mathfrak{E})/Z(\mathfrak{E})$. If $\mathfrak{E} =$ iso II or III, then by Lemma 1.5 if T is any noncentral involution of $\mathfrak{E}$ then for some $x \in \mathfrak{V}^\#$, $\mathfrak{E}_x = \langle T \rangle$. This implies that $\bar{\mathfrak{F}}$ fixes all involution vectors and $\bar{\mathfrak{F}} = \langle 1 \rangle$, a contradiction. If $\mathfrak{E} =$ iso I then by Lemma 1.5, $|\mathfrak{E}_x| = 1$ or 4. However here it is easy to see that for each such T we can find two points $x_1, x_2 \in \mathfrak{V}^\#$ with $\langle T \rangle = \mathfrak{E}_{x_1} \cap \mathfrak{E}_{x_2}$. This again implies that $\bar{\mathfrak{F}}$ fixes all involution vectors and the result follows.

LEMMA 5.3. $\mathfrak{E} =$ iso I *does not occur*.

Proof. Here $\mathfrak{E} \cong \mathfrak{Q}\mathfrak{Q}\mathfrak{Q}$ and we see easily that Aut $\mathfrak{E}$ contains $\bar{\mathfrak{F}} \sim \bar{\mathfrak{F}}$ where $|\bar{\mathfrak{F}}| = 3$ and this is a full Sylow 3-subgroup of $Sp(6, 2)$. Then any 3-group acting on $\mathfrak{E}$ can be embedded in this Sylow 3-subgroup. Let $\bar{\mathfrak{K}}$ be a Sylow 3-subgroup of Aut $\mathfrak{E}$. Then $\bar{\mathfrak{K}}$ acts faithfully on $\mathfrak{W} = \mathfrak{E}/Z(\mathfrak{E})$. As a Sylow 3-subgroup of $Sp(6, 2)$ we know that it has the following structure. We can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3$, a direct sum of orthogonal 2-dimensional nonisotropic subspaces. $\bar{\mathfrak{K}}$ has a subgroup $\bar{\mathfrak{N}}$ of index 3 with $\bar{\mathfrak{N}} = \bar{\mathfrak{L}}_1 \times \bar{\mathfrak{L}}_2 \times \bar{\mathfrak{L}}_3$. Here $|\bar{\mathfrak{L}}_i| = 3$ and $\bar{\mathfrak{L}}_i$ acts irreducibly on $\mathfrak{W}_i$ and centralizes the remaining $\mathfrak{W}_j$. Further, any element of $\bar{\mathfrak{K}} - \bar{\mathfrak{N}}$ permutes these three subspaces. Now let $\mathfrak{E}_i$

be the subgroup of $\mathfrak{E}$ with $\mathfrak{E}_i/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}_i$. Then $\mathfrak{E}_i$ is nonabelian of order 8 and admits an automorphism of order 3. Thus $\mathfrak{E}_i \cong \mathfrak{Q}$. Suppose $T = T_1T_2T_3$ is a noncentral involution of $\mathfrak{E}$ with $T_i \in \mathfrak{E}_i$. Since $\mathfrak{E}_i \cong \mathfrak{Q}$ we see that precisely one of the T_i is contained in $\boldsymbol{Z}(\mathfrak{E})$, say for example T_1. Then we can write $T = T_2T_3$. If some subgroup $\bar{\mathfrak{L}}$ of $\bar{\mathfrak{K}}$ centralizes the involution vector corresponding to T then clearly $\bar{\mathfrak{L}}$ normalizes $\mathfrak{W}_1$. Thus $\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{N}}$ so $\bar{\mathfrak{L}}$ normalizes $\mathfrak{W}_2$ and $\mathfrak{W}_3$. This clearly implies that $\bar{\mathfrak{L}}$ centralizes $\mathfrak{W}_2$ and $\mathfrak{W}_3$ and thus $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_1$. Hence the only subgroups of $\bar{\mathfrak{K}}$ which centralize involution vectors are $\bar{\mathfrak{L}}_1$, $\bar{\mathfrak{L}}_2$ and $\bar{\mathfrak{L}}_3$.

Now $\bar{\mathfrak{F}}$ is not cyclic and hence a Sylow 3-subgroup of $\mathfrak{G}$ is not cyclic. Thus $3 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{V}^\#$. By the preceding lemma again $q \neq 3$. Hence if $T \in \mathfrak{E}$ is an involution, then by Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle T \rangle$. Let $\mathfrak{L}$ be a Sylow 3-subgroup of $\mathfrak{G}_x$ so $|\mathfrak{L}| \geqq 3$ and $\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Then $\bar{\mathfrak{L}}$ acts faithfully on $\mathfrak{E}$ so we can extend $\mathfrak{L}$ to $\bar{\mathfrak{K}}$ as above. Since $\bar{\mathfrak{L}}$ normalizes the involution vector corresponding to T we see that $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_i$ for some i. Thus $|\bar{\mathfrak{L}}| = 3$ and $9 \nmid |\mathfrak{G}_x|$.

Suppose $\bar{\mathfrak{G}} = \mathfrak{G}/\mathfrak{A}\mathfrak{E}$ contains a copy of $\bar{\mathfrak{N}} \subseteq \bar{\mathfrak{K}}$. Then let $\mathfrak{S}$ be a 3-subgroup of $\mathfrak{G}$ with $\mathfrak{S}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E} = \bar{\mathfrak{N}}$. Certainly $\mathfrak{S}' \subseteq \mathfrak{A}$. Now $\mathfrak{S}$ acts on $\mathfrak{V}$, a vector space of dimension $n = 2^4$. Since $\mathfrak{S}$ is a 3-group we conclude that $\mathfrak{S}'$ is in the kernel of some irreducible constituent and hence $\mathfrak{S}'$ has a fixed point in $\mathfrak{V}^\#$. Since $\mathfrak{S}' \subseteq \mathfrak{A}$ we see that $\mathfrak{S}' = \langle 1 \rangle$ and $\mathfrak{S}$ is abelian. Now $\mathfrak{S}/\mathfrak{S} \cap \mathfrak{A}$ is abelian of type $(3, 3, 3)$ and hence $\mathfrak{S}$ contains a subgroup of type $(3, 3, 3)$. But this implies that $9 \mid |\mathfrak{G}_x|$, a contradiction. In particular we see that a Sylow 3-subgroup of $\bar{\mathfrak{G}}$ has order $\leqq 3^3$.

Let T and $\mathfrak{L}$ be as above and set $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E}$. This time embed 3-group $\bar{\mathfrak{F}}\bar{\mathfrak{L}}$ in $\bar{\mathfrak{K}}$. Again $\bar{\mathfrak{L}} = \bar{\mathfrak{L}}_i$ for some i. Now $\bar{\mathfrak{K}}$ is generated by $\bar{\mathfrak{L}}_i$ and any element outside $\bar{\mathfrak{N}}$. Since $\bar{\mathfrak{F}}\bar{\mathfrak{L}} < \bar{\mathfrak{K}}$ we must have $\bar{\mathfrak{F}} \subseteq \bar{\mathfrak{N}}$ and hence $\bar{\mathfrak{F}}\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{N}}$. Since $\bar{\mathfrak{L}}$ centralizes $\bar{\mathfrak{F}}$ we have $\bar{\mathfrak{L}} \subseteq \bar{\mathfrak{F}}$.

Now embed $\bar{\mathfrak{F}}$ alone in $\bar{\mathfrak{K}}$. We have shown that for each involution vector of $\mathfrak{W}$, $\bar{\mathfrak{F}}$ contains a subgroup of order 3 centralizing it. Thus $\bar{\mathfrak{F}} \supseteq \bar{\mathfrak{L}}_1, \bar{\mathfrak{L}}_2, \bar{\mathfrak{L}}_3$ and $\bar{\mathfrak{F}} \supseteq \bar{\mathfrak{N}}$, a contradiction since $\bar{\mathfrak{G}} \not\supseteq \bar{\mathfrak{N}}$. This completes the proof of this result.

