TRIVIALLY EXTENDING DECOMPOSITIONS OF $E^n$

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Let $G$ be a monotone decomposition of $E^n$, then $G$ can be extended in a trivial way, to the monotone decomposition $G^1$ of $E^{n+1}$, where $E^n = \{(x_1, \cdots, x_n, 0) \in E^{n+1}\}$, by adding to $G$ all points of $E^{n+1} - E^n$. If the decomposition space $E^n/G$ of $G$ is homeomorphic to $E^n$, $E^n/G$ is said to be obtained by a pseudo-isotopy if there exists a map $F: E^n \times I \to E^n \times I$, such that $F_t(=F|E^n \times t)$ is homeomorphism onto $E^n \times t$, for all $0 \leq t < 1$, $F_0$ is the identity and $F_1$ is equivalent to the projection $E^n \to E^n/G$.

The purpose of this paper is to present a relation between these two notions. It will then follow, that if $G$ is the decomposition of $E^3$ to points, circles and figure-eights, due to R. H. Bing, for which $E^3/G$ is homeomorphic to $E^3$, then $E^3/G^1$ is not homeomorphic to $E^3$.

Moreover, we will present a direct, geometric proof to this particular property.

For definitions, see [1]. See also [2].

**Theorem 1.** If $G$ is a monotone decomposition of $E^n$, such that $E^n/G$ is homeomorphic to $E^n$, then the following are equivalent:

1. $E^{n+1}/G^1$ is homeomorphic to $E^{n+1}$.
2. $E^n/G$ can be obtained by a pseudo-isotopy.

**Proof.** (1) $\Rightarrow$ (2). Let $h: E^{n+1}/G^1 \to E^{n+1}$ be a homeomorphism and let $p: E^{n+1} \to E^{n+1}/G^1$ be the projection map.

The map $H: E^n \times I \to E^{n+1}$, defined by $H(x, t) = h(p(x, 1 - t))$ for all $x \in E^n$, $t \in I$, is such that $H_t$ is a homeomorphism into for all $0 \leq t < 1$, $H_1$ is equivalent to the projection map $E^n \to E^n/G$, and $H(E^n \times I)$ is homeomorphic to $E^n \times I$, hence, up to a homeomorphism of $E^n \times I$ onto itself, $H$ is the required pseudo-isotopy.

(2) $\Rightarrow$ (1). Let $F: E^n \times I \to E^n \times I$ be the pseudo-isotopy for $E^n/G$. The map $H: E^{n+1} \to E^{n+1}$, where

\[
H(x, t) = \begin{cases} 
F(x, 1 + t) & -1 \leq t \leq 0 \\
F(x, 1 - t) & 0 \leq t \leq 1 \\
(x, t) & t \geq 1 \text{ or } t \leq -1
\end{cases}
\]

where $x \in E^n$.

is well defined, $H(E^{n+1}) = E^{n+1}$, and $H(E^{n+1})$ is homeomorphic to $E^{n+1}/G^1$, because $H_0 = F_1$ and it is equivalent to the projection map $E^n$ onto $E^n/G$. The proof is completed.
Using Theorem 1 of [4], we have the following

**COROLLARY.** If $G$ is a monotone decomposition of $E^2$, such that $E^2/G$ is homeomorphic to $E^2$, then $E^3/G^1$ is homeomorphic to $E^3$.

It is well known that the decomposition $G$ of $E^3$ to points, circles and figure-eights, as described in §4 of [3], is such that $E^3/G$ is homeomorphic to $E^3$ but $E^3/G$ cannot be obtained by a pseudo-isotopy, see [1] and [2]. Therefore, it follows from Theorem 1 that this $G$ has the property that $E^3/G^1$ is not homeomorphic to $E^4$; see our remark at the end of this paper.

However, we would like to present a direct proof for

**THEOREM 2.** Let $G$ be the decomposition of $E^3$, as described in §4 of [3], then $E^3/G^1$ is not homeomorphic to $E^4$.

*Proof.* Suppose it is not true, then let $h: E^3/G^1 \rightarrow E^4$ be a homeomorphism, and $p: E^4 \rightarrow E^4/G^1$ be the projection map.

Let $f$ be the map of the complete 2-complex, $C^2_7$, with 7 vertices, into $E^4$, which is affine on each triangle of $C^2_7$, and is almost an embedding, except for its effect $f(P) = f(Q)$ for two points $P$ and $Q$ of $C^2_7$, where $P$ and $Q$ are points in the relative interior of two disjoint (in $C^3_2$) triangles $A$ and $B$, respectively. $f$ is described in [6], see also [7].

Without loss of generality we may assume, as we do, that $f(A) \subset E^3 \subset E^4$, and $f(P) = f(Q) =$ the origin. Therefore $f(B)$ has in $E^3$ only an edge $l$, passing through the origin, as described in Figure 1, where we also describe the two disks, which are the union of all the circles and figure-eights of $G$. In order that the disk of $G$, which is perpendicular to $f(A)$, will not meet $f(A)$ except in the common radius of the two disks of $G$, we push, continuously and without touching the rest of $f(C^2_7)$, the interior of the disk $D$, which is contained in $f(A)$, so that it will have small positive values in the 4-th coordinate.

By doing this, we defined the two disks to lie in $E^3$, therefore we get an equivalent decomposition to that of §4 of [3], which we denote again by $G$, and we let $G^1$ be its extension to $E^4$.

The set $p f(A \cup B)$ in $E^4/G^1$ is homeomorphic to the union of two disjoint disks, together with a simple arc $\alpha$ joining an interior point of one disk to an interior point of the other. Therefore, $[p f(C^2_7)]$-interior $\alpha$ is homeomorphic to $C^2_7$ in $E^4/G^1$, and since $h$ is supposed to be a homeomorphism, $h[p f(C^2_7)]$-interior $\alpha$ is a subset of $E^4$, homeomorphic to $C^2_7$.

This contradicts a well known result of A. Flores, [5], therefore
In fact, $E^4/G^1$ is even not embeddable in $E^4$, (same proof).

**REMARK.** Theorem 2 was proved by M. M. Cohen in his "Simplicial structures and transverse cellularity", Ann. of Math. 85 (1967) 218–245.

**REFERENCES**


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