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NEGATIVE SECTIONAL CURVATURE OF A RIEMANNIAN
SPACE**

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The main results of this paper can be stated as follows. Let M_1, M_2 be two big open submanifolds of the Riemannian manifolds (R_1^2, h_1) and (R_2^2, h_2) , respectively. The submanifolds M_1, M_2 with the metrics h_1/M_1 and h_2/M_2 , respectively, have positive constant sectional curvature. We have constructed a special I-parameter family of Riemannian metrics $d(t)$ on $M_1 \times M_2$ which is the deformation of the product metric $h_1/M_1 \times h_2/M_2$ and it has strictly positive sectional curvature. In other words, we have proved that $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature with respect to the parameter t for $t = 0$ and for any plane which is spanned by $X \in (M_1)_p$ and $Y \in (M_2)_p$ is strictly positive.

Let S^2 be a two-dimensional sphere with the canonical metric g whose sectional curvature is positive constant. Consider the product of two manifolds $S^2 \times S^2$. It is not known, ([1], p. 287), ([4], p. 171), ([11], p. 106), if there exists a deformation of the metric $g \times g$ with strictly positive sectional curvature.

Let R^2 be a two-dimensional Euclidean space with the metric h induced from the canonical metric g of S^2 . It is obvious that the Riemannian manifold R^2 with the metric h has constant sectional curvature. Consider two such Riemannian manifolds (R_1^2, h_1) , (R_2^2, h_2) . The space $R_1^2 \times R_2^2$ with the metric $h_1 \times h_2$ has no negative sectional curvature. I do not know if there is a deformation of the metric $h_1 \times h_2$ whose sectional curvature is strictly positive.

1. Let R^2 be a Euclidean plane which is referred to a coordinate system (u_1, u_2) on which we obtain a metric defined by

$$h_1 = \{h_{11} = 1, h_{12} = h_{21} = 0, h_{22} = \sin^2 u_1\},$$

whose sectional curvature is positive constant 1.

Consider an open Riemannian submanifold M_1 of the Riemannian manifold (R^2, h_1) defined by

$$M_1 = \left\{ (u_1, u_2) \in R^2 : 0 < u_1 < \frac{\pi}{2}, -\infty < u_2 < \infty \right\},$$

whose metric is h_1/M_1 .

Let R_3^2 be also another Euclidean plane referred to a coordinate system (u_3, u_4) on which we take a metric defined by

$$h_2 = \{h_{33} = 1, h_{34} = h_{43} = 0, h_{44} = \sin^2 u_3\}.$$

We also consider an open Riemannian submanifold M_2 of \mathbf{R}_2^2 defined by

$$M_2 = \left\{ (u_3, u_4) \in \mathbf{R}^2 : 0 < u_3 < \frac{\pi}{2}, -\infty < u_4 < \infty \right\},$$

whose metric is h_2/M_2 .

Let $M_1 \times M_2$ be the product manifold of M_1, M_2 which is defined by

$$M_1 \times M_2 = \left\{ (u_1, u_2, u_3, u_4) \in \mathbf{R}_1^2 \times \mathbf{R}_2^2 : 0 < u_1 < \frac{\pi}{2}, \right. \\ \left. -\infty < u_2 < \infty, 0 < u_3 < \frac{\pi}{2}, -\infty < u_4 < \infty \right\}.$$

On the manifold $M_1 \times M_2$ we get a special 1-parameter family of Riemannian metrics defined by

$$(1.1) \quad d(t) = \begin{cases} d_{11} = 1 + tf_1, & d_{22} = \sin^2 u_1(1 + tf_2), \\ d_{33} = 1 + t\varphi_1, & d_{44} = \sin^2 u_3(1 + t\varphi_2), & d_{ij} = 0, \text{ if } i \neq j, \end{cases}$$

where

$f_1 = f_1(u_3, u_4), f_2 = f_2(u_3, u_4), \varphi_1 = \varphi_1(u_1, u_2), \varphi_2 = \varphi_2(u_1, u_2), -\varepsilon < t < \varepsilon,$
 ε is a small positive number.

It is obvious that $d(0) = h_1/M_1 \times h_2/M_2$.