LEMMA 5.4. $\mathfrak{E} = \text{iso II}$ *and* III *do not occur.*

Proof. Suppose $C_{\mathfrak{W}}(\bar{\mathfrak{F}}) = \mathfrak{W}_1 \neq \langle 1 \rangle$. Then $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$ where $\mathfrak{W}_2 = (\mathfrak{W}, \bar{\mathfrak{F}})$. Since $\mathfrak{W}_2$ has even dimension (the nonprincipal irreducible representations of a 3-group over $GF(2)$ have even dimension) so does $\mathfrak{W}_1$. One of these two subspaces, say $\mathfrak{W}_i$ has dimension equal to 4.

Let $\mathfrak{E}_i$ be the subgroup of $\mathfrak{E}$ with $\mathfrak{E}_i/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}_i$. Then $\mathfrak{E}_i \triangle \mathfrak{G}$ and $\mathfrak{G}$ is primitive so $\mathfrak{E}_i$ is of symplectic type. By the Reduction Lemma and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$, a contradiction by Lemma 5.2.

Now let $\mathfrak{E} = \text{iso II}$. By Lemma 1.3, $\bar{\mathfrak{F}}$ permutes the $i(\mathfrak{W}) = 35$ involution vectors. Hence $\bar{\mathfrak{F}}$ must fix one of these and $\boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{F}}) \neq \langle 1 \rangle$, a contradiction.

Having already eliminated $\mathfrak{E} = \text{iso I}$ and II we now eliminate iso III. $\bar{\mathfrak{F}}$ acts on $\mathfrak{E}/\mathfrak{E}' = \mathfrak{U}$ and centralizes $\boldsymbol{Z}(\mathfrak{E})/\mathfrak{E}'$. Since $\boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{F}}) = \langle 1 \rangle$ we see that $\mathfrak{U} = \mathfrak{U}_1 \oplus \mathfrak{U}_2$ where $\mathfrak{U}_1 = \boldsymbol{C}_{\mathfrak{W}}(\bar{\mathfrak{F}})$, $\mathfrak{U}_2 = (\mathfrak{U}, \bar{\mathfrak{F}})$, $|\mathfrak{U}_1| = 2$, $|\mathfrak{U}_2| = 2^4$. Let $\mathfrak{E}_2$ be a subgroup of $\mathfrak{E}$ with $\mathfrak{E}_2/\mathfrak{E}' = \mathfrak{U}_2$. Then $\mathfrak{E}_2 \triangle \mathfrak{G}$ and $\mathfrak{E}_2$ is type $E(2, 3)$ and iso I or II. By the Reduction Lemma and the above we have a contradiction.

We now consider $m = 4$. Here we have partial results in Lemmas 2.6, 2.10 and 2.12. Thus $\mathfrak{E} \neq \text{iso III}$, $q \geqq 7$ and $|\mathfrak{G}/\mathfrak{A}\mathfrak{E}| > 10^4$. We consider $\bar{\mathfrak{F}}$.

LEMMA 5.5. *All irreducible constituents of* $\bar{\mathfrak{F}}_p$ *on* $\mathfrak{W}$ *have the same degree. Thus* $\bar{\mathfrak{F}}_2 = \langle 1 \rangle$, $\bar{\mathfrak{F}}_p = \langle 1 \rangle$ *if* $p \nmid i(\mathfrak{W})$ *and* $\bar{\mathfrak{F}}_3$ *is elementary abelian.*

Proof. Suppose $\bar{\mathfrak{F}}_2 \neq \langle 1 \rangle$. Then by Lemma 4.1, $\mathfrak{G}$ has a normal subgroup $\mathfrak{E}_0$ of type $E(2, 3)$ and iso III. By the Reduction Lemma and Lemma 5.4 this is a contradiction.

If $p \neq 2$ then $\bar{\mathfrak{F}}_p$ acts in a completely reducible manner on $\mathfrak{W}$. If all its irreducible constituents do not have the same degree, then certainly we can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$ where $\mathfrak{W}_i \neq \langle 1 \rangle$ and $\mathfrak{W}_i$ is $\bar{\mathfrak{G}}$ invariant. One of these two, say $\mathfrak{W}_1$, has dimension at least 4. If $\mathfrak{E}_1/\boldsymbol{Z}(\mathfrak{E}) = \mathfrak{W}_1$ then $\mathfrak{E}_1 \triangle \mathfrak{G}$ and since $\mathfrak{G}$ is primitive, $\mathfrak{E}_1$ is type $E(2, m')$ with $m' = 2$ or 3. Since $q \geqq 7$. the Reduction Lemma and the $m = 2$ and 3 results yield a contradiction. Now if $p \nmid i(\mathfrak{W})$, then certainly $\bar{\mathfrak{F}}_p$ has a 1-dimensional constituent so they are all 1-dimensional and over $GF(2)$ this implies that $\bar{\mathfrak{F}}_p$ centralizes $\mathfrak{W}$ so $\bar{\mathfrak{F}}_p = \langle 1 \rangle$.

Finally we consider $\bar{\mathfrak{F}}_3$. If $\bar{\mathfrak{F}}_3$ is nonabelian then the degree of an irreducible representation of $\bar{\mathfrak{F}}_3$, with $\bar{\mathfrak{F}}_3'$ not in the kernel is divisible by 3. Since $3 \nmid \dim \mathfrak{W}$, $\bar{\mathfrak{F}}_3'$ is in the kernel of all constituents so $\bar{\mathfrak{F}}_3' = \langle 1 \rangle$ and $\bar{\mathfrak{F}}_3$ is abelian. Let $\mathfrak{W}_0$ be an irreducible $\bar{\mathfrak{F}}_3$-constituent of $\mathfrak{W}$ with dimension j. Then $j \mid \dim \mathfrak{W}$ so $j = 1, 2, 4$ or 8. In all these cases $9 \nmid 2^j - 1$ and hence clearly $\bar{\mathfrak{F}}_3$ is elementary abelian.

LEMMA 5.6. $\mathfrak{E} = \text{iso I}$ *does not occur.*

Proof. Here by Lemma 1.3, $i(\mathfrak{W}) = 3^3 \cdot 5$ so only $\bar{\mathfrak{F}}_5$ and $\bar{\mathfrak{F}}_3$ can be nontrivial. We show first that $\bar{\mathfrak{F}}_5 = \langle 1 \rangle$. Note that a Sylow 5-subgroup of $Sp(8, 2)$ is abelian of type $(5, 5)$.

Suppose first that $|\bar{\mathfrak{F}}_5| = 5^2$. Then $\bar{\mathfrak{F}}_5$ is elementary abelian and a Sylow 5-subgroup of $\bar{\mathfrak{G}}$. We can write $\mathfrak{W} = \mathfrak{W}_1 \oplus \mathfrak{W}_2$, $\bar{\mathfrak{F}}_5 = \bar{\mathfrak{L}}_1\bar{\mathfrak{L}}_2$ where $\dim \mathfrak{W}_i = 4$, $|\bar{\mathfrak{L}}_i| = 5$ and $\bar{\mathfrak{L}}_i$ acts irreducibly on $\mathfrak{W}_i$ and centralizes the other $\mathfrak{W}_j$. Now a Sylow 5-subgroup of $\mathfrak{G}$ is not cyclic so $5 \mid |\mathfrak{G}_x|$ for all $x \in \mathfrak{V}^\#$. We have $i(\mathfrak{W}) = 135$ and $|\mathfrak{W}_1 \cup \mathfrak{W}_2| = 31$. Hence we can find a noncentral involution $T \in \mathfrak{E}$ with $TZ(\mathfrak{E})/Z(\mathfrak{E}) \nsubseteq \mathfrak{W}_1 \cup \mathfrak{W}_2$. By Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E}_x = \langle T \rangle$ and if $\mathfrak{L} \subseteq \mathfrak{G}_x$ has order 5, then $\mathfrak{L}$ normalizes $\mathfrak{G}_x \cap \mathfrak{E} = \langle T \rangle$. Thus $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E} \subseteq \bar{\mathfrak{F}}_5$ centralizes the involution vector corresponding to T. Since $C_{\mathfrak{W}}(\bar{\mathfrak{L}}_1) = \mathfrak{W}_2$ and $C_{\mathfrak{W}}(\bar{\mathfrak{L}}_2) = \mathfrak{W}_1$ we see by our choice of T that $\bar{\mathfrak{L}} \neq \bar{\mathfrak{L}}_1$ or $\bar{\mathfrak{L}}_2$. But then $\bar{\mathfrak{L}}$ acts irreducibly on $\mathfrak{W}_1$ and $\mathfrak{W}_2$ so by the Jordan-Holder Theorem, $C_{\mathfrak{W}}(\bar{\mathfrak{L}}) = \langle 1 \rangle$, a contradiction.

Now let $|\bar{\mathfrak{F}}_5| = 5$. By the preceding lemma $\bar{\mathfrak{F}}$ is abelian. Since the irreducible nonprincipal representations of $\bar{\mathfrak{F}}_5$ over $GF(2)$ have degree 4 we see that either $\bar{\mathfrak{F}}$ is irreducible or it has two irreducible constituents of dimension 4. Thus $\bar{\mathfrak{F}}$ has two generators and $\bar{\mathfrak{F}}$ is abelian of type (5), $(3, 5)$ or $(3, 3, 5)$. Hence

$$|\bar{\mathfrak{G}}| \leqq 3^2\,|GL(2, 3)| \cdot 5 \cdot 4 = 8640 < 10^4 ,$$

a contradiction.

Thus $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_3$ is elementary abelian. If $|\bar{\mathfrak{F}}_3| \leqq 3^2$, then

$$|\bar{\mathfrak{G}}| \leqq 3^2\,|GL(2, 3)| = 432 < 10^4 ,$$

a contradiction. If $|\bar{\mathfrak{F}}_3| = 3^3$, then $|\bar{\mathfrak{G}}|$ divides both $|\bar{\mathfrak{F}}_3|\,|GL(3, 3)| = 2^5 \cdot 3^6 \cdot 13$ and $|Sp(8, 2)| = 2^{16} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17$ so $|\bar{\mathfrak{G}}|$ divides $2^5 \cdot 3^5 = 7776 < 10^4$, a contradiction. Since the Sylow 3-subgroup of $Sp(8, 2)$ is nonabelian of order 3^5 this leaves only $|\bar{\mathfrak{F}}_3| = 3^4$.

Let $\mathfrak{S}$ be a 3-subgroup of $\mathfrak{G}$ with $\mathfrak{S}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E} = \bar{\mathfrak{F}}_3$. Clearly $\mathfrak{S}' \subseteq \mathfrak{A}$. The action of $\mathfrak{S}$ on $\mathfrak{V}$ is completely reducible since $q \neq 3$ and since $\dim \mathfrak{V} = 2^4$ is not divisible by 3 it follows that $\mathfrak{S}'$ is in the kernel of some constituent so $\mathfrak{S}'$ has a fixed point in $\mathfrak{V}^\#$. Since $\mathfrak{A}$ acts semiregularly, $\mathfrak{S}' = \langle 1 \rangle$. Now $\mathfrak{S}$ is abelian and $\mathfrak{S}/(\mathfrak{S} \cap \mathfrak{A})$ is abelian of type $(3, 3, 3, 3)$. Thus $\mathfrak{S}$ contains a subgroup of type $(3, 3, 3, 3)$ and hence $3^3 \mid |\mathfrak{G}_x|$.

Now $\mathfrak{E} \cong \mathfrak{Q}\mathfrak{Q}\mathfrak{Q}\mathfrak{Q}$ so it is clear that the automorphism group of $\mathfrak{E}$ contains $\bar{\mathfrak{K}} = \bar{\mathfrak{J}} \times (\bar{\mathfrak{J}} \sim \bar{\mathfrak{J}})$ where $|\bar{\mathfrak{J}}| = 3$. This group is a Sylow 3-subgroup of $Sp(8, 2)$ and hence is a Sylow 3-subgroup of Aut $\mathfrak{E}$. We describe it more precisely. Write $\mathfrak{E} = \mathfrak{E}_0\mathfrak{E}_1\mathfrak{E}_2\mathfrak{E}_3$ where each $\mathfrak{E}_i \cong \mathfrak{Q}$. Then $\bar{\mathfrak{K}}$ has an elementary abelian subgroup $\bar{\mathfrak{N}}$ of index 3 with

$\bar{\mathfrak{N}} = \bar{\mathfrak{L}}_0\bar{\mathfrak{L}}_1\bar{\mathfrak{L}}_2\bar{\mathfrak{L}}_3$. Here $\bar{\mathfrak{L}}_i$ acts nontrivially on $\mathfrak{E}_i$ and centralizes the remaining $\mathfrak{E}_j$. Every element of $\bar{\mathfrak{K}} - \bar{\mathfrak{N}}$ normalizes $\mathfrak{E}_0$ and cyclically permutes $\mathfrak{E}_1, \mathfrak{E}_2$ and $\mathfrak{E}_3$. Let $\mathfrak{W} = \mathfrak{W}_0 \oplus \mathfrak{W}_1 \oplus \mathfrak{W}_2 \oplus \mathfrak{W}_3$ be the corresponding decomposition of $\mathfrak{W}$.

Let T be a noncentral involution of $\mathfrak{E}$. Then there exists $x \in \mathfrak{V}^\#$ by Lemma 1.5 with $\mathfrak{E}_x = \langle T \rangle$. Since $3^3 \mid |\mathfrak{G}_x|$ let $\mathfrak{L}$ be a subgroup of $\mathfrak{G}_x$ of order 3^3. Then $\mathfrak{L}$ normalizes $\mathfrak{G}_x \cap \mathfrak{E} = \mathfrak{E}_x = \langle T \rangle$. Since $\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$, $\mathfrak{L}$ acts faithfully on $\mathfrak{E}$. Thus a suitable conjugate $\bar{\mathfrak{L}}$ of $\mathfrak{L}$ in Aut $\mathfrak{E}$ is contained in $\bar{\mathfrak{K}}$ and clearly $\bar{\mathfrak{L}}$ also centralizes an involution vector of $\mathfrak{W}$. Let $W = W_0 + W_1 + W_2 + W_3 \in \mathfrak{W}$ with $W_i \in \mathfrak{W}_i$. Then we see easily that W is an involution vector if and only if either none or two of the W_i are zero. Suppose two of the W_i are zero. Then clearly $C_{\bar{\mathfrak{K}}}(W) \subseteq \bar{\mathfrak{N}}$ and then $|C_{\bar{\mathfrak{K}}}(W)| \leqq 3^2$. If none of the W_i are zero, then $C_{\bar{\mathfrak{K}}}(W) \cap \bar{\mathfrak{N}} = \langle 1 \rangle$ so $|C_{\bar{\mathfrak{K}}}(W)| \leqq 3$. This contradicts the fact that $|\bar{\mathfrak{L}}| = 3^3$ and $\bar{\mathfrak{L}}$ fixes an involution vector.

LEMMA 5.7. $\mathfrak{E} =$ iso II *does not occur.*

Proof. Here $i(\mathfrak{W}) = 7 \cdot 17$ by Lemma 1.3. Hence only $\bar{\mathfrak{F}}_7$ and $\bar{\mathfrak{F}}_{17}$ can be nontrivial. If $\bar{\mathfrak{F}}_7 \neq \langle 1 \rangle$ then since $7^2 \nmid |Sp(8, 2)|$, $|\bar{\mathfrak{F}}_7| = 7$. But the nonprincipal irreducible representations of this group over $GF(2)$ all have degree 3. Since $3 \nmid \dim \mathfrak{W}$ we have a contradiction. Then $\bar{\mathfrak{F}} = \bar{\mathfrak{F}}_{17}$ has order 1 or 17 and $|\bar{\mathfrak{G}}| \leqq 17 \cdot 16 < 10^4$, a contradiction.