2. Let P be any point of $M_1 \times M_2$. As is known, the sectional curvature of a plane spanned two vectors X, Y of the tangent space $(M_1 \times M_2)_P$ is given by

$$\sigma(X, Y)(t) = - \frac{\langle R(X, Y)X, Y \rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y \rangle^2}.$$

If we apply Taylor's expansion theorem for the function $\sigma(X, Y)(t)$, we get

$$\sigma(X, Y)(t) = \sigma(X, Y)(0) + \sigma'_t(X, Y)(0) \frac{t}{1} + \sigma''_t(X, Y)(0) \frac{t^2}{2!} + \dots$$

From the above formula we conclude that the sign of $\sigma(X, Y)(t)$ depends on the sign of $\sigma(X, Y)(0)$, if t is a small positive number and $\sigma(X, Y)(0) \neq 0$, but if $\sigma(X, Y) = 0$, then its sign depends on $t\sigma'_t(X, Y)(0)$.

As is known ([1], p. 287), $\sigma(X, Y)(0) = 0$, if $X \in (M_1)_P$ and $Y \in (M_2)_P$. In this case we estimate $\sigma(X, Y)(t)$ which is given by the formula

$$(2.1) \quad \sigma(X, Y)(t) = - \frac{A(t)}{B(t)},$$

where

$$(2.2) \quad \begin{aligned} A(t) = \langle R(X, Y)X, Y \rangle &= R_{1313}(X^1)^2(Y^3)^2 + R_{1414}(X^1)^2(Y^4)^2 \\ &+ R_{2323}(X^2)^2(Y^3)^2 + R_{2424}(X^2)^2(Y^4)^2 + 2R_{1323}X^1X^2(Y^3)^2 \\ &+ 2R_{1314}(X^1)^2Y^3Y^4 + 2R_{2324}(X^2)^2Y^3Y^4 + 2R_{1424}X^1X^2(Y^4)^2 \\ &+ 2(R_{1324} + R_{1423})X^1X^2Y^3Y^4 . \end{aligned}$$

$$(2.3) \quad B(t) = \{d_{11}(X^1)^2 + d_{22}(X^2)^2\}\{d_{33}(Y^3)^2 + d_{44}(Y^4)^2\} > 0 ,$$

because, in this case, $\langle X, Y \rangle = 0$.

From relation (2.1), we obtain

$$\sigma(X, Y)(0) = -\frac{A(0)}{B(0)} = 0 ,$$

or

$$(2.4) \quad A(0) = 0 .$$

If we differentiate the same relation (2.1) with respect to t , we obtain

$$\sigma'_i(X, Y)(0) = -\frac{A'(0)B(0) - A(0)B'(0)}{B^2(0)} ,$$

which, by virtue of (2.4), takes the form

$$(2.5) \quad \sigma'_i(X, Y)(0) = -\frac{A'(0)}{B(0)} .$$

From the formula (2.2), we obtain

$$(2.6) \quad \begin{aligned} A'(0) &= R'_{1313}(0)(X^1)^2(Y^3)^2 + R'_{2323}(0)(X^2)^2(Y^3)^2 + R'_{1414}(0)(X^1)^2(Y^4)^2 \\ &+ R'_{2424}(0)(X^2)^2(Y^4)^2 + 2R'_{1323}(0)X^1X^2(Y^3)^2 + 2R'_{1314}(0)(X^1)^2Y^3Y^4 \\ &+ 2R'_{2324}(0)(X^2)^2Y^3Y^4 + 2R'_{1424}(0)X^1X^2(Y^4)^2 \\ &+ 2\{R'_{1324}(0) + R'_{1423}(0)\}X^1X^2Y^3Y^4 . \end{aligned}$$

We shall estimate the coefficients of the Riemannian tensor which appear in the formula (2.6). As is known, R_{ijkl} is given by ([18], p. 18)

$$(2.7) \quad \begin{aligned} R_{ijkl} &= \frac{1}{2} \left\{ \frac{\partial^2 d_{ik}}{\partial u_j \partial u_l} + \frac{\partial^2 d_{jl}}{\partial u_i \partial u_k} - \frac{\partial^2 d_{jk}}{\partial u_i \partial u_l} - \frac{\partial^2 d_{il}}{\partial u_j \partial u_k} \right\} \\ &- d_{rs} \left\{ \Gamma_{jk}^r \Gamma_{il}^s - \Gamma_{ji}^r \Gamma_{ik}^s \right\} , \end{aligned}$$

where $\Gamma_{jk}^r, \Gamma_{il}^s, \Gamma_{ji}^r, \Gamma_{ik}^s$ are the Christoffel symbols of second kind.

From (1.1) and (2.7), if we make the calculations, we obtain

$$\begin{aligned}
R_{1313} &= \frac{t}{2} \left(\frac{\partial^2 f_1}{\partial u_3^2} + \frac{\partial^2 \varphi_1}{\partial u_1^2} \right) - \frac{t^2}{4} \left\{ \frac{(\partial f_1 / \partial u_3)^2}{1 + t f_1} + \frac{(\partial \varphi_1 / \partial u_1)^2}{1 + t \varphi_1} \right\}, \\
R_{1414} &= \frac{t}{2} \left(\frac{\partial^2 f_1}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_1^2} + \frac{\sin 2u_3 (\partial f_1 / \partial u_3)}{2(1 + t \varphi_1)} \right) \\
&\quad - \frac{t^2}{4} \left\{ \frac{(\partial f_1 / \partial u_4)^2}{1 + t f_1} + \frac{\sin^2 u_3 (\partial \varphi_2 / \partial u_1)^2}{1 + t f_2} - \frac{\sin 2u_3 (\partial f_1 / \partial u_3) \varphi_2}{1 + t \varphi_1} \right\}, \\
R_{2323} &= \frac{t}{2} \left(\sin^2 u_1 \frac{\partial^2 f_2}{\partial u_3^2} + \frac{\partial^2 \varphi_1}{\partial u_2^2} + \frac{\sin 2u_1 (\partial \varphi_1 / \partial u_1)}{2(1 + t f_1)} \right) \\
&\quad - \frac{t^2}{4} \left\{ \frac{\sin^2 u_1 (\partial f_2 / \partial u_3)^2}{1 + t f_2} + \frac{(\partial \varphi_1 / \partial u_2)^2}{1 + t \varphi_1} - \frac{\sin 2u_1 f_2 (\partial \varphi_1 / \partial u_1)}{1 + t f_1} \right\}, \\
R_{2424} &= \frac{t}{2} \left(\sin^2 u_1 \frac{\partial^2 f_2}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_2^2} + \frac{\sin 2u_1 \sin^2 u_3 (\partial \varphi_2 / \partial u_1)}{2(1 + t f_1)} \right) \\
(2.8) \quad &+ \frac{\sin 2u_3 \sin^2 u_1 (\partial f_2 / \partial u_3)}{2(1 + t \varphi_1)} - \frac{t^2}{4} \left\{ \frac{\sin^2 u_1 (\partial f_2 / \partial u_4)^2}{1 + t f_2} \right. \\
&+ \frac{\sin^2 u_3 (\partial \varphi_2 / \partial u_3)^2}{1 + t \varphi_2} - \frac{\sin 2u_1 \sin^2 u_3 f_2 (\partial \varphi_2 / \partial u_1)}{1 + t f_1} \\
&\left. - \frac{\sin 2u_3 \sin^2 u_1 \varphi_2 (\partial f_2 / \partial u_3)}{1 + t \varphi_1} \right\},
\end{aligned}$$