We have therefore shown in this section that if $\mathfrak{G}$ is solvable then $m = 3$ and 4 do not occur.

6. Theorem B.

The following assumption holds throughout this section.

ASSUMPTION. Group $\mathfrak{G}$ acts faithfully on vector space $\mathfrak{V}$ of order q^n, q a prime, and acts half-transitively but not semiregularly on $\mathfrak{V}^\#$. Further $\mathfrak{G}$ is primitive as a linear group and $\mathfrak{G}$ is solvable.

Let $\mathscr{T}(q^n)$ denote the group of all semilinear transformations on $GF(q^n)$ of the form $x \longrightarrow ax^\sigma$ where $a \in GF(q^n)^\#$ and σ is a field automorphism. Thus $\mathscr{T}(q^n)$ is the stabilizer in the permutation group $\mathscr{S}(q^n)$ of the point 0.

LEMMA 6.1. *Let* $\mathfrak{F} = F(\mathfrak{G})$ *and set* $\mathfrak{A} = Z(C_{\mathfrak{F}}(\Phi(\mathfrak{F})))$. *Then* $\mathfrak{A}$ *is a normal cyclic subgroup of* $\mathfrak{G}$

(i) *If* $\mathfrak{A} = C_{\mathfrak{F}}(\Phi(\mathfrak{F}))$, *then with suitable identification we have* $\mathfrak{G} \subseteq \mathscr{T}(q^n)$.

(ii) *If* $\mathfrak{A} \neq C_{\mathfrak{F}}(\Phi(\mathfrak{F}))$, *then* $C_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{E}$ *where* $\mathfrak{E}$ *is a group of type* $E(2, m)$ *and* $\mathfrak{E} \triangle \mathfrak{G}$. *Moreover* $m = 1$ *or* 2.

(iii) *In the above if* $m = 1$ *and* $4 \nmid |\mathfrak{A}|$, *then either* $\mathfrak{G} \subseteq \mathscr{T}(q^n)$ *or* $q^n = 3^2, 7^2$ *or* 11^2.

Proof. Let $\mathfrak{F}_p$ be the normal Sylow p-subgroup of $\mathfrak{F}$. By Theorem A $\mathfrak{F}_p$ is cyclic for p > 2 and $\mathfrak{F}_2$ is a group of symplectic type. Since $\mathfrak{A} = \boldsymbol{Z}(\boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F})))$ is a normal abelian subgroup of a primitive group it is cyclic.

From the structure of 2-groups of symplectic type we see that if $\mathfrak{A} = \boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F}))$, then $\mathfrak{F}_2$ is either cyclic or maximal class of order at least 16. Now $\mathfrak{F} = \mathfrak{A}\mathfrak{F}_2$ so $\boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A})/\boldsymbol{Z}(\mathfrak{F})$ acts faithfully on $\mathfrak{F}_2$. Since Aut $\mathfrak{F}_2$ is a 2-group and $\boldsymbol{Z}(\mathfrak{F}) \subseteq \mathfrak{A}$ we see that $\boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A})$ is a normal nilpotent subgroup of $\mathfrak{G}$ and hence $\boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A}) \subseteq \mathfrak{F}$. This yields easily $\boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A}) = \mathfrak{A}$. By Proposition 1.2 of [5] we see that $\mathfrak{G} \subseteq \mathscr{T}(q^n)$ and (i) follows.

Suppose $\mathfrak{A} \neq \boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F}))$. Then as we pointed out in § 1, $\boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{E}$ where $\mathfrak{E}$ is a group of type $E(2, m)$ and $\mathfrak{E} \triangle \mathfrak{G}$. By Theorem A and the results of § 5, $m = 1$ or 2.

Let $m = 1$ and suppose $4 \nmid |\mathfrak{A}|$. Then $\mathfrak{F}_2 \cong \mathfrak{D}$ or $\mathfrak{Q}$. If $\mathfrak{F}_2 \cong \mathfrak{D}$ then $\mathfrak{F}$ has a characteristic cyclic subgroup $\mathfrak{B}$ of index 2. Since Aut $\mathfrak{D}$ is a 2-group, the above argument yields $\mathfrak{G} \subseteq \mathscr{T}(q^n)$ again. If $\mathfrak{F}_2 \cong \mathfrak{Q}$, then by Proposition 1.10 of [5] $q^n = 3^2, 7^2$ or 11^2. This completes the proof.

We assume now that $\mathfrak{A} \neq \boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F}))$.

LEMMA 6.2. *Let* $\mathfrak{B} = \boldsymbol{C}_{\mathfrak{G}}(\mathfrak{A})/\mathfrak{A}\mathfrak{E}$. *Then* $\boldsymbol{O}_2(\mathfrak{B}) = \langle 1 \rangle$, $\mathfrak{B}$ *acts faithfully on* $\mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ *and* $\mathfrak{B} \subseteq Sp(2m, 2)$.

Proof. Let $\mathfrak{L}/\mathfrak{A}\mathfrak{E} = \boldsymbol{O}_2(\mathfrak{B})$. Since $\mathfrak{A}$ is central in $\mathfrak{L}$ and $\mathfrak{L}/\mathfrak{A}$ is a 2-group, we see that $\mathfrak{L}$ is a normal nilpotent subgroup of $\mathfrak{G}$ and hence $\mathfrak{L} \subseteq \mathfrak{F}$. Now $\Phi(\mathfrak{F}) \subseteq \mathfrak{A}$ and $\boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{E}$. Hence

$$\mathfrak{L} \subseteq \boldsymbol{C}_{\mathfrak{F}}(\mathfrak{A}) \subseteq \boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{E}$$

so $\mathfrak{L} = \mathfrak{A}\mathfrak{E}$ and $\boldsymbol{O}_2(\mathfrak{B}) = \langle 1 \rangle$.

Let $\mathfrak{H} = \mathfrak{C}_{\mathfrak{G}}(\mathfrak{A})$ and let $\mathfrak{K} = \boldsymbol{C}_{\mathfrak{H}}(\mathfrak{W})$ where $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. We have of course $\mathfrak{K} \supseteq \mathfrak{A}\mathfrak{E}$. First $\mathfrak{K}$ centralizes $\boldsymbol{O}_{2'}(\mathfrak{F}) \subseteq \mathfrak{A}$. If $\mathfrak{F}_2 = \boldsymbol{O}_2(\mathfrak{F})$, then since clearly $[\mathfrak{F}_2 : \mathfrak{A}_2\mathfrak{E}] = 2$, where $\mathfrak{A}_2 = \mathfrak{A} \cap \mathfrak{F}_2$, we see that $\mathfrak{K}$ stabilizes the chain $\mathfrak{F}_2 \supseteq \mathfrak{A}_2\mathfrak{E} \supseteq \mathfrak{A}_2 \supseteq \langle 1 \rangle$. Thus $\mathfrak{K}/\boldsymbol{C}_{\mathfrak{K}}(\mathfrak{F})$ is a 2-group. Since $\mathfrak{K} \supseteq \boldsymbol{Z}(\mathfrak{F})$, $\boldsymbol{C}_{\mathfrak{K}}(\mathfrak{F}) = \boldsymbol{Z}(\mathfrak{F})$ and hence $\mathfrak{K}/\boldsymbol{Z}(\mathfrak{F})$ is a 2-group. But $\boldsymbol{Z}(\mathfrak{F}) \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is central in $\mathfrak{K}$ so $\mathfrak{K}$ is a normal nilpotent subgroup of $\mathfrak{G}$ and $\mathfrak{K} \subseteq \mathfrak{F}$. This yields easily $\mathfrak{K} = \mathfrak{A}\mathfrak{E}$ and thus $\mathfrak{B} = \mathfrak{H}/\mathfrak{K}$ acts faithfully on $\mathfrak{W}$. It now follows immediately that $\mathfrak{B} \subseteq Sp(2m, 2)$.