$$\begin{aligned}
R_{1323} &= \frac{t}{2} \left(\frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2} - 2 \frac{\cos u_1}{\sin u_1} \frac{\partial \varphi_1}{\partial u_2} \right) - \frac{t^2}{4} \frac{(\partial \varphi_1 / \partial u_1) (\partial \varphi_1 / \partial u_2)}{1 + t \varphi_1}, \\
R_{1314} &= \frac{t}{2} \left(\frac{\partial^2 f_1}{\partial u_3 \partial u_4} - 2 \frac{\cos u_3}{\sin u_3} \frac{\partial f_1}{\partial u_4} \right) - \frac{t^2}{4} \frac{(\partial f_1 / \partial u_3) (\partial f_1 / \partial u_4)}{1 + t f_1}, \\
R_{2324} &= \frac{t}{2} \sin^2 u_1 \left(\frac{\partial^2 f_2}{\partial u_3 \partial u_4} - \frac{\cos u_3}{\sin u_3} \frac{\partial f_2}{\partial u_4} \right) \\
(2.9) \quad &- \frac{t^2}{4} \frac{\sin^2 u_1 (\partial f_2 / \partial u_3) (\partial f_2 / \partial u_4)}{1 + t f_2}, \\
R_{1424} &= \frac{t}{2} \sin^2 u_3 \left(\frac{\partial^2 \varphi_2}{\partial u_1 \partial u_2} - \frac{\cos u_1}{\sin u_1} \frac{\partial \varphi_2}{\partial u_2} \right) \\
&- \frac{t^2}{4} \frac{\sin^2 u_3 (\partial \varphi_2 / \partial u_1) (\partial \varphi_2 / \partial u_2)}{1 + t \varphi_2}.
\end{aligned}$$

$$(2.10) \quad R_{1324} = R_{1423} = 0.$$

If we choose the functions φ_1 , f_1 , f_2 , φ_2 such that they satisfy the partial differential equations

$$(2.11) \quad \begin{aligned} \frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2} - 2 \frac{\cos u_1}{\sin u_1} \frac{\partial \varphi_1}{\partial u_2} &= 0, \\ \frac{\partial^2 f_1}{\partial u_3 \partial u_4} - 2 \frac{\cos u_3}{\sin u_3} \frac{\partial f_1}{\partial u_4} &= 0, \\ \frac{\partial^2 f_2}{\partial u_3 \partial u_4} - \frac{\cos u_3}{\sin u_3} \frac{\partial f_2}{\partial u_4} &= 0, \\ \frac{\partial^2 \varphi_2}{\partial u_1 \partial u_2} - \frac{\cos u_1}{\sin u_1} \frac{\partial \varphi_2}{\partial u_2} &= 0, \end{aligned}$$

then the formulas (2.9) take the form

$$(2.12) \quad \begin{aligned} R_{1323} &= -\frac{t^2}{4} \frac{(\partial \varphi_1 / \partial u_1)(\partial \varphi_1 / \partial u_2)}{1 + t \varphi_1}, \\ R_{1314} &= -\frac{t^2}{4} \frac{(\partial f_1 / \partial u_3)(\partial f_1 / \partial u_4)}{1 + t f_1}, \\ R_{2324} &= -\frac{t^2}{4} \frac{\sin^2 u_1 (\partial f_2 / \partial u_3)(\partial f_2 / \partial u_4)}{1 + t f_2}, \\ R_{1424} &= -\frac{t^2}{4} \frac{\sin^2 u_3 (\partial \varphi_2 / \partial u_1)(\partial \varphi_2 / \partial u_2)}{1 + t \varphi_2}. \end{aligned}$$

From the relations (2.8) and (2.12) we obtain

$$(2.13) \quad \begin{aligned} R'_{1313}(0) &= \frac{1}{2} \left(\frac{\partial^2 f_1}{\partial u_3^2} + \frac{\partial^2 \varphi_1}{\partial u_1^2} \right), \\ R'_{1414}(0) &= \frac{1}{2} \left(\frac{\partial^2 f_1}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_1^2} + \frac{\sin 2u_3}{2} \frac{\partial f_1}{\partial u_3} \right), \\ R'_{2323}(0) &= \frac{1}{2} \left(\frac{\partial^2 \varphi_1}{\partial u_1^2} + \sin^2 u_1 \frac{\partial^2 f_2}{\partial u_3^2} + \frac{\sin 2u_1}{2} \frac{\partial \varphi_1}{\partial u_1} \right), \\ R'_{2424}(0) &= \frac{1}{2} \left(\sin^2 u_1 \frac{\partial^2 f_2}{\partial u_4^2} + \sin^2 u_3 \frac{\partial^2 \varphi_2}{\partial u_2^2} + \frac{\sin 2u_1 \sin^2 u_3}{2} \frac{\partial \varphi_2}{\partial u_1} \right. \\ &\quad \left. + \frac{\sin 2u_3 \sin^2 u_1}{2} \frac{\partial f_2}{\partial u_3} \right). \end{aligned}$$

$$(2.14) \quad R'_{1323}(0) = R'_{1314}(0) = R'_{2324}(0) = R'_{1424}(0) = 0.$$

The first partial differential equation of (2.11) can be written

$$\frac{\partial^2 \varphi_1}{\partial u_1 \partial u_2} - \frac{\partial}{\partial u_1} \log \sin^2 u_1 \frac{\partial \varphi_1}{\partial u_2} = 0,$$

or

$$\frac{\partial^2 \varphi_1 / \partial u_1 \partial u_2}{\partial \varphi_1 / \partial u_2} = \frac{\partial}{\partial u_1} \log \sin^2 u_1,$$

or

$$\frac{\partial \varphi_1}{\partial u_2} = Z(u_2) \sin^2 u_1,$$

whose general solution is

$$(2.15) \quad \varphi_1 = V_1(u_2) \sin^2 u_1 + T_1(u_1) ,$$

where $V_1(u_2)$ and $T_1(u_1)$ are arbitrary functions of u_2 and u_1 , respectively.

We can find the general solutions of the rest of partial differential equations (2.11) in the same way. The general solutions of these equations are

$$(2.16) \quad \begin{aligned} f_1 &= \sin^2 u_3 \lambda_1(u_4) + \mu_1(u_3) , \\ \varphi_2 &= \sin u_1 V_2(u_2) + T_2(u_1) , \\ f_2 &= \sin u_3 \lambda_2(u_4) + \mu_2(u_3) , \end{aligned}$$

where $\lambda_1(u_4)$, $\mu_1(u_3)$, $V_2(u_2)$, $T_2(u_1)$, $\lambda_2(u_4)$, $\mu_2(u_3)$ are arbitrary functions of u_4 , u_3 , u_2 , u_1 , u_4 , u_3 , respectively.

The formulas (2.13) by virtue of (2.15) and (2.16) take the form

$$(2.17) \quad \begin{aligned} R'_{1313}(0) &= \frac{1}{2} \left\{ 2 \cos 2u_1 V_1(u_2) + T_1''(u_1) \right\} + \frac{1}{2} \left\{ 2 \cos 2u_3 \lambda_1(u_4) \right. \\ &\quad \left. + \mu_1''(u_3) \right\} , \\ R'_{1414}(0) &= \frac{1}{2} \left\{ \sin^2 u_3 (\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2} \lambda_1(u_4) \right. \\ &\quad \left. + \frac{\sin 2u_3}{2} \mu_1'(u_3) - \sin^2 u_3 \sin u_1 V_2(u_2) \right\} , \\ R'_{2323}(0) &= \frac{1}{2} \left\{ \sin^2 u_1 (\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2} V_1(u_2) \right. \\ &\quad \left. + \frac{\sin 2u_1}{2} T_1'(u_1) - \sin^2 u_1 \sin u_3 \lambda_2(u_4) \right\} , \\ R'_{2424}(0) &= \frac{\sin^2 u_1 \sin u_3}{2} \left\{ \lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) \right. \\ &\quad \left. + \frac{\sin u_1 \sin^2 u_3}{2} \left\{ V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2) \right\} \right\} . \end{aligned}$$

The relation (2.6) by means of (2.10) and (2.14) takes the form

$$(2.18) \quad \begin{aligned} A'(0) &= R'_{1313}(0)(X^1)^2(Y^3)^2 + R'_{2323}(0)(X^2)^2(Y^3)^2 + R'_{1414}(0)(X^1)^2(Y^4)^2 \\ &\quad + R'_{2424}(0)(X^2)^2(Y^4)^2 . \end{aligned}$$