LEMMA 6.3. *Let* $\mathfrak{A} = \langle A \rangle$ *and let* ζ *be an eigenvalue of* A *with* $GF(q)(\zeta) = GF(q^r)$. *Then*

(i) $C_{\mathfrak{G}}(\mathfrak{A}) \subseteq GL(n/r, q^r)$, $|\mathfrak{A}| \mid (q^r - 1)$
(ii) $\mathfrak{G}/C_{\mathfrak{G}}(\mathfrak{A})$ *is cyclic of order dividing* r.
(iii) $n = w2^m r$ *for some integer* w.

Proof. Parts (i) and (ii) follow from Lemma 1.1 of [5]. Now all irreducible constituents of $\mathfrak{E}$ are faithful and the same is clearly true if we view $\mathfrak{E} \subseteq GL(n/r, q^r)$. Thus n/r is divisible by 2^m, the degree of the nonlinear absolutely irreducible representations of $\mathfrak{E}$.

LEMMA 6.4. *If* $m = 1$ *and* $4 \mid |\mathfrak{A}|$, *then* $q^n = 5^2$ *or* 17^2.

Proof. We can assume that $|Z(\mathfrak{E})| = 4$ so $\mathfrak{E} \cong \mathfrak{Z}\mathfrak{Q}$. By the Reduction Lemma and Lemma 3.4, $q = 3,5$ or 17. Set $\mathfrak{H} = C_{\mathfrak{G}}(\mathfrak{A})$. Then by the above $\mathfrak{H}/\mathfrak{A}\mathfrak{E} = \mathfrak{B}$ is contained isomorphically in $Sp(2, 2) = SL(2, 2) \cong \mathrm{Sym}_3$. Since $O_2(\mathfrak{B}) = \langle 1 \rangle$, $|\mathfrak{B}| = 1$, 3 or 6.

Suppose $|\mathfrak{B}| = 1$. Now $2 \mid |\mathfrak{G}_x|$ so we can apply Lemma 3.1 with $p = 2$. Note that $\mathfrak{G}/\mathfrak{A}\mathfrak{E}$ is cyclic of order dividing r and $k = n/r = 2w$. If r is odd, then $\lambda_1 \leqq 3$, $\lambda_2 = 0$ so by Lemma 3.1, (ii) and (iii), we have $q^r < 6$ so $r = 1$. If r is even, then $\lambda_1 \leqq 3$, $\lambda_2 \leqq 4$ so we get easily $q^{r/2} \leqq 5$ and hence $r = 2$. Now $\mathfrak{E}$ has precisely three normal abelian subgroups of type $(2, 2)$. Since $\mathfrak{G}/\mathfrak{A}\mathfrak{E}$ is a 2-group one of these three abelian groups will be normal in $\mathfrak{G}$, a contradiction since $\mathfrak{G}$ is primitive. Thus $|\mathfrak{B}| = 3$ or 6.

Suppose $3 \mid |\mathfrak{G}_x|$. We again apply Lemma 3.1. If $3 \nmid r$ then $\lambda_1 \leqq 4$, $\lambda_2 = 0$ while if $3 \mid r$, then $\lambda_1 \leqq 4$ and we see easily that $\lambda_2 \leqq 9$. Let $3 \nmid r$ so by Lemma 3.1 we have $q^r < 8$. Since $4 \mid q^r - 1$, $q^r = 5$ and then by Lemma 3.1 (i) we have $k = 2$ and $n = 2$. But $3 \nmid q - 1$ so no element of $GL(2, 5)$ of order 3 can have a nonzero fixed point, a contradiction. Let $3 \mid r$. Then Lemma 3.1, (ii) and (iii), yields $q^{r/3} < 4$ so $q^r = 3^3$. This is a contradiction since $4 \nmid (3^3 - 1)$. Now $3 \mid |\mathfrak{G}|$ so we see also that $q \neq 3$ and thus $q = 5$ or 17. We assume that $q^n \neq 5^2$ or 17^2 and derive a contradiction.

Suppose first that r is odd. We apply Lemma 3.1 with $p = 2$. Then $\lambda_1 \leqq 9$, $\lambda_2 = 0$ so we have $q^r < 18$. Thus $q^r = 5$ or 17 and $r = 1$. By Lemma 1.5, there exists $x \in \mathfrak{B}^{\#}$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Since $r = 1$ and $3 \nmid |\mathfrak{G}_x|$ we have $|\mathfrak{G}_x| = 2$. Hence by Lemma 1.9, $I(\mathfrak{G}) = q^{n/2} + 1$. Now $\mathfrak{A}$ is central in $\mathfrak{G}$ and cyclic so each involution of $\mathfrak{G}/\mathfrak{A}$ corresponds to at most two noncentral involutions of $\mathfrak{G}$. Thus

$$q^{n/2} + 1 = I(\mathfrak{G}) \leqq 2 \cdot 9 = 18$$

so $q^n = 5^2$ or 17^2, a contradiction.

Now let r be even. We have easily $\lambda_1 \leqq 9$, $\lambda_2 \leqq 10$. Thus if $k > 2$ then Lemma 3.1 (iii) yields $q^r = 5^2$ and then by Lemma 3.1 (i) with $k > 2$ we have a contradiction. Thus $k = 2$ and by Lemma 3.1 (ii), $q^r + 1 \leqq 18 + 10(q^{r/2} + 1)$ so $q^{r/2} < 13$. Since r is even $q^r = 5^2$. By Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{G}_x \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$ and hence since $3 \nmid |\mathfrak{G}_x|$ we have $|\mathfrak{G}_x| = 2$ or 4.

Suppose $|\mathfrak{G}_x| = 4$. Since $[\mathfrak{G} : \mathfrak{H}] = 2$ where $\mathfrak{H} = \boldsymbol{C}(\mathfrak{A})$ we see that $2 \mid |\mathfrak{H}_x|$ for all $x \in \mathfrak{V}^\#$. Clearly $\mathfrak{H}$ acts irreducibly on $\mathfrak{V}$ so by Lemma 3.1 applied to $\mathfrak{H}$ with $p = 2$ we have $\lambda_1 \leqq 9$, $\lambda_2 = 0$ so $25 = q^r < 18$, a contradiction. Thus $|\mathfrak{G}_x| = 2$.

Now here $n = kr = 4$. By Lemma 1.9, we have $I(\mathfrak{G}) = 1 + q^{n/2} = 26$. Let $\mathfrak{L}$ be a Sylow 3-subgroup of $\mathfrak{G}$. Since $3 \nmid |\mathfrak{G}_x|$, $\mathfrak{L}$ is cyclic and acts semiregularly so $|\mathfrak{L}| \mid 5^4 - 1$ and $|\mathfrak{L}| = 3$. Since $3 \mid |\mathfrak{B}|$ we have $\mathfrak{L} \cap \mathfrak{A}\mathfrak{E} = \langle 1 \rangle$. Now $\mathfrak{L}$ permutes by conjugation the noncentral involutions of $\mathfrak{G}$ and since $3 \nmid I(\mathfrak{G})$ we see that $\mathfrak{L}$ centralizes a noncentral involution of $\mathfrak{G}$. The group $\mathfrak{G}/\mathfrak{A}\mathfrak{E}$ acts on $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$. If the action is faithful then clearly $\mathfrak{G}/\mathfrak{A} \subseteq \mathrm{Sym}_4$. Since subgroups of order 3 of Sym_4 are self-centralizing we have a contradiction. Hence the action is not faithful so say $\mathfrak{K}/\mathfrak{A}\mathfrak{E}$ is the kernel with $\mathfrak{K} > \mathfrak{A}\mathfrak{E}$. Now $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ does act faithfully so $[\mathfrak{K} : \mathfrak{A}\mathfrak{E}] = 2$. Note that $\mathfrak{K} \triangle \mathfrak{G}$. Also $3 \nmid |\mathfrak{A}|$ and $|\mathfrak{A}| \mid 5^2 - 1$ implies $\mathfrak{A}$ is a 2-group and hence $\mathfrak{K}$ is a 2-group. Since $\mathfrak{G}$ is primitive, $\mathfrak{K}$ is of symplectic type. Moreover $\boldsymbol{Z}(\mathfrak{E}) \triangle \mathfrak{G}$, $|\boldsymbol{Z}(\mathfrak{E})| = 4$ and $4 \mid q - 1$. Hence $\boldsymbol{Z}(\mathfrak{E})$ is central in $\mathfrak{G}$ so $\mathfrak{K}$ must be the central product of a cyclic group with a nonabelian group of order 8. Now $\mathfrak{K} \subseteq \mathfrak{F}$ and since $[\mathfrak{F} : \mathfrak{A}\mathfrak{E}] \leqq 2$ we have $\mathfrak{K} = \mathfrak{F}$. Then $\Phi(\mathfrak{F})$ is central in $\mathfrak{F}$ and $\mathfrak{F} = \boldsymbol{C}_{\mathfrak{F}}(\Phi(\mathfrak{F})) = \mathfrak{A}\mathfrak{E}$, a contradiction. This completes the proof of the lemma.

LEMMA 6.5. *If $m = 2$, then $q^n = 3^4$.*

Proof. By the Reduction Lemma and Lemmas 4.3, 4.4 and 4.5 we have $q = 3$ and $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}$. Hence $4 \nmid |\mathfrak{A}|$. We consider $\bar{\mathfrak{K}} = F(\mathfrak{H}/\mathfrak{A}\mathfrak{E})$. By Lemma 6.2, $\bar{\mathfrak{K}}_2 = \boldsymbol{O}_2(\bar{\mathfrak{K}}) = \langle 1 \rangle$. Suppose $\bar{\mathfrak{K}}_3 = \boldsymbol{O}_3(\bar{\mathfrak{K}}) \neq \langle 1 \rangle$. Since $q = 3$, a Sylow 3-subgroup of $\mathfrak{G}$ has a fixed point in $\mathfrak{V}^\#$ and hence by half-transitivety $\mathfrak{G}_x$ contains a Sylow 3-subgroup of $\mathfrak{G}$ for all $x \in \mathfrak{V}^\#$ Let T be a noncentral involution of $\mathfrak{E}$. By Lemma 1.5 there exists $x \in \mathfrak{V}^\#$ with $\mathfrak{E} = \langle T \rangle$. Now we can find 3-subgroup $\mathfrak{L}$ of $\mathfrak{G}_x$ such that $\bar{\mathfrak{L}} = \mathfrak{L}\mathfrak{A}\mathfrak{E}/\mathfrak{A}\mathfrak{E} = \bar{\mathfrak{K}}_3$. Since $\mathfrak{L}$ normalizes $\mathfrak{E} \cap \mathfrak{G}_x = \mathfrak{E}_x$ we see that $\bar{\mathfrak{K}}_3$ centralizes the involution vector corresponding to T. Thus $\bar{\mathfrak{K}}_3$ centralizes all the involution vectors of $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ so by Lemma 6.3, $\bar{\mathfrak{K}}_3 = \langle 1 \rangle$.

Now $\mathfrak{B} \subseteq Sp(4, 2)$ and $|Sp(4, 2)| = 2^4 \cdot 3^2 \cdot 5$. Since $\bar{\mathfrak{K}} = \boldsymbol{O}_5(\mathfrak{B})$ by the above we have $|\bar{\mathfrak{K}}| = 1$ or 5 and hence $|\mathfrak{B}| \leqq 20$ and $|\mathfrak{H}/\mathfrak{A}| \leqq 16 \cdot 20 = 320$. We use Lemma 3.1 with $p = 2$. Note that $k = n/r \geqq 2^m = 4$ so Lemma 3.1 (iii) always applies. Certainly $\lambda_2 \leqq 320$. From the structure of $\bar{\mathfrak{H}} = \mathfrak{H}/\mathfrak{A}$ we see that $\lambda_1 \leqq 15 + 5 \cdot 4 = 35$. Hence

$$q^r < 2(\lambda_1 + \lambda_2) = 710 .$$

Since $q = 3$, this yields $r \leqq 5$. However if $r = 5$, then $[\mathfrak{G} : \mathfrak{H}]$ is odd so $\lambda_2 = 0$ and then $q^r < 2\lambda_1 = 70$, a contradiction. Thus $r \leqq 4$.

Since $r \leqq 4$ we see that $\bar{\mathfrak{K}}$ is a Sylow 5-subgroup of $\mathfrak{G}/\mathfrak{A}\mathfrak{E}$. Hence if $5 \mid |\mathfrak{G}_x|$, then as in the preceding argument with $\bar{\mathfrak{K}}_3$ we conclude that $\bar{\mathfrak{K}}$ fixes all involution vectors of $\mathfrak{W} = \mathfrak{E}/\boldsymbol{Z}(\mathfrak{E})$ and thus $\bar{\mathfrak{K}} = \langle 1 \rangle$. This certainly contradicts $5 \mid |\mathfrak{G}_x|$. Hence $5 \nmid |\mathfrak{G}_x|$. Let T be a noncentral involution of $\mathfrak{G}$. We show that $|\boldsymbol{C}_{\mathfrak{V}}(T)| \leqq q^{n/2}$. This is certainly the case if $T \in \mathfrak{A}\mathfrak{E}$. Let $T \in \mathfrak{H} - \mathfrak{A}\mathfrak{E}$. Then $|\bar{\mathfrak{K}}| = 5$ since $\mathfrak{H}/\mathfrak{A}\mathfrak{E} \neq \langle 1 \rangle$. Clearly there exists $K \in \mathfrak{H}$ so that the image of $\langle T, T^K \rangle$ in $\mathfrak{H}/\mathfrak{A}\mathfrak{E}$ contains $\bar{\mathfrak{K}}$. Since $5 \nmid |\mathfrak{G}_x|$ we see that $\boldsymbol{C}_{\mathfrak{V}}(T) \cap \boldsymbol{C}_{\mathfrak{V}}(T^K) = \{0\}$. Thus the result follows here. Finally if $T \in \mathfrak{G} - \mathfrak{H}$, then there exists $A \in \mathfrak{A}$ with $\langle T, T^A \rangle \cap \mathfrak{A} \neq \langle 1 \rangle$. Since $\mathfrak{A}$ acts semiregularly the result follows.

We show that r is not even. If r is even, then $r = 2$ or 4. If $r = 2$ then $|\mathfrak{A}| \mid q^r - 1$ and $q^r - 1 = 8$. Since $4 \nmid |\mathfrak{A}|$ we have $|\mathfrak{A}| = 2$ and $|\mathfrak{A}| \mid q - 1$. This violates the definition of r and hence $r = 4$. Here $|\mathfrak{A}| \mid q^r - 1$ and $q^r - 1 = 2^4 \cdot 5$ so $|\mathfrak{A}| \mid 10$. Since $|\mathfrak{A}| \leqq 10$ each involution of $(\mathfrak{G} - \mathfrak{H})/\mathfrak{A}$ comes from at most 10 of $\mathfrak{G} - \mathfrak{H}$. Thus

$$I(\mathfrak{G}) \leqq 2 \cdot 35 + 10 \cdot 320 = 3270 .$$

Since $2 \mid |\mathfrak{G}_x|$ we have $\mathfrak{V} = \bigcup C_{\mathfrak{V}}(T)$ over involutions T and hence

$$q^n = |\mathfrak{V}| \leqq I(\mathfrak{G}) q^{n/2} \leqq 3270 q^{n/2}$$

so $q^{n/2} \leqq 3270$. Thus $n < 16$. But $r = 4$ and $n \geqq 2^m r = 16$ so we have a contradiction. Thus r is odd.