In order that $\sigma'(X, Y)(0) = -A'(0)/B(0)$ be positive on the Riemannian manifold $M_1 \times M_2$, it must be

$$(2.19) \quad A'(0) < 0 .$$

From the formula (2.18) we conclude that (2.19) is valid when we have

$$R'_{1313}(0) < 0, \quad R'_{1414}(0) < 0, \quad R'_{2323}(0) < 0, \quad R'_{2424}(0) < 0,$$

which, by virtue of (2.17), take the form

$$\begin{aligned} & \frac{1}{2}\{2 \cos 2u_1 V_1(u_2) + T_1''(u_1)\} + \frac{1}{2}\{2 \cos 2u_3 \lambda_1(u_4) + \mu_1''(u_3)\} < 0, \\ & \frac{1}{2}\left\{\sin^2 u_3 (\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2} \lambda_1(u_4) + \frac{\sin 2u_3}{2} \mu_1'(u_3) \right. \\ & \quad \left. - \sin^2 u_3 \sin u_1 V_2(u_2)\right\} < 0, \\ & \frac{1}{2}\left\{\sin^2 u_1 (\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2} V_1(u_2) + \frac{\sin 2u_1}{2} T_1'(u_1) \right. \\ & \quad \left. - \sin^2 u_1 \sin u_3 \lambda_2(u_4)\right\} < 0, \\ & \frac{\sin^2 u_1 \sin u_3}{2}\left\{\lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4)\right\} \\ & \quad + \frac{\sin u_1 \sin^2 u_3}{2}\left\{V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2)\right\} < 0, \end{aligned}$$

which must be valid on the Riemannian manifold $M_1 \times M_2$.

The above inequalities hold if we have

$$\begin{aligned} & 2 \cos 2u_1 V_1(u_2) + T_1''(u_1) < 0, \\ (2.20) \quad & \sin^2 u_1 (\mu_2''(u_3) + V_1''(u_2)) + \frac{\sin^2 2u_1}{2} V_1(u_2) + \frac{\sin 2u_1}{2} T_1'(u_1) \\ & \quad - \sin^2 u_1 \sin u_3 \lambda_2(u_4) < 0, \\ & \lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) < 0, \end{aligned}$$

$$\begin{aligned} & 2 \cos 2u_3 \lambda_1(u_4) + \mu_1''(u_3) < 0, \\ (2.21) \quad & \sin^2 u_3 (\lambda_1''(u_4) + T_2''(u_1)) + \frac{\sin^2 2u_3}{2} \lambda_1(u_4) + \frac{\sin 2u_3}{2} \mu_1'(u_3) \\ & \quad - \sin^2 u_3 \sin u_1 V_2(u_2) < 0, \\ & V_2''(u_2) + \cos u_1 T_2'(u_1) + \cos^2 u_1 V_2(u_2) < 0. \end{aligned}$$

The inequalities (2.21) are similar to the inequalities (2.20); for this reason we shall only study the inequalities (2.20).

The factor $\cos 2u_1$ changes sign when $0 < u_1 < 2/\pi$; from this and from the fact that $V_1(u_2)$ and $V_1''(u_2)$ must have constant sign and bounded when $-\infty < u_2 < \infty$, we conclude that $V_1(u_2)$ must be a constant negative number $-\alpha$.

From the above remark, the inequalities (2.20) take the form

$$\begin{aligned}
& -2\alpha \cos 2u_1 + T_1''(u_1) < 0, \\
(2.22) \quad & \sin^2 u_1 \mu_2''(u_3) - \alpha \frac{\sin^2 2u_1}{2} + \frac{\sin 2u_1}{2} T_1'(u_1) - \sin^2 u_1 \sin u_3 \lambda_2(u_4) < 0, \\
& \lambda_2''(u_4) + \cos u_3 \mu_2'(u_3) + \cos^2 u_3 \lambda_2(u_4) < 0.
\end{aligned}$$

In order for the second and the third inequalities of (2.22) to be valid, the function $\lambda_2(u_4)$ must be a positive constant number β .

Therefore the above inequalities become

$$\begin{aligned}
& -2\alpha \cos 2u_1 + T_1''(u_1) < 0, \\
(2.23) \quad & \sin^2 u_1 \mu_2''(u_3) - \frac{\alpha \sin^2 2u_1}{2} + \frac{\sin 2u_1}{2} T_1'(u_1) - \beta \sin^2 u_1 \sin u_3 < 0, \\
& \mu_2'(u_3) + \beta \cos u_3 < 0.
\end{aligned}$$

If the functions $T_1(u_1)$, $\mu_2(u_3)$ are chosen such that

$$\begin{aligned}
& T_1'(u_1) < 0, \quad \max\{T_1''(u_1)\} < -2\alpha, \quad 0 < u_1 < \frac{\pi}{2}, \\
& \max\{\mu_2'(u_3)\} < -\beta, \quad \mu_2''(u_3) < 0, \quad 0 < u_3 < \frac{\pi}{2},
\end{aligned}$$

then the inequalities (2.23) hold.