Since r is odd, all involutions of $\mathfrak{G}$ are contained in $\mathfrak{H}$. Now $\mathfrak{A}$ is cyclic and central in $\mathfrak{H}$ so each involution of $\mathfrak{H}/\mathfrak{A}$ comes from at most two of $\mathfrak{H}$. Hence $I(\mathfrak{G}) \leqq 2 \cdot 35 = 70$ and since $2 \mid |\mathfrak{G}_x|$ we have

$$q^n = |\mathfrak{V}| \leqq I(\mathfrak{G}) q^{n/2} \leqq 70 q^{n/2}$$

or $q^{n/2} \leqq 70$. Since $4 \mid n$ we have $n = 4$ and thus $r = 1$. This completes the proof of the lemma.

Combining Lemmas 6.1, 6.4 and 6.5 we obtain

THEOREM 6.6. *Let $\mathfrak{G}$ act faithfully on vector space $\mathfrak{V}$ of order q^n and let $\mathfrak{G}$ act half-transitively but not semiregularly on $\mathfrak{V}^\#$. If $\mathfrak{G}$ is primitive as a linear group and if $\mathfrak{G}$ is solvable, then $\mathfrak{G}$ satisfies one of the following.*

(i) $\mathfrak{G} \subseteqq \mathscr{T}(q^n)$.
(ii) $q^n = 3^2, 5^2, 7^2, 11^2, 17^2$ *or* 3^4.

The proof of the main theorem now follows easily.

Proof of Theorem B. Let $\mathfrak{G}$ be the given solvable 3/2-transitive permutation group and assume that $\mathfrak{G}$ is not a Frobenius group. By Theorem 10.4 of [11], $\mathfrak{G}$ is primitive. Let $\mathfrak{V}$ be a minimal normal subgroup of $\mathfrak{G}$. Since $\mathfrak{G}$ is solvable, $\mathfrak{V}$ is elementary abelian of order q^n. Since $\mathfrak{G}$ is primitive, $\mathfrak{V}$ is transitive and hence regular. If α is a point being permuted, then by Theorem 11.2 of [11], $\mathfrak{G}_\alpha$ is an automorphism group of $\mathfrak{V}$ which acts half-transitively but not semiregularly on $\mathfrak{V}^\#$. By Theorems 1.1 and 6.6 we have $\mathfrak{G}_\alpha = \mathscr{T}_0(q^{n/2})$, $\mathfrak{G}_\alpha \subseteqq \mathscr{T}(q^n)$ or $q^n = 3^2, 5^2, 7^2, 11^2, 17^2, 3^4$. Note that the exception of Theorem 1.1 of degree 2^6 is a subgroup of $\mathscr{T}(2^6)$. Since deg $\mathfrak{G} = q^n$ and $\mathfrak{G} = \mathfrak{V}\mathfrak{G}_\alpha$, the result follows.

7. Theorem C.

We can now obtain several easy corollaries.

COROLLARY 7.1. *Let $\mathfrak{G}$ be a solvable 3/2-transitive permutation group. Then for all points $\alpha \neq \beta$ the stabilizers $\mathfrak{G}_{\alpha\beta}$ are isomorphic. In fact if $q^n \neq 3^2$, then $\mathfrak{G}_{\alpha\beta}$ is cyclic, while if $q^n = 3^2$, then $\mathfrak{G}_{\alpha\beta} \subseteqq$ Sym$_3$.*

Proof. The result is clear if $\mathfrak{G}$ is a Frobenius group, $\mathfrak{G} \subseteqq \mathscr{S}(q^n)$ or $\mathfrak{G} = \mathscr{S}_0(q^n)$. Thus we need only consider the exceptions. Here $\mathfrak{G}_\alpha$ acts on $\mathfrak{V}$ and $\mathfrak{G}_{\alpha\beta}$ is the stabilizer of $\beta \in \mathfrak{V}^\#$. Suppose $q^n = 5^2, 7^2, 11^2$ or 17^2. Since we see easily that $|\mathfrak{G}_\alpha|$ is prime to q it follows by complete reducibility that $\mathfrak{G}_{\alpha\beta}$ has a faithful 1-dimensional representation and hence is cyclic. Suppose $q^n = 3^2$. Since $\mathfrak{G}_\alpha \supseteqq \mathfrak{E} \cong \mathfrak{Q}$ we see that $\mathfrak{G}_\alpha$ is transitive on $\mathfrak{V}^\#$. Also $\mathfrak{G}_\alpha/\mathfrak{E} \subseteqq$ Sym$_3$ and $\mathfrak{G}_{\alpha\beta} \cap \mathfrak{E} = \langle 1 \rangle$ so the result follows here. Finally let $q^n = 3^4$ so that $\mathfrak{E} \triangle \mathfrak{G}_\alpha$ with $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}$. Then $\mathfrak{E} = O_2(\mathfrak{G}_\alpha)$. If $\mathfrak{E} = \mathfrak{G}_\alpha$ then $|\mathfrak{G}_{\alpha\beta}| = 2$. If $\mathfrak{G}_\alpha > \mathfrak{E}$ then as we have seen $5 \mid |\mathfrak{G}_\alpha/\mathfrak{E}|$. This implies that $\mathfrak{G}_\alpha$ acts transitively on $\mathfrak{V}^\#$. The result now follows by Lemma 2.4 of [5].

COROLLARY 7.2. *Let $\mathfrak{G}$ be a solvable linear group acting on $GF(q)$-vector space $\mathfrak{V}$. Suppose $\mathfrak{G}$ acts half-transitively on $\mathfrak{V}^\#$. If $q \neq 2$ and $|\mathfrak{G}|$ is even, then $\mathfrak{G}$ has a central involution.*

Proof. The result is well known if $\mathfrak{G}$ acts semi-regularly and obvious in all of the remaining cases with the exception of $\mathfrak{G} \subseteqq \mathscr{T}(q^n)$. Here the argument of Step 1 of the proof of Proposition 2.7 of [8] yields the result.

Finally we consider the transitive extensions of these exceptional 3/2-transitive groups.

Proof of Theorem C. Let $\mathfrak{G}$ be a 5/2-transitive permutation group on the set Ω and assume that $\mathfrak{G}$ is not a Zassenhaus group. Let $\infty, 0 \in \Omega$ and assume that $\mathfrak{G}_\infty$ is solvable. Thus $\mathfrak{G}_\infty$ is a solvable 3/2-transitive group which is not a Frobenius group. If $\mathfrak{G}_\infty \subseteqq \mathscr{S}(q^n)$ or $\mathfrak{G}_\infty = \mathscr{S}_0(q^{n/2})$ then by the results of [8], $\bar{\Gamma}(q^n) < \mathfrak{G} \subseteqq \Gamma(q^n)$. Hence we need only consider the exceptional groups. We show that these have no transitive extensions.

Set $\mathfrak{H} = \mathfrak{G}_{\infty 0}$ so that $\mathfrak{G}_\infty = \mathfrak{H}\mathfrak{B}$ where $\mathfrak{B}$ is a regular normal elementary abelian subgroup of order q^n. Let Z denote the central involution of $\mathfrak{H}$. Then Z fixes 0 and ∞ and moves all the rest. Since $\mathfrak{G}$ is doubly transitive we can find a suitable conjugate T of Z with $T = ((0, \infty)) \cdots$. Thus T normalizes $\mathfrak{H}$. By Lemma 1.3 of [8], $|\mathfrak{H}| \geqq (q^n - 1)/2$. If $q^n = 17^2$, then by Lemma 3.5

$$96 = |\mathfrak{H}| \geqq (17^2 - 1)/2 \,.$$

a contradiction.

We will use results of § 3 and § 4 about these exceptional groups which were not explicitly stated. Let $\mathfrak{E} = O_2(\mathfrak{H})$ so that T normalizes $\mathfrak{E}$. Suppose T fixes the point α. Since T centralizes Z we see that $(\alpha Z)T = \alpha TZ = \alpha Z$ so T also fixes $\beta = \alpha Z$ and these must be the two points of Ω fixed by T. Since T is conjugate to Z and Z is central in $\mathfrak{G}_{\infty 0}$ we see that T is central in $\mathfrak{G}_{\alpha\beta}$. Thus T centralizes $\mathfrak{H}_{\alpha\beta}$. Note that $\mathfrak{H}_{\alpha\beta} = \mathfrak{H}_\alpha = \mathfrak{H}_\beta$ since $\alpha Z = \beta$. Conversely let T centralize $H \in \mathfrak{H}$. Then $(\alpha H)T = \alpha TH = \alpha H$ so $\alpha H = \alpha$ or β. Hence $H \in \langle Z, \mathfrak{H}_\alpha \rangle$ and hence $C_{\mathfrak{H}}(T) = \langle Z, \mathfrak{H}_\alpha \rangle$.

Suppose $3 \mid |\mathfrak{H}_x|$ for $x \in \mathfrak{B}^\#$. This implies easily that $q^n = 3^2$ or 7^2 and $\mathfrak{E} \cong \mathfrak{Q}$. Since $\mathfrak{E}$ acts semiregularly on $\mathfrak{B}^\#$, $C_{\mathfrak{E}}(T) = \langle Z \rangle$ and thus T acts nontrivially on $\mathfrak{E}/Z(\mathfrak{E})$. Let $\mathfrak{J}$ be a subgroup of $\mathfrak{H}_\alpha$ of order 3. Then $\langle T, \mathfrak{J} \rangle$ is cyclic of order 6 and acts faithfully on $\mathfrak{E}/Z(\mathfrak{E})$, a contradiction. Thus $|\mathfrak{H}_x|$ is a cyclic 2-group. Note that if $q^n = 3^2$, then $3 \nmid |\mathfrak{H}|$ so clearly $\mathfrak{H} \subseteqq \mathscr{T}(3^2)$ and $\mathfrak{G}_\infty$ is not exceptional.

Set $\mathfrak{K} = \mathfrak{G} \cap \operatorname{Alt} \Omega$. Since $\mathfrak{K} \supseteqq \mathfrak{B}, T, Z$ $\mathfrak{K}$ is doubly transitive and $\mathfrak{K}_{\infty 0}$ has a central involution. Also $[\mathfrak{G} : \mathfrak{K}] \leqq 2$. Let $q^n = 7^2$ or 11^2. Then $|\mathfrak{H}_x| \mid q - 1$ so clearly $|\mathfrak{H}_x| = 2$. If H is a noncentral involution of $\mathfrak{H}$ then H moves $q^2 - q$ points and hence H is a product of $q(q - 1)/2$ transpositions. Thus with $q = 7$ or 11, $H \notin \mathfrak{K}$ and therefore $\mathfrak{K}$ is a Zassenhaus group. Since $\mathfrak{K}_{\infty 0}$ has a central involution the results of [12] yield $\mathfrak{K} \subseteqq \mathscr{T}(q^2)$ and hence $\mathfrak{K}_{\infty 0}$ has a normal Sylow 3-subgroup, a contradiction. This leaves only $q^n = 5^2$ and 3^4.

Let $q^n = 5^2$. Suppose $H \in \mathfrak{H}$ has order 4 and fixes a point of $\mathfrak{B}^\#$.

Since H and H^2 fix the same set of points here, we see that H is a product of $(5^2-5)/4 = 5$ 4-cycles. Thus $H \notin \mathfrak{K}$. Now $\mathfrak{G}_\infty$ is exceptional so $3 \mid |\mathfrak{G}_{\infty 0}|$ and hence by the above remarks $|\mathfrak{K}_{\infty 0}| = 16 \cdot 3 = 48$. Thus $|\mathfrak{K}| = 26 \cdot 25 \cdot 48$. Let $\mathfrak{P}$ be a Sylow 13-subgroup of $\mathfrak{K}$. Then $[\mathfrak{K} : \mathfrak{P}] = 2 \cdot 25 \cdot 48 \equiv 8 \bmod 13$. If $\mathfrak{N} = N_{\mathfrak{K}}(\mathfrak{P})$, then by Sylow's theorem, $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \bmod 13$. We see easily that $\mathfrak{P}$ has two orbits of size 13. If $\mathfrak{A}$ is an abelian subgroup of $\mathfrak{K}$ containing $\mathfrak{P}$, then either $\mathfrak{A}$ has two orbits and then $\mathfrak{A} = \mathfrak{P}$ or $\mathfrak{A}$ is transitive. In the latter case $\mathfrak{A}$ is regular so if $A \in \mathfrak{A}$ has order 2, then A is a product of 13 transpositions and $A \notin \operatorname{Alt} \Omega$, a contradiction. Hence $\mathfrak{A} = \mathfrak{P}$ and $\mathfrak{P} = C_{\mathfrak{K}}(\mathfrak{P})$. Thus $\mathfrak{N}/\mathfrak{P} \subseteqq \operatorname{Aut} \mathfrak{P}$ so $[\mathfrak{N} : \mathfrak{P}] \mid 12$. Since $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \bmod 13$, we have a contradiction.

Finally let $q^n = 3^4$ so that $\mathfrak{G}$ has degree $3^4 + 1 = 2 \cdot 41$. Now $|\mathfrak{H}| \geqq (q^n - 1)/2 = 40$ so we cannot have $\mathfrak{H} \cong \mathfrak{D}\mathfrak{Q}$. Hence we must have $5 \mid |\mathfrak{H}|$ so $\mathfrak{H}$ is transitive on $\mathfrak{V}^{\sharp}$ and we thus see easily that $\mathfrak{K}$ is triply transitive. Now $|\mathfrak{H}_x| = 2,4$ or 8 so write $|\mathfrak{K}_x| = 2 \cdot 2^\delta$ where $2^\delta = 1,2$ or 4. Then

$$|\mathfrak{K}| = 82(82-1)(82-2) \cdot 2 \cdot 2^\delta .$$

Let $\mathfrak{P}$ be a Sylow 41-subgroup of $\mathfrak{K}$ so that $[\mathfrak{K} : \mathfrak{P}] \equiv 8 \cdot 2^\delta \bmod 41$. Hence if $\mathfrak{N} = N_{\mathfrak{K}}(\mathfrak{P})$, then $[\mathfrak{N} : \mathfrak{P}] \equiv 8 \cdot 2^\delta \bmod 41$. As in the $q^n = 5^2$ case we see easily that $\mathfrak{P}$ is self-centralizing so $\mathfrak{N}/\mathfrak{P} \subseteqq \operatorname{Aut} \mathfrak{P}$ and $[\mathfrak{N} : \mathfrak{P}] \mid 40$. Since $2^\delta \leqq 4$ this yields $2^\delta = 1$ and $[\mathfrak{N} : \mathfrak{P}] = 8$.

The fact that $2^\delta = 1$ implies that $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}$ is normal in $\mathfrak{K}_{\infty 0}$ and $[\mathfrak{K}_{\infty 0} : \mathfrak{E}] = 5$. Since $\mathfrak{N}/\mathfrak{P}$ is cyclic, let $\mathfrak{L} = \langle L \rangle$ be a subgroup of $\mathfrak{N}$ of order 8. $\mathfrak{L}$ permutes the two orbits of $\mathfrak{P}$. If it fixes each then L clearly has fixed points in each orbit. Thus some conjugate of L is contained in $\mathfrak{K}_{\infty 0}$, a contradiction since $\mathfrak{E} \cong \mathfrak{D}\mathfrak{Q}$ has period 4. Thus $\mathfrak{L}$ interchanges the two orbits. This implies easily that L is a product of ten 8-cycles and one transposition. Hence L is an odd permutation, a contradiction. This completes the proof of the theorem.

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