We also conclude that if the functions $\lambda_1(u_4)$, $V_2(u_2)$, $\mu_1(u_3)$, $T_2(u_1)$ satisfy the conditions

$$\begin{aligned}
& \lambda_1(u_4) = -\gamma, \quad V_2(u_2) = \delta, \\
& \mu_1'(u_3) < 0, \quad \max\{\mu_1''(u_3)\} < -2\gamma, \quad 0 < u_3 < \frac{\pi}{2}, \\
& \max\{T_2'(u_1)\} < -\delta, \quad T_2''(u_1) < 0, \quad 0 < u_1 < \frac{\pi}{2},
\end{aligned}$$

then the inequalities (2.21) hold.

Therefore, if the functions $\varphi_1, f_1, \varphi_2, f_2$ have the form

$$\begin{aligned}
(2.24) \quad & \varphi_1 = -\alpha \sin^2 u_1 + T_1(u_1), \quad \alpha > 0, \\
& f_1 = -\gamma \sin^2 u_3 + \mu_1(u_3), \quad \gamma > 0, \\
& \varphi_2 = \delta \sin u_1 + T_2(u_1), \quad \delta > 0, \\
& f_2 = \beta \sin u_3 + \mu_2(u_3), \quad \beta > 0,
\end{aligned}$$

such that the functions $T_1(u_1)$, $\mu_1(u_3)$, $T_2(u_1)$ and $\mu_2(u_3)$ satisfy the conditions

$$\begin{aligned}
 & T_1'(u_1) < 0, \quad \max\{T_1''(u_1)\} < -2\alpha, \quad 0 < u_1 < \frac{\pi}{2}, \\
 & \max\{\mu_2'(u_3)\} < -\beta, \quad \mu_2''(u_3) < 0, \quad 0 < u_3 < \frac{\pi}{2}, \\
 (2.25) \quad & \mu_1'(u_3) < 0, \quad \max\{\mu_1''(u_3)\} < -2\gamma, \quad 0 < u_3 < \frac{\pi}{2}, \\
 & \max\{T_2'(u_1)\} < -\delta, \quad T_2''(u_1) < 0, \quad 0 < u_1 < \frac{\pi}{2},
 \end{aligned}$$

then $\sigma'_i(X, Y)(0) > 0$ for $X \in (M_1)_P, Y \in (M_2)_P$.

Hence we have the following theorem.

THEOREM. *Let M_1, M_2 be two Riemannian spaces with positive constant sectional curvature defined in §1. If we consider a special 1-parameter family of Riemannian metrics $d(t)$ on $M_1 \times M_2$ defined by (1.1) where the functions $f_1, f_2, \varphi_1, \varphi_2$ have the form (2.24) in which the functions $T_1(u_1), \mu_1(u_3), T_2(u_1)$ and $\mu_2(u_3)$ must satisfy the conditions (2.25), then $\forall P \in M_1 \times M_2$ the derivative of the sectional curvature of any plane spanned by $X \in (M_1)_P$ and $Y \in (M_2)_P$ with respect to t for $t = 0$ is strictly positive.*

From the above, we conclude that if the parameter t is positive and small enough, then the corresponding Riemannian metric $d(t)$ defined by (1.1) on $M_1 \times M_2$, where the functions $f_1, f_2, \varphi_1, \varphi_2$ have the form (2.24) in which the functions $T_1(u_1), \mu_1(u_3), T_2(u_1)$ and $\mu_2(u_3)$ must satisfy the conditions (2.25), has strictly positive sectional curvature.

3. We can extend the manifold $M_1 \times M_2$ to a manifold

$$N_1 \times N_2 \supset M_1 \times M_2$$

such that there is a deformation of another product metric on $N_1 \times N_2$ which has strictly positive sectional curvature.

This method can be stated as follows. On the Euclidean plane \mathbf{R}_2^2 we obtain a metric which is given by

$$\omega_1 = \left\{ \omega_{11} = 1, \quad \omega_{12} = \omega_{21} = 0, \quad \omega_{22} = \sin^2 \frac{u_1}{n} \right\},$$

where n is an integer > 1 . The sectional curvature of this metric is $1/n^2$.

Now, consider an open Riemannian submanifold N_1 of the Riemannian manifold $(\mathbf{R}_1^2, \omega_1)$ defined by

$$N_1 = \{(u_1, u_2) \in \mathbf{R}_1^2 : 0 < u_1 < n \frac{\pi}{2}, -\infty < u_2 < \infty\},$$

whose metric is ω_1/N_1 .

Similarly, on the Euclidean plane \mathbf{R}_2^2 , we obtain a metric which is given by

$$\omega_2 = \left\{ \omega_{33} = 1, \quad \omega_{34} = \omega_{43} = 0, \quad \omega_{44} = \sin^2 \frac{u_3}{n} \right\},$$

whose sectional curvature is $1/n^2$.

Let N_2 be an open Riemannian submanifold of the Riemannian manifold $(\mathbf{R}_2^2, \omega_2)$ which is defined by

$$N_2 = \{(u_3, u_4) \in \mathbf{R}_2^2 : 0 < u_3 < n \frac{\pi}{2}, \quad -\infty < u_4 < \infty\},$$

whose metric is ω_2/N_2 .

We consider the product manifold $N_1 \times N_2$ of N_1, N_2 defined by

$$N_1 \times N_2 = \{(u_1, u_2, u_3, u_4) \in \mathbf{R}_1^2 \times \mathbf{R}_2^2 : 0 < u_1 < n \frac{\pi}{2}, \quad -\infty < u_2 < \infty, \\ 0 < u_3 < n \frac{\pi}{2}, \quad -\infty < u_4 < \infty\}.$$

It is obvious that $(N_1 \times N_2) \supset (M_1 \times M_2)$ and with the same technique as in §2 we can prove that there is a deformation of the metric $\omega_1/N_1 \times \omega_2/N_2$ which has strictly positive sectional curvature on the manifold $N_1 \times N_2$.

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Pacific Journal of Mathematics

Vol. 29, No. 3

July, 1969

Herbert James Alexander, <i>Extending bounded holomorphic functions from certain subvarieties of a polydisc</i>	485
Edward T. Cline, <i>On an embedding property of generalized Carter subgroups</i>	491
Roger Cuppens, <i>On the decomposition of infinitely divisible characteristic functions with continuous Poisson spectrum. II</i>	521
William Richard Emerson, <i>Translation kernels on discrete Abelian groups</i>	527
Robert William Gilmer, Jr., <i>Power series rings over a Krull domain</i>	543
Julien O. Hennefeld, <i>The Arens products and an imbedding theorem</i>	551
James Secord Howland, <i>Embedded eigenvalues and virtual poles</i>	565
Bruce Ansgar Jensen, <i>Infinite semigroups whose non-trivial homomorphs are all isomorphic</i>	583
Michael Joseph Kascic, Jr., <i>Polynomials in linear relations. II</i>	593
J. Gopala Krishna, <i>Maximum term of a power series in one and several complex variables</i>	609
Renu Chakravarti Laskar, <i>Eigenvalues of the adjacency matrix of cubic lattice graphs</i>	623
Thomas Anthony Mc Cullough, <i>Rational approximation on certain plane sets</i>	631
T. S. Motzkin and Ernst Gabor Straus, <i>Divisors of polynomials and power series with positive coefficients</i>	641
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix.</i>	653
Graciano de Oliveira, <i>Matrices with prescribed characteristic polynomial and a prescribed submatrix. II</i>	663
Donald Steven Passman, <i>Exceptional 3/2-transitive permutation groups</i>	669
Grigorios Tsagas, <i>A special deformation of the metric with no negative sectional curvature of a Riemannian space</i>	715
Joseph Zaks, <i>Trivially extending decompositions of E^n</i>	727